In the process of showing

$$\delta_{N,\infty} A_\mu = \xi_N \partial_\mu C + \eta_N [A_\mu, C]$$

$$\delta_{N,\infty} \psi = -\eta_N C \psi \qquad (\xi_N, \eta_N \text{ constant})$$

$$\delta_{N,\infty} C = -\frac{1}{2} \eta_N [C, C]$$

from global gauge symmetry & Zinn-Justin eqn,

we needed the following

FACT (*)

G a compact Lie group, simple. $\mathfrak{g} := \mathrm{Lie}(G)$.

$\varphi : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ antisymmetric $\mathbb{R}$-bilinear map s.t.

(φ_0) G-equivariant : $\varphi(gX, gY) = g\varphi(X, Y)$,

(φ_1) $\varphi([X,Y],Z) + \varphi([Y,Z],X) + \varphi([Z,X],Y) = 0$.

Then, $\varphi(X,Y) = \eta [X,Y]$ for some $\eta \in \mathbb{R}$.

We show that this follows from a fact on Lie algebra cohomology (which is also explained).

Rmk The infinitesimal version of (φ0) :

$$(\varphi 0)_* \quad \varphi([Z,X],Y) + \varphi(X,[Z,Y]) = [Z, \varphi(X,Y)].$$

$(\varphi 0)_* + (\varphi 1) \Longrightarrow$

$$(\varphi 2) \quad [\varphi(X,Y),Z] + [\varphi(Y,Z),X] + [\varphi(Z,X),Y] = 0.$$

Lie algebra cohomology

Let $\mathfrak{g}$ be a Lie algebra and M be a $\mathfrak{g}$-module (i.e. a representation of $\mathfrak{g}$) over a field k.

Put $C^p(\mathfrak{g}, M) = \left\{ \alpha : \mathfrak{g}^p \to M \;\middle|\; \begin{array}{c} \text{antisymmetric, } k\text{-linear} \\ \text{in each entry} \end{array} \right\}$

and define $d : C^p(\mathfrak{g}, M) \to C^{p+1}(\mathfrak{g}, M)$ by

$$(d\alpha)(X_0, \cdots, X_p) = \sum_{i=0}^{p} (-1)^i X_i \, \alpha(X_0, \cdots \hat{i} \cdots X_p)$$

$$- \sum_{i<j} (-1)^{i+j-1} \alpha([X_i, X_j], X_0, \cdots \hat{i} \cdots \hat{j} \cdots, X_p)$$

Then we have $d \circ d = 0$ and we can define

$$H^p(\mathfrak{g}, M) = \frac{\mathrm{Ker}\left(d : C^p(\mathfrak{g}, M) \to C^{p+1}(\mathfrak{g}, M)\right)}{\mathrm{Im}\left(d : C^{p-1}(\mathfrak{g}, M) \to C^p(\mathfrak{g}, M)\right)}$$

(p-th cohomology group of $\mathfrak{g}$ with values in M).

E.g.

- $m \in C^0(\mathfrak{g}, M) = M$: $(dm)(X) = Xm$.

- $f \in C^1(\mathfrak{g}, M) = \mathrm{Hom}_k(\mathfrak{g}, M)$:

$$(df)(X, Y) = Xf(Y) - Yf(X) - f([X, Y])$$

- $\varphi \in C^2(\mathfrak{g}, M)$:

$$(d\varphi)(X, Y, Z) = X\varphi(Y, Z) - Y\varphi(X, Z) + Z\varphi(X, Y)$$
$$- \varphi([X, Y], Z) + \varphi([X, Z], Y) - \varphi([Y, Z], X).$$

<u>Rmk</u> Our φ is in $C^2(\mathfrak{g}, \mathfrak{g})$ and satisfies $d\varphi = 0$;

The 1st line $= 0$ by $(\varphi 2)$,

the 2nd line $= 0$ by $(\varphi 1)$.

We have a

<u>Theorem ✫</u>

When $\mathfrak{g}$ is semi-simple and M is irreducible and non-trivial, $H^p(\mathfrak{g}, M) = 0 \quad \forall p$.

A proof will be given in a moment.

Proof of FACT (*)

Let us apply Thm ✡ to $\mathfrak{g} = M = \mathrm{Lie}(G)$ in our setup.

Since G is simple, $\mathfrak{g}$ is (semi)simple, also $M = \mathfrak{g}$ is irreducible and non-trivial.

Thus $H^p(\mathfrak{g}, \mathfrak{g}) = 0 \quad \forall p$. In particular for $p=2$.

Since our $\varphi \in C^2(\mathfrak{g}, \mathfrak{g})$ satisfies $d\varphi = 0$,

$\exists f \in C^1(\mathfrak{g}, \mathfrak{g}) = \mathrm{Hom}_{\mathbb{R}}(\mathfrak{g}, \mathfrak{g})$ s.t. $\varphi = df$, i.e.

$$\varphi(X, Y) = [X, f(Y)] - [Y, f(X)] - f([X, Y]).$$

Since φ is G-equivariant, $\varphi = \bar{\varphi} = d\bar{f}$, where the average $\alpha \mapsto \bar{\alpha}$ is defined by

$$\bar{\alpha}(X_1, \ldots, X_p) = \int_G dg \; g\alpha(g^{-1}X_1, \ldots, g^{-1}X_p).$$

Note that $\bar{\alpha}$ is always G-equivariant:

$$\bar{\alpha}(gX_1, \ldots, gX_p) = \int_G dg' \; g'\alpha(g'^{-1}gX_1, \ldots, g'^{-1}gX_p)$$

$$\overset{g' = gg''}{=} \int_G dg'' \; gg''\alpha(g''^{-1}X_1, \ldots, g''^{-1}X_p) = g\bar{\alpha}(X_1, \ldots, X_p).$$

Thus $\bar{f} : \mathfrak{g} \to \mathfrak{g}$ is also G-equivariant, i.e.,

commute with G-action.

Since $\mathfrak{g}$ is simple, it is irreducible as a representation of G.

By Schur's lemma, $\bar{f} \propto \mathrm{id}_{\mathfrak{g}}$.

(To be precise, we need to $\otimes_{\mathbb{R}} \mathbb{C}$ at intermediate stages.)

$\therefore \bar{f} = \eta \cdot \mathrm{id}_{\mathfrak{g}}$ for some $\eta \in \mathbb{R}$.

$$\begin{aligned}
\therefore \varphi(X, Y) &= [X, \bar{f}(Y)] - [Y, \bar{f}(X)] - \bar{f}([X, Y]) \\
&= [X, \eta \cdot Y] - [Y, \eta \cdot X] - \eta \cdot [X, Y] \\
&= \eta \cdot [X, Y].
\end{aligned}$$

FACT ($*$) is proved. $\square$

A proof of Theorem ✩

To be precise, this holds for $\mathfrak{g}$ & M over a field k of characteristic 0.

The basic reference is

C. Chevalley & S. Eilenberg "Cohomology theory of Lie groups and Lie algebras" Trans. Amer. Math. Soc. 63 (1948) 85–124.

Two proofs are given there

- algebraic & direct
- topological.

The topological proof is easy for "us". An outline is as follows. (For this we may consider $k = \mathbb{R}$ and we may assume $\mathfrak{g} = \mathrm{Lie}(G_0)$ for a compact connected G_0)

(i) $H^p(\mathfrak{g}, M) \cong H^p_{equiv}(G_0, M)$ = de Rham cohomology for equivariant forms

$$\Omega^p_{equiv}(G_0, M) = \{ \omega \in \Omega^p(G_0, M) \mid g^*\omega = g(\omega), \forall g \in G_0 \}$$

M-valued p-forms on G_0.

$$\Omega^p_{equiv}(G_0, M) \cong C^p(\mathfrak{g}, M) \quad \text{by}$$

$$\omega \longmapsto \{\omega\}$$

$$\hat{\eta} \longleftarrow\!\shortmid \ \eta$$

$$\{\omega\}(X_1, \cdots, X_p) := \omega(eX_1, \cdots, eX_p)$$

for $X_0, \cdots, X_p \in \mathfrak{g}$.

$$\hat{\eta}(gX_1, \cdots, gX_p) := g(\eta(X_1, \cdots, X_p))$$

($e \in G_0$ is the identity ; For $g \in G_0$ & $X \in \mathfrak{g}$, $gX \in T_g G_0$ is the vector tangent to $g_t = g e^{tX}$ at $t=0$.)

$$\{d\omega\}(X_0, \cdots, X_p) = d\omega(\tilde{X}_0, \cdots, \tilde{X}_p)\big|_e$$

$$= \sum_{i=0}^{p} (-1)^i \tilde{X}_i \, \omega(\tilde{X}_0, \cdots \hat{i} \cdots \tilde{X}_p)\big|_e$$

$$- \sum_{i<j} (-1)^{i+j-1} \omega([\tilde{X}_i, \tilde{X}_j], \tilde{X}_0, \cdots \hat{i} \cdots \hat{j} \cdots, \tilde{X}_p)\big|_e$$

where $\tilde{X}_i$ is (any) vector field in a neighborhood of e s.t. $\tilde{X}_i(e) = eX_i$. Let us choose it to be the left invariant vector field : $\tilde{X}_i(g) = gX_i$.

Then $[\tilde{X}_i, \tilde{X}_j] = \widetilde{[X_i, X_j]}$. Thus

$$\omega([\tilde{X}_i, \tilde{X}_j], \tilde{X}_0, \cdots \hat{i} \cdots \hat{j} \cdots \tilde{X}_p)\big|_e = \{\omega\}([X_i, X_j], X_0, \cdots \hat{i} \cdots \hat{j} \cdots X_p).$$

Also $\tilde{X}_i \, \omega(\tilde{X}_0, \cdots \hat{i} \cdots \tilde{X}_j)\big|_e = \left(\frac{d}{dt}\right)_0 \omega(\tilde{X}_0, \cdots \hat{i} \cdots \tilde{X}_p)\big|_{e^{tX_i}}$

$$\omega(\tilde{X}_0 \cdots \hat{i} \cdots \tilde{X}_p)\big|_{e^{tX_i}} = \omega(e^{tX_i} X_0, \cdots \hat{i}, \cdots, e^{tX_i} X_p)$$

$$\underset{\text{equivariance}}{=} e^{tX_i} \omega(eX_0, \cdots \hat{i} \cdots eX_p)$$

$$= e^{tX_i} \{\omega\}(X_0, \cdots \hat{i} \cdots X_p)$$

$$= X_i \{\omega\}(X_0, \cdots \hat{i} \cdots X_p).$$

$\therefore \{d\omega\}(X_0, \cdots, X_p)$

$$= \sum_{i=0}^{p} (-1)^i X_i \{\omega\}(X_0, \cdots \hat{i} \cdots X_p)$$

$$- \sum_{i<j} (-1)^{i+j-1} \{\omega\}([X_i, X_j], X_0, \cdots \hat{i} \cdots \hat{j} \cdots X_p)$$

$$= d\{\omega\}(X_0, \cdots, X_p).$$

i.e. $\{d\omega\} = d\{\omega\}$.

$\therefore H^p_{equiv}(G_0, M) \cong H^p(\mathfrak{g}, M)$. //

<u>Rmk</u> This holds as long as $\mathfrak{g} = \operatorname{Lie} G_0$.
(No need of G_0 being compact nor connected.)

(ii) $H^p_{equiv}(G_0, M) = 0 \quad \forall p$

if G_0 is compact connected

M is irreducible & non-trivial.

$\because$ Suppose $[\omega] \in H^p_{equiv}(G_0, M)$.

Take any p-cycle Z in G_0.

$$\forall g \in G_0, \quad g\left(\int_Z \omega\right) = \int_Z g(\omega) \overset{\text{equiv}}{=} \int_Z g^* \omega$$

$$= \int_{g_* Z} \omega \underset{\substack{\uparrow \\ G_0 \text{ conn}}}{=} \int_Z \omega.$$

M irreducible & non-trivial $\Rightarrow \int_Z \omega = 0$.

$\omega = d\alpha$ for some $\alpha \in \Omega^{p-1}(G_0, M)$

G_0 compact $\Rightarrow \omega = \bar{\omega} = d\bar{\alpha}, \ \bar{\alpha} \in \Omega^{p-1}_{equiv}(G_0, M)$

$\therefore [\omega] = 0$ in $H^p_{equiv}(G_0, M)$. //.

Combining (i) & (ii), we get

$$H^p(\mathfrak{g}, M) \underset{\substack{\uparrow \\ \mathfrak{g} = \text{Lie } G_0}}{=} H^p_{equiv}(G_0, M) = 0$$

$\Leftarrow$ G_0 compact connected

M nontrivial irrep.

$\boxtimes$