Some notes on Lie groups and Lie algebras

1 Compact Lie algebras

Let G be a compact Lie group and let $\mathfrak{g}$ be its Lie algebra. There exists an adjoint invariant positive definite inner product $(\ ,\ )$ on $\mathfrak{g}$,

$$(\mathrm{Ad}_g X, \mathrm{Ad}_g Y) = (X, Y), \quad \forall g \in G,\ \forall X, \forall Y \in \mathfrak{g}. \tag{1.1}$$

This can be constructed using an invariant measure $\mathrm{d}g$ which exists on a compact group G [1]. Let $(\ ,\ )_1$ be an arbitrary positive definite inner prouct on $\mathfrak{g}$ which may not be adjoint invariant. Let $(\ ,\ )$ be its average:

$$(X, Y) := \int_G \mathrm{d}g\, (\mathrm{Ad}_g X, \mathrm{Ad}_g Y)_1. \tag{1.2}$$

Then, it is also a positive definite inner product on $\mathfrak{g}$ (**Exercise:** Show this), and is adjoint invariant:

$$\begin{aligned} (\mathrm{Ad}_g X, \mathrm{Ad}_g Y) &= \int_G \mathrm{d}g'\, (\mathrm{Ad}_{g'} \mathrm{Ad}_g X, \mathrm{Ad}_{g'} \mathrm{Ad}_g Y)_1 \\ &= \int_G \mathrm{d}g'\, (\mathrm{Ad}_{g'g} X, \mathrm{Ad}_{g'g} Y)_1 \\ &= \int_G \mathrm{d}g'' (\mathrm{Ad}_{g''} X, \mathrm{Ad}_{g''} Y)_1 = (X, Y), \end{aligned} \tag{1.3}$$

where invariance of $\mathrm{d}g'$ under right translation $g' \mapsto g'g$ is used in the third equality. The infinitesimal form of (1.1) is

$$([Z, X], Y) + (X, [Z, Y]) = 0, \quad \forall X, \forall Y, \forall Z \in \mathfrak{g}. \tag{1.4}$$

A positive definite inner product $(\ ,\ )$ on a real Lie algebra $\mathfrak{g}$ with the property (1.4) is called an *invariant inner product*. What we have seen above is that the Lie algebra of a compact Lie group has an invariant inner product. Conversely, a real Lie algebra with an invariant inner product is the Lie algebra of a compact Lie group [1]. For this reason, a real Lie algebra with an invariant inner product is called a *compact Lie algebra*.

Let $\{e^a\} \subset \mathfrak{g}$ be an orthonormal basis of a compact Lie algebra $\mathfrak{g}$ with respect to an invariant inner product, $(e^a, e^b) = \delta^{ab}$. Let f^{abc} be the structure constant with respect to this basis,

$$[e^a, e^b] = \sum_c e^c f^{cab}. \tag{1.5}$$

Then of course, $f^{cab} = (e^c, [e^a, e^b])$. By the antisymmetry of the Lie algebra bracket, $[X, Y] = -[Y, X]$, it is antisymmetric in the latter two indices,

$$f^{cab} = -f^{cba}. \tag{1.6}$$

By the adjoint invariance (1.4) of the inner product, it is also antisymmetric in the first and the last indices,

$$f^{cba} + f^{abc} = ([e^b, e^a], e^c) + (e^a, [e^b, e^c]) = 0. \tag{1.7}$$

Then, it is also antisymmetric in the first two indices, $f^{bac} = -f^{bca} = f^{acb} = -f^{abc}$. That is, f^{abc} is totally antisymmetric. This also means that it is invariant under cyclic rotation of the three indices, $f^{abc} = f^{bca} = f^{cab}$.

2 Simple Lie algebras

A non-Abelian Lie algebra $\mathfrak{g}$ is *simple* when it has no nonzero proper ideal, or equivalently, when the adjoint representation is irreducible. When $\mathfrak{g}$ is compact (in the above sense), $\mathfrak{g}$ is simple as a Lie algebra over $\mathbb{R}$ if and only if its complexification $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ is simple as a Lie algebra over $\mathbb{C}$. Simple Lie algebras over $\mathbb{C}$ are classified. There are four infinite series and five exceptional ones, with names

$$\mathrm{A}_\ell\,(\ell \geq 1),\ \mathrm{B}_\ell\,(\ell \geq 2),\ \mathrm{C}_\ell\,(\ell \geq 3),\ \mathrm{D}_\ell\,(\ell \geq 4),\ \mathrm{E}_6,\ \mathrm{E}_7,\ \mathrm{E}_8,\ \mathrm{F}_4,\ \mathrm{G}_2. \tag{2.1}$$

See for example [2, 3]. Each of these is the complexification of a unique compact Lie algebra. Thus, this is also the classification of compact simple Lie algebras. In what follows in this section, $\mathfrak{g}$ is a compact simple Lie algebra.

Let $\mathfrak{t} \subset \mathfrak{g}$ be a maximal Abelian subalgebra. The dimension of $\mathfrak{t}$ is called the *rank* of $\mathfrak{g}$. The subscript of each member in the list (2.1) (e.g. ℓ for A_ℓ) shows the rank. When acting on $\mathfrak{g}_{\mathbb{C}}$ via the adjoint representation, elements of $\mathfrak{t}_{\mathbb{C}}$ commute with each other and can be simultaneously diagonalized. This defines the decomposition of $\mathfrak{g}_{\mathbb{C}}$ into the simultaneous eigenspaces,

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha. \tag{2.2}$$

$\mathfrak{t}_{\mathbb{C}}$ is the subspace of eigenvalue zero. Each non-trivial eigenspace $\mathfrak{g}_\alpha$ is one dimensional. Its label $\alpha \in \Phi$ is called a *root*, and is used also to describe the eigenvalue: $X \in \mathfrak{t}_{\mathbb{C}}$ has eigenvalue $\alpha(X) \in \mathbb{C}$ on $\mathfrak{g}_\alpha$. The map $\alpha : X \mapsto \alpha(X)$ is of course $\mathbb{C}$-linear. $\alpha(X)$ is pure imaginary for $X \in \mathfrak{t}$ and real for $X \in i\mathfrak{t}$. Thus a root α is an $\mathbb{R}$-linear form on $i\mathfrak{t}$. That is, $\alpha \in i\mathfrak{t}^* = \mathrm{Hom}_{\mathbb{R}}(i\mathfrak{t}, \mathbb{R})$. It can be shown that (a) the only scalar multple of a root α are $\pm\alpha$ and (b) the set Φ of all roots spans $i\mathfrak{t}^*$ over $\mathbb{R}$.

A symmetric bilinear form $B : \mathfrak{g}_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}} \to \mathbb{C}$ is said to be *adjoint invariant* when it satisfies

$$B([Z,X],Y) + B(X,[Z,Y]) = 0, \quad \forall X, \forall Y, \forall Z \in \mathfrak{g}_{\mathbb{C}}. \tag{2.3}$$

We show that such a form is unique up to scalar multiplication. Recall that there is an invariant inner product $(\ ,\ )$ on $\mathfrak{g}$, and it can be extended to an adjoint invariant and non-degenerate symmetric bilinear form on $\mathfrak{g}_{\mathbb{C}}$ denoted by the same symbol $(\ ,\ )$. By the non-degeneracy of $(\ ,\ )$, there is a linear map $f_B : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$ such that $B(X,Y) = (X, f_B(Y))$ for all $X, Y \in \mathfrak{g}_{\mathbb{C}}$. Then, it commutes with $\mathrm{ad}X$,

$$f_B \circ \mathrm{ad}X = \mathrm{ad}X \circ f_B, \quad \forall X \in \mathfrak{g}_{\mathbb{C}}. \tag{2.4}$$

Indeed,

$$\begin{aligned}(Y, (\mathrm{ad}X \circ f_B)Z) &= (Y, [X, f_B(Z)]) = -([X,Y], f_B(Z)) = -B([X,Y],Z) \\ &= B(Y,[X,Z]) = (Y, f_B[X,Z]) = (Y, (f_B \circ \mathrm{ad}X)Z),\end{aligned} \tag{2.5}$$

where we used the adjoint invariance of $(\ ,\ )$ and B in the second and the fourth equalities. Since this holds for any $Y, Z \in \mathfrak{g}_{\mathbb{C}}$, by the non-degeneracy of $(\ ,\ )$, this shows (2.4). Since the adjoint representation of $\mathfrak{g}_{\mathbb{C}}$ on $\mathfrak{g}_{\mathbb{C}}$ is irreducible, by Schur's lemma, this means that f_B is a scalar multiplication, $f_B = c \times \mathrm{id}_{\mathfrak{g}_{\mathbb{C}}}$. That is,

$$B(X,Y) = c \times (X,Y) \quad \text{for some } c \in \mathbb{C}. \tag{2.6}$$

Adjoint invariant symmetric bilinear forms on $\mathfrak{g}_{\mathbb{C}}$ are all proportional to $(\ ,\ )$ and hence are related to one another by scalar multiplication. This is what we wanted to show. As a consequence, we find that invariant inner product on $\mathfrak{g}$ is unique up to positive scalar multiplication.

An invariant inner product on $\mathfrak{g}$ induces a positive definite inner product on $\mathfrak{t}$ and hence on its dual $\mathfrak{t}^*$, or a negative definite inner product $(\ ,\ )$ on $i\mathfrak{t}^*$. The set of roots $\Phi \subset i\mathfrak{t}^*$ has the following properties with respect to this inner product: (i) Φ is invariant under the reflections by roots, $s_\alpha : \beta \longmapsto \beta - \alpha \cdot 2(\alpha,\beta)/(\alpha,\alpha)$, (ii) $2(\alpha,\beta)/(\alpha,\alpha)$ are integers for all pairs α, β of roots, and (iii) Φ cannot be separated into two subsets which are orthogonal to each other. Note that the inner product is determined only up to scalar multiplication but the properties (i), (ii), (iii) are invariant under the scalar multiplication. A finite subset Φ of an Euclidean space with the properties (a), (b), (i), (ii), (iii) is called an *irreducible root system*. Conversely, given an irreducible root system Φ, there is a unique simple Lie algebra over $\mathbb{C}$ which has Φ as the set of roots. The classification (2.1) is obtained via this correspondence by the classification of root systems. For A_ℓ, D_ℓ, E_6, E_7 and E_8 (simply laced Lie algebras), all the roots $\alpha \in \Phi$ have the same length $|\alpha| = \sqrt{|(\alpha,\alpha)|}$, whereas for the remaing ones B_ℓ, C_ℓ, F_4 and G_2 (non-simply-laced Lie algebras), there are two lengths — roots are separated into two classes, long roots and short roots.

3 Normalization of the Yang-Mills action

We can now describe the standard normalization of the gauge kinetic term in theories whose gauge group is simple, that is, a compact Lie group G whose Lie algebra $\mathfrak{g}$ is simple. In the lecture, the Yang-Mills action is denoted by (in the Euclidean signature):

$$\mathcal{L}_{\mathrm{YM}} = \frac{1}{4e^2}\sum_{\mu,\nu=1}^{4} F_{\mu\nu}\cdot F_{\mu\nu}. \tag{3.1}$$

Here "$\cdot$" is an invariant inner product on $\mathfrak{g}$. As we have seen above, such an inner product is unique up to positive scalar multiplication. The problem is how to normalize it. The standard choice is:

With respect to the inner product on $i\mathfrak{t}^*$ induced from "$\cdot$" on $\mathfrak{g}$, the longer roots[1] have length 1. — (∗)

4 Examples: classical groups

Examples of compact Lie groups are provided by the classical groups: special unitary groups, special orthogonal groups and unitary symplectic groups,

$$SU(n) = \{\, g \in \mathrm{M}_n(\mathbb{C}) \,|\, g^\dagger g = 1,\ \det g = 1 \,\}, \tag{4.1}$$

$$SO(n) = \{\, g \in \mathrm{M}_n(\mathbb{R}) \,|\, g^T g = 1,\ \det g = 1 \,\}, \tag{4.2}$$

$$USp(n) = \{\, g \in \mathrm{M}_n(\mathbb{C}) \,|\, g^\dagger g = 1,\ g^T J_n g = J_n \,\}. \tag{4.3}$$

In the above expressions, $\mathrm{M}_n(k)$ is the set of $n\times n$ matrices with entries from a field k. For $SU(n)$ and $SO(n)$, n is a non-negative integer. For $USp(n)$, n is an even integer $2\nu \geq 2$ and J_n is the symplectic form

$$J_n = \begin{pmatrix} \mathbf{0}_\nu & -\mathbf{1}_\nu \\ \mathbf{1}_\nu & \mathbf{0}_\nu \end{pmatrix}. \tag{4.4}$$

[1] I.e. all the roots for a simply laced Lie algebra and long roots for a non-simply-laced Lie algebra

The Lie algebras of these groups are

$$\mathfrak{su}(n) = \{ X \in \mathrm{M}_n(\mathbb{C}) \mid X^\dagger + X = 0,\ \mathrm{tr}X = 0 \}, \tag{4.5}$$

$$\mathfrak{so}(n) = \{ X \in \mathrm{M}_n(\mathbb{R}) \mid X^T + X = 0 \}, \tag{4.6}$$

$$\mathfrak{usp}(n) = \{ X \in \mathrm{M}_n(\mathbb{C}) \mid X^\dagger + X = 0,\ X^T J_n + J_n X = 0 \}. \tag{4.7}$$

Their complexifications can be realized as subspaces in $\mathrm{M}_n(\mathbb{C})$ with the constraints $\mathrm{tr}(X) = 0$, $X^T + X = 0$ and $X^T J_n + J_n X = 0$, respectively. This tells that the dimensions are $n^2 - 1$, $n(n-1)/2$ and $n(n+1)/2$ respectively. A, B, C, D in the list (2.1) are given by most of these:

$$\mathrm{A}_\ell \cong \mathfrak{su}(\ell+1)_\mathbb{C},\ \ \mathrm{B}_\ell \cong \mathfrak{so}(2\ell+1)_\mathbb{C},\ \ \mathrm{C}_\ell \cong \mathfrak{usp}(2\ell)_\mathbb{C},\ \ \mathrm{D}_\ell \cong \mathfrak{so}(2\ell)_\mathbb{C}. \tag{4.8}$$

For low values of n, there are relations as Lie algebras: $\mathfrak{su}(1) \cong \mathfrak{so}(1) \cong 0$, $\mathfrak{so}(2) \cong \mathbb{R}$, $\mathfrak{so}(3) \cong \mathfrak{usp}(2) = \mathfrak{su}(2)$, $\mathfrak{so}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, $\mathfrak{usp}(4) \cong \mathfrak{so}(5)$, $\mathfrak{so}(6) \cong \mathfrak{su}(4)$.

As maximal Alebian subalgebras $\mathfrak{t}$, we may take

$$\mathfrak{g} = \mathfrak{su}(n): \qquad \mathfrak{t} = \left\{ \begin{pmatrix} ix_1 & & \\ & \ddots & \\ & & ix_n \end{pmatrix} \middle|\, x_1 + \cdots + x_n = 0 \right\}, \tag{4.9}$$

$$\mathfrak{g} = \mathfrak{so}(2\ell): \qquad \mathfrak{t} = \left\{ \begin{pmatrix} 0 & -x_1 & & & \\ x_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & -x_\ell \\ & & & x_\ell & 0 \end{pmatrix} \right\}, \tag{4.10}$$

$$\mathfrak{g} = \mathfrak{so}(2\ell+1): \qquad \mathfrak{t} = \left\{ \begin{pmatrix} 0 & -x_1 & & & & \\ x_1 & 0 & & & & \\ & & \ddots & & & \\ & & & 0 & -x_\ell & \\ & & & x_\ell & 0 & \\ & & & & & 0 \end{pmatrix} \right\}, \tag{4.11}$$

$$\mathfrak{g} = \mathfrak{usp}(2\ell): \qquad \mathfrak{t} = \left\{ \begin{pmatrix} ix_1 & & & & & \\ & \ddots & & & & \\ & & ix_\ell & & & \\ & & & -ix_1 & & \\ & & & & \ddots & \\ & & & & & -ix_\ell \end{pmatrix} \right\}. \tag{4.12}$$

The sets of roots are

$$\begin{aligned}
\mathfrak{g} = \mathfrak{su}(n) : &\quad \Phi = \{\, \varepsilon_i - \varepsilon_j \,|\, 1 \le i \ne j \le n \,\}, && (4.13)\\
\mathfrak{g} = \mathfrak{so}(2\ell) : &\quad \Phi = \{\pm\varepsilon_i \pm \varepsilon_j \,|\, 1 \le i < j \le \ell \,\}, && (4.14)\\
\mathfrak{g} = \mathfrak{so}(2\ell+1) : &\quad \Phi = \{\pm\varepsilon_i \pm \varepsilon_j \,|\, 1 \le i < j \le \ell \,\} \cup \{\, \pm\varepsilon_i \,|\, 1 \le i \le \ell \,\}, && (4.15)\\
\mathfrak{g} = \mathfrak{usp}(2\ell) : &\quad \Phi = \{\pm\varepsilon_i \pm \varepsilon_j \,|\, 1 \le i < j \le \ell \,\} \cup \{\, \pm 2\varepsilon_i \,|\, 1 \le i \le \ell \,\}. && (4.16)
\end{aligned}$$

In the above expressions, ε_i is the linear function on $\mathfrak{t}_{\mathbb{C}}$ which sends the matrix inside the parenthesis $\{\ \}$ of (4.9)-(4.12) to $\sqrt{-1}x_i$. The signs of $\pm\varepsilon_i \pm \varepsilon_j$ in (4.14)-(4.16) can take all four possible combinations.

The inner product obeying the condition $(*)$ is given by

$$\begin{aligned}
X, Y \in \mathfrak{su}(n) : &\quad X \cdot Y = -2\,\mathrm{tr}(XY), && (4.17)\\
X, Y \in \mathfrak{so}(n) : &\quad X \cdot Y = -\mathrm{tr}(XY), && (4.18)\\
X, Y \in \mathfrak{usp}(n) : &\quad X \cdot Y = -2\,\mathrm{tr}(XY), && (4.19)
\end{aligned}$$

where "tr" is the trace over the defining (or fundamental) representation of each group or Lie algebra: $\mathbb{C}^n$ for $\mathfrak{su}(n)$, $\mathbb{R}^n$ (or its complexification $\mathbb{C}^n$) for $\mathfrak{so}(n)$ and $\mathbb{C}^n$ for $\mathfrak{usp}(n)$.

Exercise: Confirm that the longer roots indeed have length 1 with respect to the inner product induced on $i\mathfrak{t}^*$ from this "$\cdot$".

5 Invariants of representations

Let $\rho : G \to GL(V)$ be a representation of a Lie group on a finite dimensional complex vector space V: Each $g \in G$ determines an automorphism $\rho(g) : V \to V$ in such a way that $\rho(gh) = \rho(g)\rho(h)$. This induces a representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ of the Lie algebra: Each $X \in \mathfrak{g}$ determines an endomorphism $\rho(X) : V \to V$ in such a way that $\rho([X,Y]) = \rho(X)\rho(Y) - \rho(Y)\rho(X)$.[1] Suppose G is a compact Lie group. Then, there is a G-invariant positive definite hermitian inner product $(\ ,\ )$ on V,

$$(\rho(g)v, \rho(g)w) = (v, w), \quad \forall g \in G,\ \forall v, \forall w \in V. \tag{5.1}$$

Exercise: Show this. (Hint: use again the invariant measure $\mathrm{d}g$ on G.)
The infinitesimal form of this is

$$(\rho(X)v, w) + (v, \rho(X)w) = 0, \quad \forall X \in \mathfrak{g}, \forall v, \forall w \in V. \tag{5.2}$$

[1] In the lecture, I omitted "ρ" so that I just wrote $g : V \to V$ and $X : V \to V$ instead of $\rho(g) : V \to V$ and $\rho(X) : V \to V$.

If $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is a representation of a compact Lie algebra on a finite dimensional complex vector space V, it lifts to a representation $\rho : G \to GL(V)$ of a compact Lie group G whose Lie algebra is $\mathfrak{g}$. Then there is a positive definite hermitian inner product $(\ ,\ )$ on V with property (5.1) and hence with property (5.2).

Let $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of a compact simple Lie algebra $\mathfrak{g}$ on a finite dimensional complex vector space V.[2] This determines a bilinear form B_V on $\mathfrak{g}_{\mathbb{C}}$

$$B_V(X,Y) := -\mathrm{tr}_V(\rho(X)\rho(Y)). \tag{5.3}$$

This is adjoint invariant in the sense of (2.3). (**Exercise:** Show this.) As we have seen, such a form is unique up to a constant multiplication, and in particular proportional to the bilinear form $(X,Y) \mapsto X \cdot Y$ induced from the invariant inner product "$\cdot$" on $\mathfrak{g}$: there is some $T_V \in \mathbb{C}$ such that $B_V(X,Y) = T_V \times X \cdot Y$, that is,

$$\mathrm{tr}_V(\rho(X)\rho(Y)) = -T_V\, X \cdot Y, \quad \forall X, \forall Y \in \mathfrak{g}_{\mathbb{C}}. \tag{5.4}$$

This T_V is a non-negative real number,

$$T_V \geq 0, \tag{5.5}$$

where $T_V = 0$ if and only if the representation is trivial, $\rho(X) = 0$ for all $X \in \mathfrak{g}$. To see this, recall that there is a positive definite hermitian inner product $(\ ,\ )$ on V with property (5.2). Let $\{e_i\} \subset V$ be an orthonormal basis, $(e_i, e_j) = \delta_{i,j}$. Then, for $X \in \mathfrak{g}$,

$$\begin{aligned} T_V\ X \cdot X &= -\mathrm{tr}_V(\rho(X)\rho(X)) = -\sum_i (e_i, \rho(X)\rho(X)e_i) \\ &\overset{(5.2)}{=} \sum_i (\rho(X)e_i, \rho(X)e_i) = \sum_i ||\rho(X)e_i||^2 \geq 0, \end{aligned} \tag{5.6}$$

where the last equality $= 0$ holds if and only if $\rho(X)e_i = 0$ for all i, that is, $\rho(X) = 0$. Since $X \cdot X$ is real positive for a non-zero $X \in \mathfrak{g}$, this is what we wanted to show.

Let $\{e^a\} \subset \mathfrak{g}$ be an orthonormal basis, $e^a \cdot e^b = \delta^{ab}$. Then, (5.4) reads

$$\mathrm{tr}_V(\rho(e^a)\rho(e^b)) = -T_V\, \delta^{ab}. \tag{5.7}$$

The *quadratic Casimir operator* of the representation is defined by

$$C_2(V) := -\sum_a \rho(e^a)\rho(e^a). \tag{5.8}$$

[2] By the correspondence mentioned in Section 2, this is the same as a representation of a simple Lie algebra $\mathfrak{g}_{\mathbb{C}}$ on a finite dimensional complex vector space V.

It commute with $\rho(X)$ for all $X \in \mathfrak{g}_{\mathbb{C}}$. (**Exercise:** Show this.) Suppose $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is an irreducible representation. By Schur's lemma, the commutativity means that $C_2(V)$ is a scalar multiplication,

$$C_2(V) = c_{2,V}\,\mathrm{id}_V \tag{5.9}$$

for some $c_{2,V} \in \mathbb{C}$. Taking the contraction of (5.7) with $-\delta_{ab}$, we find $\mathrm{tr}_V C_2(V) = T_V \dim_{\mathbb{R}} \mathfrak{g}$, that is,

$$c_{2,V} \dim_{\mathbb{C}} V = T_V \dim_{\mathbb{R}} \mathfrak{g}. \tag{5.10}$$

This in particular means that $c_{2,V}$ is a positive real number.

Examples

In view of (4.17), (4.18), (4.19), we see that these numbers for the defining (or fundamental) representation $\mathbb{C}^n$ of the classical Lie algebras are

$$\mathfrak{su}(n): \qquad T_{\mathbb{C}^n} = \frac{1}{2}, \quad c_{2,\mathbb{C}^n} = \frac{n^2-1}{2n}, \tag{5.11}$$

$$\mathfrak{so}(n): \qquad T_{\mathbb{C}^n} = 1, \quad c_{2,\mathbb{C}^n} = \frac{n-1}{2}, \tag{5.12}$$

$$\mathfrak{usp}(n): \qquad T_{\mathbb{C}^n} = \frac{1}{2}, \quad c_{2,\mathbb{C}^n} = \frac{n+1}{4}. \tag{5.13}$$

6 The dual Coxeter number

Let $\mathfrak{g}$ be a compact simple Lie algebra, i.e., let $\mathfrak{g}_{\mathbb{C}}$ be a simple Lie algebra over $\mathbb{C}$. The quadratic Casimir operator of the adjoint representation, which is irreducible, is multiplication by a positive real number $c_{2,\mathfrak{g}_{\mathbb{C}}}$. By (5.10), we have

$$c_{2,\mathfrak{g}_{\mathbb{C}}} = T_{\mathfrak{g}_{\mathbb{C}}}. \tag{6.1}$$

This number is called the *dual Coxeter number* of $\mathfrak{g}_{\mathbb{C}}$ and is denoted by $h^\vee$. It is shown in the table below, along with the dimension:

$\mathfrak{g}_{\mathbb{C}}$	A_ℓ	B_ℓ	C_ℓ	D_ℓ	E_6	E_7	E_8	F_4	G_2
dimension	$\ell^2+2\ell$	$2\ell^2+\ell$	$2\ell^2+\ell$	$2\ell^2-\ell$	78	133	248	52	14
$h^\vee$	$\ell+1$	$2\ell-1$	$\ell+1$	$2\ell-2$	12	18	30	9	4

Exercise: For A_ℓ, B_ℓ, C_ℓ and D_ℓ, show that the dual Coxeter number is as in the table.

References

[1] L. Pontrjagin, *Topological Groups*, (Princeton University Press, 1939); Japanese version: *Renzoku Gun Ron*, (Iwanami Shoten, 1957 or 1974); Physical copies are available in U Tokyo Libraries.

[2] H. Georgi, *Lie Algebras in Particle Physics*, (Benjamin, 1982; and newer editions); E-book is available via OPAC.

[3] J.E. Humphreys, *Introcudtion to Lie Algebras and Representation Theory*, (Springer 1972); E-book is available via OPAC.