Axial anomaly as a chiral anomaly

The axial anomaly (G ungauged) must be the same as the chiral anomaly for $G_{tot} = U(1)_5 \times G$ with

variation $\epsilon_{tot} = (\epsilon, 0)$

background $A_{tot} = (0, A)$.

Let us check if that is indeed the case.

$$a^5_\epsilon[A] = \int \frac{-1}{4\pi^2} \epsilon \, tr_V (F_A \wedge F_A)$$

$$= \int \frac{-1}{4\pi^2} \epsilon \, d \, tr_V \left(AdA + \frac{2}{3} A^3\right)$$

$$a^{tot}_{(\epsilon, 0)}[0, A] = \int \frac{i}{24\pi^2} \Big\{ tr_V \left[i\epsilon \, d\left(AdA + \frac{1}{2} A^3\right)\right.$$

$$\left. - tr_V \left[(-i\epsilon) \, d\left(AdA + \frac{1}{2} A^3\right)\right] \right\}$$

$$= \int \frac{-1}{12\pi^2} \epsilon \, d \, tr_V \left(AdA + \frac{1}{2} A^3\right)$$

They DO NOT match!

However, we recall that there is a freedom to modify $W[A]$ by addition of local counter terms. In the present case, we modify $W_{tot}[A_5, A]$.

Note that $W_{tot}[A_5, A]$ is not invariant under G-gauge transform:

$$a^{tot}_{(0,\epsilon)}[A_5, A] = \int \frac{i}{24\pi^2} \Big\{ \mathrm{tr}_V \Big[\epsilon\, d\Big((A+A_5)\, d(A+A_5) + \frac{1}{2}(A+A_5)^3 \Big) \Big]$$
$$- \mathrm{tr}_V \Big[\epsilon\, d\Big((A-A_5)\, d(A-A_5) + \frac{1}{2}(A-A_5)^3 \Big) \Big] \Big\}$$

use $A_5 A + A A_5 = 0$, $A_5^2 = 0$

$$= \int \frac{i}{12\pi^2} \mathrm{tr}_V \Big[\epsilon\, d\Big(A\, dA_5 + A_5\, dA + \frac{1}{2} A_5 A^2 \Big) \Big]$$

$$= \int \frac{i}{12\pi^2} \mathrm{tr}_V \Big[\epsilon \Big(2\, dA_5\, dA + \frac{1}{2} d(A_5 A^2) \Big) \Big] \neq 0.$$

However, this is the G-gauge variation of a local expression:

$$a^{tot}_{(0,\epsilon)}[A_5, A] = \delta_\epsilon \int \frac{i}{12\pi^2} A_5\, \mathrm{tr}_V \Big(2 A\, dA + \frac{3}{2} A^3 \Big).$$

(Exercise: Show this.)

Adding the local counter term

$$\Delta W_{tot}[A_5, A] = \int \frac{-i}{12\pi^2} A_5\, \mathrm{tr}_V \Big(2 A\, dA + \frac{3}{2} A^3 \Big),$$

$$W'_{tot}[A_5, A] = W_{tot}[A_5, A] + \Delta W_{tot}[A_5, A]$$

is invariant under G-gauge transformations.

Since G is going to be gauged, there is no choice but to do this modification.

Now, let us see how $W'_{tot}[A_5, A]$ changes under the $U(1)_5$ transformations: $\delta^5_\epsilon A_5 = i d\epsilon$, $\delta^5_\epsilon A = 0$.

$$\delta^5_\epsilon W'_{tot}[A_5, A]\Big|_{A_5=0}$$

$$= a^{tot}_{(\epsilon,0)}[0, A] + \delta^5_\epsilon \Delta W_{tot}[A_5, A]\Big|_{A_5=0}$$

$$= \int \frac{-1}{12\pi^2} \epsilon \, d\, tr_V\left(A dA + \frac{1}{2} A^3\right) + \underbrace{\int \frac{-i}{12\pi^2} i d\epsilon \, tr_V\left(2 A dA + \frac{3}{2} A^3\right)}_{-\int \frac{1}{12\pi^2} \epsilon \, d\, tr_V\left(2AdA + \frac{3}{2} A^3\right)}$$

by partial integration

$$= \int \frac{-1}{12\pi^2} \epsilon \, d\, tr_V\left(3 A dA + 2 A^3\right)$$

$$= \int \frac{-1}{4\pi^2} \epsilon \, d\, tr_V\left(A dA + \frac{2}{3} A^3\right)$$

$$= a^5_\epsilon[A].$$

match ✓