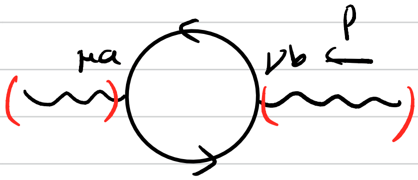
The one-loop diagrams: evaluation in DR



$$= \mu_{DR}^{4-d}\int\frac{d^dk}{(2\pi)^d}\,\mathrm{tr}_{V_f\otimes S}\left(\gamma^\mu e^a \xleftarrow{k} \gamma^\nu e^b \xleftarrow{k-p}\right)$$

$$= \mu_{DR}^{4-d}\int\frac{d^dk}{(2\pi)^d}\,\mathrm{tr}_{V_f\otimes S}\left(\gamma^\mu e^a \frac{1}{-\cancel{k}+m_f}\,\gamma^\nu e^a \frac{1}{-(\cancel{k}-\cancel{p})+m_f}\right)$$

$$= \underbrace{\mathrm{tr}_{V_f}(e^a e^b)}_{-T_{V_f}\delta^{ab}}\;\mu_{DR}^{4-d}\int\frac{d^dk}{(2\pi)^d}\,\frac{\mathrm{tr}_S\left(\gamma^\mu(\cancel{k}+m_f)\gamma^\nu((\cancel{k}-\cancel{p})+m_f)\right)}{(k^2+m_f^2)((k-p)^2+m_f^2)}$$

$$\text{numerator} = \mathrm{tr}_S(\gamma^\mu\gamma^\nu)\,m_f^2 + \mathrm{tr}_S(\gamma^\mu\cancel{k}\gamma^\nu(\cancel{k}-\cancel{p}))$$

- $\mathrm{tr}_S(\gamma^\mu\gamma^\nu) = -\delta^{\mu\nu}\,\mathrm{tr}_S(1)$
- $\mathrm{tr}_S(\gamma^\mu\gamma^\rho\gamma^\nu\gamma^\lambda) = \left(\delta^{\mu\rho}\delta^{\nu\lambda} - \delta^{\mu\nu}\delta^{\rho\lambda} + \delta^{\mu\lambda}\delta^{\rho\nu}\right)\mathrm{tr}_S(1)$

Let us use $\mathrm{tr}_S(1) = 4$. then

$$= 4\left(-\delta^{\mu\nu}(m_f^2 + k\cdot(k-p)) + k^\mu(k-p)^\nu + k^\nu(k-p)^\mu\right)$$

$$= -4T_{V_f}\delta^{ab}\,\mu_{DR}^{4-d}\int\frac{d^dk}{(2\pi)^d}\,\frac{-\delta^{\mu\nu}(m_f^2+k\cdot(k-p)) + k^\mu(k-p)^\nu + k^\nu(k-p)^\mu}{(k^2+m_f^2)((k-p)^2+m_f^2)}$$

$$= -4T_{V_f}\delta^{ab} I_{DR}\left(-\delta^{\mu\nu}\left(m_f^2 + k\cdot(k-p)\right) + k^\mu (k-p)^\nu + k^\nu (k-p)^\mu\right)$$

with $m=\mu=m_f$; $\Delta = x(1-x)p^2 + m_f^2$ below

$$= -4T_{V_f}\delta^{ab} \frac{1}{(4\pi)^2}\int_0^1 dx \left(\frac{4\pi \mu_{DR}^2}{\Delta}\right)^{2-\frac{d}{2}} \Gamma\left(2-\frac{d}{2}\right) \times$$

$$\left[\cancel{-\delta^{\mu\nu} m_f^2} \underbrace{- \delta^{\mu\nu} \widehat{k\cdot(k-p))}}_{-\delta^{\mu\nu}\frac{d\Delta}{2-d} + \delta^{\mu\nu} x(1-x)p^2} + \underbrace{\widehat{k^\mu (k-p)^\nu} + \widehat{k^\nu (k-p)^\mu}}_{2\left(\delta^{\mu\nu}\frac{\Delta}{2-d} - x(1-x)p^\mu p^\nu\right)}\right]$$

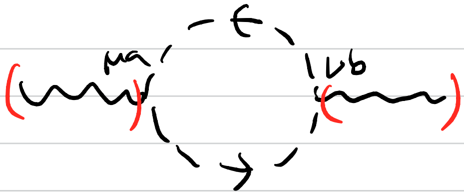
$$\delta^{\mu\nu}\Delta = \delta^{\mu\nu} x(1-x)p^2 + \cancel{\delta^{\mu\nu} m_f^2}$$

$$= -4T_{V_f}\delta^{ab} \frac{1}{(4\pi)^2}\int_0^1 dx \left(\frac{4\pi \mu_{DR}^2}{\Delta}\right)^{2-\frac{d}{2}} \Gamma\left(2-\frac{d}{2}\right) \times$$

$$\left[2\delta^{\mu\nu} x(1-x)p^2 - 2x(1-x)p^\mu p^\nu\right]$$

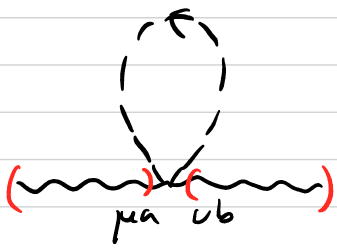
$$= -\delta^{ab}\left(\delta^{\mu\nu}p^2 - p^\mu p^\nu\right) \times$$

$$T_{V_f} \frac{1}{(4\pi)^2}\int_0^1 dx \left(\frac{4\pi \mu_{DR}^2}{\Delta}\right)^{2-\frac{d}{2}} \Gamma\left(2-\frac{d}{2}\right) 8x(1-x)$$



$$= \mu_{DR}^{4-d}\int\frac{d^dk}{(2\pi)^d}\,\mathrm{tr}_{V_b}\Big(\big(-ik^\mu - i(k-p)^\mu\big)\,e^a \xleftarrow{k} \big(-i(k-p)^\nu - ik^\nu\big)\,e^b \xleftarrow{k-p}\Big)$$

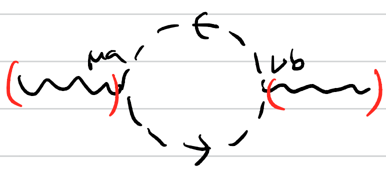
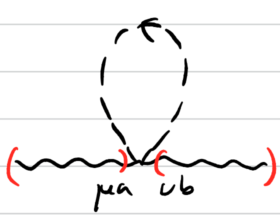
$$= -\,\mathrm{tr}_{V_b}(e^a e^b)\,\mu_{DR}^{4-d}\int\frac{d^dk}{(2\pi)^d}\,\frac{(2k-p)^\mu(2k-p)^\nu}{(k^2+m_b^2)\big((k-p)^2+m_b^2\big)}$$



$$= 2\,\mu_{DR}^{4-d}\int\frac{d^dk}{(2\pi)^d}\,\delta^{\mu\nu}\,\mathrm{tr}_{V_b}\Big(e^a e^b \xleftarrow{k}\Big)$$

$$= 2\,\mathrm{tr}_{V_b}(e^a e^b)\,\mu_{DR}^{4-d}\int\frac{d^dk}{(2\pi)^d}\,\frac{\delta^{\mu\nu}}{k^2+m_b^2} \quad \text{or} \rightsquigarrow \quad \frac{\delta^{\mu\nu}}{(k-p)^2+m_b^2}$$

$$= \mathrm{tr}_{V_b}(e^a e^b)\,\mu_{DR}^{4-d}\int\frac{d^dk}{(2\pi)^d}\left(\frac{\delta^{\mu\nu}}{k^2+m_b^2} + \frac{\delta^{\mu\nu}}{(k-p)^2+m_b^2}\right)$$

 + 

$$= \underbrace{\mathrm{tr}_{V_b}(e^a e^b)}_{-T_{V_b}\delta^{ab}}\,\mu_{DR}^{4-d}\int\frac{d^dk}{(2\pi)^d}\,\frac{-(2k-p)^\mu(2k-p)^\nu + \delta^{\mu\nu}\big((k-p)^2+m_b^2+k^2+m_b^2\big)}{(k^2+m_b^2)\big((k-p)^2+m_b^2\big)}$$

$$= - T_{V_b} \delta^{ab}\, I_{DR}\Big(-(2k-p)^\mu (2k-p)^\nu + \delta^{\mu\nu}\big((k-p)^2 + k^2 + 2m_b^2\big)\Big)$$

with $m = \mu = m_b$; $\Delta = x(1-x)p^2 + m_b^2$ below

$$= - T_{V_b}\delta^{ab} \frac{1}{(4\pi)^2}\int_0^1 dx \left(\frac{4\pi\mu_{DR}^2}{\Delta}\right)^{2-\frac{d}{2}} \Gamma\left(2-\frac{d}{2}\right) \times$$

$$\Big[\underbrace{-(\widehat{2k-p)^\mu (2k-p})^\nu}_{-4\hat{k}^\mu \hat{k}^\nu + 2\hat{k}^\mu p^\nu + 2p^\mu \hat{k}^\nu - p^\mu p^\nu} + \underbrace{\delta^{\mu\nu}\big((\widehat{k-p})^2 + \hat{k}^2}_{\delta^{\mu\nu}\left(2\frac{d\Delta}{2-d} + (1-x)^2p^2 + x^2p^2\right)} + \cancel{2m_b^2}\big)\Big]$$

$$= -4\left(\delta^{\mu\nu}\frac{\Delta}{2-d} + x^2 p^\mu p^\nu\right) + (4x-1)p^\mu p^\nu$$

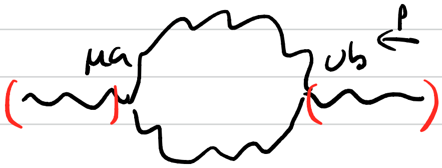
$$-2\delta^{\mu\nu}\Delta = -2\delta^{\mu\nu}x(1-x)p^2 - \cancel{2\delta^{\mu\nu}m_b^2}$$

$$\begin{cases} -4x^2 + (4x-1) = -(2x-1)^2 \\ -2x(1-x) + (1-x)^2 + x^2 = (1-x-x)^2 = (2x-1)^2 \end{cases}$$

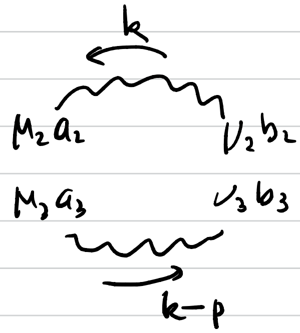
$$= - T_{V_b}\delta^{ab}\frac{1}{(4\pi)^2}\int_0^1 dx\left(\frac{4\pi\mu_{DR}^2}{\Delta}\right)^{2-\frac{d}{2}}\Gamma\left(2-\frac{d}{2}\right)(2x-1)^2\left(-p^\mu p^\nu + \delta^{\mu\nu}p^2\right)$$

$$= -\delta^{ab}\left(\delta^{\mu\nu}p^2 - p^\mu p^\nu\right) \times$$

$$T_{V_b}\frac{1}{(4\pi)^2}\int_0^1 dx\left(\frac{4\pi\mu_{DR}^2}{\Delta}\right)^{2-\frac{d}{2}}\Gamma\left(2-\frac{d}{2}\right)(2x-1)^2$$



$$= \frac{1}{2}\,\mu_{DR}^{4-d}\int\frac{d^dk}{(2\pi)^d}\, V^{\mu a\;\mu_2 a_2\;\mu_3 a_3}_{p,\,-k,\,k-p}$$



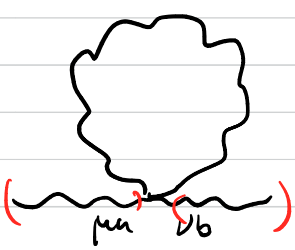
$$V^{\nu_2 b_2\;\nu_3 b_3\;\nu b}_{k,\,-(k-p),\,-p}$$

$$\overset{!}{=} \frac{1}{2}\,\mathrm{tr}_g(\mathrm{ad}\,e^a\,\mathrm{ad}\,e^b)\,\mu_{DR}^{4-d}\int\frac{d^dk}{(2\pi)^d}\,\frac{1}{k^2(k-p)^2}\times$$

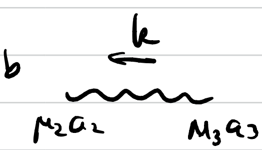
$$\Big(-\delta^{\mu\nu}(k+p)^2 + (k+p)^\mu(2k-p)^\nu - (k-2p)^\mu(k+p)^\nu$$

$$+(2k-p)^\mu(k+p)^\nu - d\,(2k-p)^\mu(2k-p)^\nu + (2k-p)^\mu(k-2p)^\nu$$

$$-(k+p)^\mu(k-2p)^\nu + (k-2p)^\mu(2k-p)^\nu - \delta^{\mu\nu}(k-2p)^2 \Big)$$

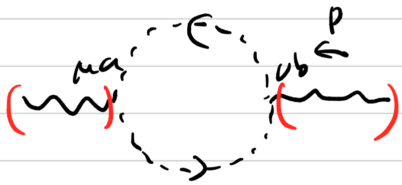


$$= \frac{1}{2}\,\mu_{DR}^{4-d}\int\frac{d^dk}{(2\pi)^d}\, V^{\mu a,\,\mu_2 a_2,\,\mu_3 a_3,\,\nu b}$$



$$\overset{!}{=} \frac{1}{2}\,\mathrm{tr}_g(\mathrm{ad}\,e^a\,\mathrm{ad}\,e^b)\,\mu_{DR}^{4-d}\int\frac{d^dk}{(2\pi)^d}\,\delta^{\mu\nu}(d-1)\,\frac{2}{k^2}$$

$$\frac{2}{k^2} \sim \frac{1}{k^2} + \frac{1}{(k-p)^2} = \frac{(k-p)^2 + k^2}{k^2(k-p)^2}$$

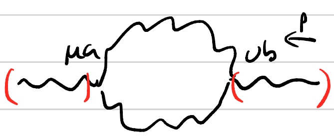


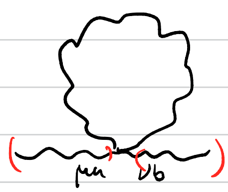
$$= {\color{red}(-1)}\, \mu_{DR}^{4-d} \int \frac{d^dk}{(2\pi)^d} \operatorname{tr}_{\mathfrak{g}}\left(\operatorname{ad} e^a \overset{k}{\dashleftarrow} i k^\nu \operatorname{ad} e^b \overset{k-p}{\dashleftarrow} i(k-p)^\mu \right)$$

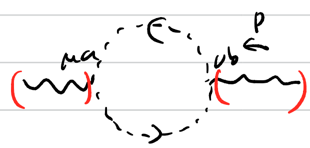
$$= \operatorname{tr}_{\mathfrak{g}}(\operatorname{ad} e^a \operatorname{ad} e^b)\, \mu_{DR}^{4-d} \int \frac{d^dk}{(2\pi)^d} \frac{(k-p)^\mu k^\nu}{k^2 (k-p)^2}$$

$$\frac{(k-p)^\mu k^\nu}{k^2(k-p)^2} \sim \frac{1}{2} \frac{(k-p)^\mu k^\nu + k^\mu (k-p)^\nu}{k^2(k-p)^2}$$

Using $\operatorname{tr}_{\mathfrak{g}}(\operatorname{ad} e^a \operatorname{ad} e^b) = -h^\vee \delta^{ab}$, we find



$+$


$+$


$$\begin{aligned} = -\frac{h^\vee}{2} \delta^{ab} I_{DR}\Big(& -\delta^{\mu\nu}(k+p)^2 + (k+p)^\mu (2k-p)^\nu - (k-2p)^\mu (k+p)^\nu \\ & + (2k-p)^\mu (k+p)^\nu - d\,(2k-p)^\mu (2k-p)^\nu + (2k-p)^\mu (k-2p)^\nu \\ & - (k+p)^\mu (k-2p)^\nu + (k-2p)^\mu (2k-p)^\nu - \delta^{\mu\nu}(k-2p)^2 \\ & + \delta^{\mu\nu}(d-1)\left((k-p)^2 + k^2\right) \\ & + (k-p)^\mu k^\nu + k^\mu (k-p)^\nu \Big) \end{aligned}$$

$$= (2-d)(2k-p)^\mu (2k-p)^\nu$$

$$= -\frac{h^\vee}{2}\delta^{ab}\, I_{DR}\Big(\delta^{\mu\nu}\big(\overbrace{-(k+p)^2-(k-2p)^2}^{-2k(k-p)-5p^2} + (d-1)(k^2+(k-p)^2)\big)$$

$$+ (2-d)(2k-p)^\mu(2k-p)^\nu + 4p^\mu p^\nu \Big)$$

with $m=\mu=0$; $\Delta = x(1-x)p^2$ below

$$= -\frac{h^\vee}{2}\delta^{ab}\frac{1}{(4\pi)^2}\int_0^1 dx\left(\frac{4\pi\mu_{DR}^2}{\Delta}\right)^{2-\frac{d}{2}}\Gamma\left(2-\frac{d}{2}\right)\times$$

$$\Big\{\Big[\delta^{\mu\nu}\Big(-2\widehat{k(k-p)} - 5p^2 + (d-1)\big(\widehat{k^2}+\widehat{(k-p)^2}\big)\Big)$$

$$+ (2-d)\widehat{(2k-p)^\mu(2k-p)^\nu} + 4p^\mu p^\nu\Big]$$

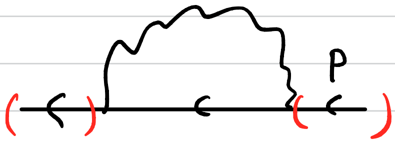
$$\delta^{\mu\nu}\left(-2\left(\frac{d\Delta}{2-d} - x(1-x)p^2\right) - 5p^2 + (d-1)\left(\frac{2d\Delta}{2-d} + (x^2+(1-x)^2)p^2\right)\right)$$

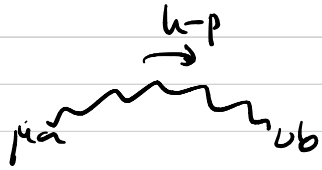
$$+ (2-d)\left(4\delta^{\mu\nu}\frac{\Delta}{2-d} + (2x-1)^2p^\mu p^\nu\right) + 4p^\mu p^\nu$$

$$= -\left(\delta^{\mu\nu}p^2 - p^\mu p^\nu\right)\left((2-d)(2x-1)^2+4\right)$$

$$= \delta^{ab}\left(\delta^{\mu\nu}p^2 - p^\mu p^\nu\right)\times$$

$$h^\vee\frac{1}{(4\pi)^2}\int_0^1 dx\left(\frac{4\pi\mu_{DR}^2}{\Delta}\right)^{2-\frac{d}{2}}\Gamma\left(2-\frac{d}{2}\right)\left(\left(1-\frac{d}{2}\right)(2x-1)^2+2\right)$$





$$= \mu_{DR}^{4-d} \int \frac{d^dk}{(2\pi)^d} \, i\gamma^\mu e^a \xleftarrow{k} i\gamma^\nu e^b$$

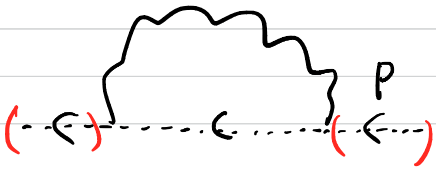
$$= \mu_{DR}^{4-d} \int \frac{d^dk}{(2\pi)^d} \, i\gamma^\mu e^a \frac{1}{-\not{k}+m_f} i\gamma^\nu e^b \times e^2 \delta_{ab} \frac{\delta_{\mu\nu}}{(k-p)^2}$$

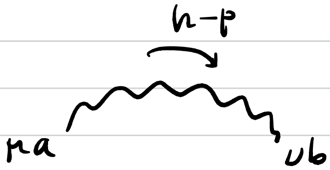
$$= \underbrace{-\sum_a e^a e^a}_{C_2(V_f)} \cdot e^2 \mu_{DR}^{4-d} \int \frac{d^dk}{(2\pi)^d} \frac{\gamma^\mu(\not{k}+m_f)\gamma_\mu}{(k^2+m_f^2)(k-p)^2} \qquad = (d-2)\not{k} - dm$$

$$= C_2(V)\, e^2 \, I_{DR}\left((d-2)\not{k} - dm\right)$$

with $m = m_f$, $\mu = 0$; $\Delta = x(1-x)p^2 + (1-x)m^2$ below

$$= C_2(V_f) \frac{e^2}{(4\pi)^2} \int_0^1 dx \left(\frac{4\pi\mu_{DR}^2}{\Delta}\right)^{2-\frac{d}{2}} \Gamma\left(2-\frac{d}{2}\right)\left((d-2)x\not{p} - dm\right)$$





$$= \mu_{DR}^{4-d} \int \frac{d^d k}{(2\pi)^d} \; i p^\mu \, \mathrm{ad}\, e^a \overset{k}{\cdots\leftarrow\cdots} i k^\nu \, \mathrm{ad}\, e^b$$

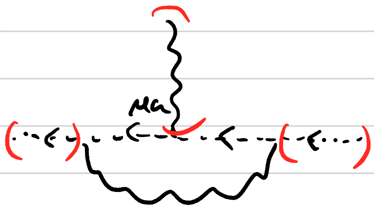
$$= \mu_{DR}^{4-d} \int \frac{d^d k}{(2\pi)^d} \; i p^\mu \, \mathrm{ad}\, e^a \frac{1}{-k^2} i k^\nu \, \mathrm{ad}\, e^b \times e^2 \delta_{ab} \frac{\delta_{\mu\nu}}{(k-p)^2}$$

$$= \underbrace{\sum_a \mathrm{ad}\, e^a \, \mathrm{ad}\, e^a}_{-h^\vee} \cdot e^2 \mu_{DR}^{4-d} \int \frac{d^d k}{(2\pi)^d} \frac{p \cdot k}{k^2 (k-p)^2}$$

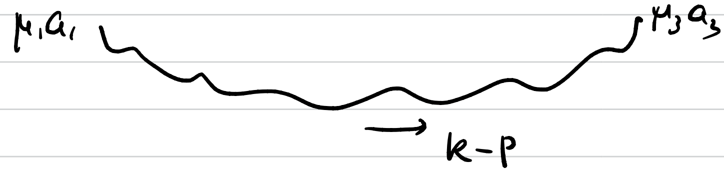
$$= -h^\vee e^2 I_{DR}(p \cdot k)$$

with $m = \mu = 0$; $\Delta = x(1-x) p^2$ below

$$= -h^\vee \frac{e^2}{(4\pi)^2} \int_0^1 dx \left(\frac{4\pi \mu_{DR}^2}{\Delta} \right)^{2-\frac{d}{2}} \Gamma\!\left(2 - \frac{d}{2}\right) x p^2$$



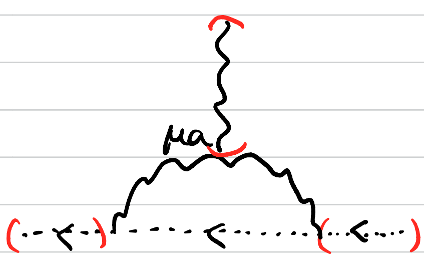
$$= \mu_{DR}^{4-d}\int\frac{d^dk}{(2\pi)^d}\; i(p+q)^{\mu_1}\,\mathrm{ad}\,e^{a_1} \xleftarrow{k+q} i(k+q)^{\mu}\,\mathrm{ad}\,e^{a} \xleftarrow{k} i k^{\mu_3}\,\mathrm{ad}\,e^{a_3}$$



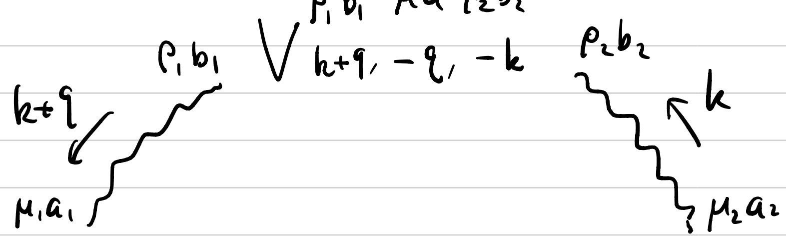
$$= \mu_{DR}^{4-d}\int\frac{d^dk}{(2\pi)^d}\; i(p+q)^{\mu_1}\,\mathrm{ad}\,e^{a_1}\frac{1}{-(k+q)^2}\, i(k+q)^{\mu}\,\mathrm{ad}\,e^{a}\,\frac{1}{-k^2}\, i k^{\mu_3}\,\mathrm{ad}\,e^{a_3}$$

$$\times\; e^2\,\delta_{a_1a_3}\,\frac{\delta_{\mu_1\mu_3}}{(k-p)^2}$$

$$= \underbrace{i^3\sum_b \mathrm{ad}\,e^b\,\mathrm{ad}\,e^a\,\mathrm{ad}\,e^b}_{-\frac{1}{2}h^\vee\,\mathrm{ad}\,e^a}\cdot e^2\,\mu_{DR}^{4-d}\int\frac{d^dk}{(2\pi)^d}\,\frac{(p+q)\cdot k\,(k+q)^{\mu}}{k^2(k+q)^2(k-p)^2}$$



$$V^{\rho_1 b_1\;\mu a\;\rho_2 b_2}_{k+q,\;-q,\;-k}$$



$$= \mu_{DR}^{4-d}\int\frac{d^dk}{(2\pi)^d}\; i(p+q)^{\mu_1}\,\mathrm{ad}\,e^{a_1} \xleftarrow{p-k} i(p-k)^{\mu_2}\,\mathrm{ad}\,e^{a_2}$$

$$= \mu_{DR}^{4-d}\int \frac{d^dk}{(2\pi)^d}\; i(p+q)^{\mu_1}\,\mathrm{ad}\, e^{a_1}\frac{1}{-(p-k)^2}\, i(p-k)^{\mu_2}\,\mathrm{ad}\, e^{a_2}\;\times$$

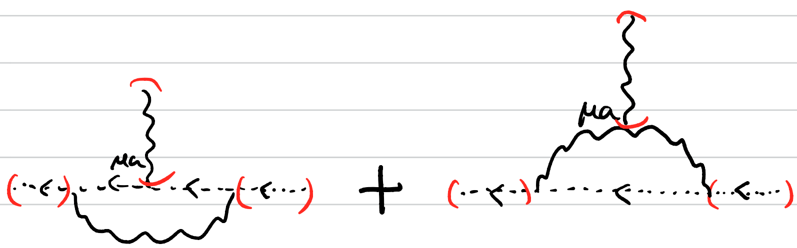
$$\frac{i}{e^2} f^{b_1 a b_2}\left(\delta^{\rho_1\mu}(k+2q)^{\rho_2} + \delta^{\mu\rho_2}(-q+k)^{\rho_1} + \delta^{\rho_2\rho_1}(-k-k-q)^{\mu}\right)\times$$

$$e^2\delta_{a_1b_1}\frac{\delta_{\mu_1\rho_1}}{(k+q)^2}\cdot e^2\delta_{b_2a_2}\frac{\delta_{\rho_2\mu_2}}{k^2}$$

$$= i\sum_{b_1,b_2} f^{b_1ab_2}\,\mathrm{ad}\,e^{b_1}\,\mathrm{ad}\,e^{b_2}\; e^2\mu_{DR}^{4-d}\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2(k+q)^2(k-p)^2}$$

$$\Big((p+q)^\mu (p-k)\cdot(k+2q) + (p-k)^\mu (p+q)\cdot(-q+k) + (-2k-q)^\mu (p+q)\cdot(p-k)\Big)$$

$$\sum_{b_1,b_2} f^{b_1ab_2}\,\mathrm{ad}\,e^{b_1}\,\mathrm{ad}\,e^{b_2} = -\tfrac{1}{2}h^\vee\,\mathrm{ad}\,e^a \quad (\text{exercise})$$



$$= -\frac{i}{2}h^\vee\,\mathrm{ad}\,e^a\cdot e^2 J_{DR}\Big((p+q)^\mu \overbrace{(p-k)\cdot(k+2q)}^{-k^2+(p-2q)\cdot k+2p\cdot q}$$

$$+(p+q)\cdot\Big\{\underbrace{-k(k+q)^\mu + (-q+k)(p-k)^\mu + (k-p)(2k+q)^\mu}_{2qk^\mu - pq^\mu + (k-q)\cdot p^\mu}\Big\}\Big)$$

$$= i\,h^\vee\,\mathrm{ad}\,e^a\cdot e^2 J_{DR}\left((p+q)^\mu\tfrac{1}{2}k^2 + \text{at most linear in } k\right)$$

$$= i h^{\nu} a d e^{q} \frac{e^2}{(4\pi)^2} \int_{\triangle} dy\, dz \left(\frac{4\pi \mu_{DR}^2}{\Delta}\right)^{2-\frac{d}{2}} \Gamma\left(2-\frac{d}{2}\right)$$

$$\left[(p+q)^{\mu} \frac{d}{4} + (4-d)\left(p^{\mu} \frac{R}{\Delta} + q^{\nu} \frac{S}{\Delta}\right) \right]$$

$$\Delta = y(1-y)q^2 + z(1-z)p^2 + 2yz\, qp$$

R, S : also linear in q^2, p^2 & $q \cdot p$

and at most linear in y & z.