Axial anomaly and index of Dirac operator $(d=2n)$

The axial anomaly for a constant ϵ is proportional to the index of the Dirac operator.

To avoid technical subtleties, instead of non-compact $\mathbb{R}^d$ it is better to consider a compact manifold X (with a metric g and a spin structure) and consider Dirac fermion with values in a vector bundle E with a connection A on X.

Thus the anomaly to consider is

$$a_\epsilon^{d+1}[g,A] = 2\epsilon \, \mathrm{Tr}_{S(E)}\left[\gamma_{d+1}\, e^{-\not{D}_{g,A}^2/\Lambda^2}\right]$$

where $S(E)$ is the space of spinors with values in E.

To simplify the expression, we suppress dependence on (g,A) below. For example, we shall simply write $\not{D}$ for $\not{D}_{g,A}$.

In Fujikawa's computation on $\mathbb{R}^d$, we used the plane wave basis to evaluate the Trace. Here, we shall use the basis of $S(E)$ consisting of eigenvectors of $\not{D}^2$.

By compactness of X, the spectrum of $\cancel{D}^2$ is discrete and the eigenvectors are square normalizable.

Suppose $\cancel{D}^2 \varphi = \lambda \varphi$.

$\cancel{D}^\dagger = \cancel{D}$

$$\lambda \|\varphi\|^2 = (\varphi, \lambda\varphi) = (\varphi, \cancel{D}^2 \varphi) = (\cancel{D}\varphi, \cancel{D}\varphi) = \|\cancel{D}\varphi\|^2$$

$\therefore$ $\lambda \geq 0$ and $\lambda = 0 \Leftrightarrow \cancel{D}\varphi = 0$.

Also, as $\gamma_{d+1} \cancel{D} = -\cancel{D}\gamma_{d+1}$, γ_{d+1} commutes with $\cancel{D}^2$.

Of course, $\cancel{D}$ commutes with $\cancel{D}^2$.

$\therefore$ $\gamma_{d+1}\varphi$ and $\cancel{D}\varphi$ are also eigenvectors of $\cancel{D}^2$ with the same eigenvalue λ.

Suppose $\lambda > 0$. If φ is right handed ($\gamma_{d+1} = 1$), then $\cancel{D}\varphi$ is left handed ($\gamma_{d+1} = -1$). Furthermore,

$$\cancel{D}(\cancel{D}\varphi) = \cancel{D}^2\varphi = \lambda\varphi \propto \varphi.$$

I.e. there is a one-to-one correspondence between righ handed & left handed eigenvectors, provided $\lambda > 0$.

Summary: Let $S(E) = \bigoplus_\lambda S_\lambda(E)$ be the decomposition to subspaces of $\not{D}^2$-eigenvectors, and let $S_\lambda(E) = S_\lambda^R(E) \oplus S_\lambda^L(E)$ be the decomposition according to $\gamma_{d+1} = +1$ or -1. Then, for $\lambda > 0$, $\not{D} : S_\lambda^R(E) \xrightarrow{\cong} S_\lambda^L(E)$ is an isomorphism.

Therefore,

$$\mathrm{Tr}_{S(E)}\left(\gamma_{d+1} e^{-\not{D}^2/\Lambda^2}\right) = \sum_\lambda \mathrm{Tr}_{S_\lambda(E)}(\gamma_{d+1})\, e^{-\lambda/\Lambda^2}$$

and
$$\mathrm{Tr}_{S_\lambda(E)}(\gamma_{d+1}) = \mathrm{Tr}_{S_\lambda^R(E)}(+1) + \mathrm{Tr}_{S_\lambda^L(E)}(-1)$$
$$= \dim S_\lambda^R(E) - \dim S_\lambda^L(E)$$
$$= 0 \text{ if } \lambda > 0$$

$$\therefore \mathrm{Tr}_{S(E)}\left(\gamma_{d+1} e^{-\not{D}^2/\Lambda^2}\right) = \mathrm{Tr}_{S_0(E)}(\gamma_{d+1})$$
$$= \dim S_0^R(E) - \dim S_0^L(E).$$

Note: $S_0^R(E) = \mathrm{Ker}(\not{D} : S^R(E) \to S^L(E))$

$$S_0^L(E) = \mathrm{Ker}(\not{D} : S^L(E) \to S^R(E))$$

$(\varphi_R, \not{D}\varphi_L) = (\not{D}\varphi_R, \varphi_L)$

$$= \mathrm{Im}(\not{D} : S^R(E) \to S^L(E))^{\perp}$$

$$= S^L(E) / \mathrm{Im}(\not{D} : S^R(E) \to S^L(E))$$

$$=: \mathrm{Coker}(\not{D} : S^R(E) \to S^L(E))$$

$$\therefore \mathrm{Tr}_{S(E)}(\gamma_{d+1} e^{-\not{D}^2/\Lambda^2})$$

$$= \dim \mathrm{Ker}(\not{D} : S^R(E) \to S^L(E))$$

$$- \dim \mathrm{Coker}(\not{D} : S^R(E) \to S^L(E))$$

The number on the right hand side is called the <u>index</u> of the operator $\not{D} : S^R(E) \to S^L(E)$

<u>This is obviously an integer.</u>

Thus for a constant ϵ,

$$a_\epsilon^{d+1}[g, A] = 2\epsilon \cdot \mathrm{index}(\not{D} : S^R(E) \to S^L(E)).$$

We can also see this directly by looking at how the measure $\mathcal{D}\bar{\psi}\mathcal{D}\psi$ changes under the axial rotation.

Let us first define the measure by mode expansion of the fields $\psi, \bar{\psi}$.

Let $\{\varphi^R_{0i}\}_{i=1}^{n_R} \subset S_0^R(E)$, $\{\varphi^L_{0j}\}_{j=1}^{n_L} \subset S_0^L(E)$ be orthonormal basis ($n_R = \dim S_0^R(E)$, $n_L = \dim S_0^L(E)$).

Let $\{\varphi^R_n\}_{n=1}^{\infty}$ & $\{\varphi^L_n\}_{n=1}^{\infty}$ be the orthonormal basis of $\bigoplus_{\lambda>0} S^R_\lambda(E)$ & $\bigoplus_{\lambda>0} S^L_\lambda(E)$ consisting of $\not{D}^2$ eigen-vectors. By the discussion above we may assume that they form pairs $(\varphi^R_n, \varphi^L_n)$ s.t.

$$\not{D}\varphi^R_n = \sqrt{\lambda_n}\,\varphi^L_n \quad \text{and} \quad \not{D}\varphi^L_n = \sqrt{\lambda_n}\,\varphi^L_n$$

where λ_n is the common eigenvalue of $\not{D}^2$.

Then

$$\left.\begin{aligned} &\{\varphi^R_{0i}\}_{i=1}^{n_R} \cup \{\varphi^R_n\}_{n=1}^{\infty} \subset S^R(E) \\ &\{\varphi^L_{0j}\}_{j=1}^{n_L} \cup \{\varphi^L_n\}_{n=1}^{\infty} \subset S^L(E) \end{aligned}\right\} \text{orthonormal basis.}$$

The fields $\Psi = \begin{pmatrix}\psi_R \\ \psi_L\end{pmatrix}$, $\bar{\Psi} = (\bar{\psi}_L, \bar{\psi}_R)$ can be expanded as

$$\psi_R = \sum_{i=1}^{n_R} b_{0i}^R \varphi_{0i}^R + \sum_{n=1}^{\infty} b_n^R \varphi_n^R, \quad \psi_L = \sum_{j=1}^{n_L} b_{0j}^L \varphi_{0j}^L + \sum_{n=1}^{\infty} b_n^L \varphi_n^L$$

$$\bar{\psi}_L = \sum_{i=1}^{n_R} \bar{b}_{0i}^L \varphi_{0i}^{R\dagger} + \sum_{n=1}^{\infty} \bar{b}_n^L \varphi_n^{R\dagger}, \quad \bar{\psi}_R = \sum_{j=1}^{n_L} \bar{b}_{0j}^R \varphi_{0j}^{L\dagger} + \sum_{n=1}^{\infty} \bar{b}_n^R \varphi_n^{L\dagger}.$$

Then, the path-integral measure is defined by

$$\mathcal{D}\bar{\Psi}\,\mathcal{D}\Psi = \prod_{i=1}^{n_R} d\bar{b}_{0i}^L\, db_{0i}^R \prod_{j=1}^{n_L} d\bar{b}_{0j}^R\, db_{0j}^L \prod_{n=1}^{\infty} d\bar{b}_n^L\, d\bar{b}_n^R\, db_n^R\, db_n^L$$

Under the axial rotation

$$\begin{array}{cccccc} \Psi & \to & e^{i\epsilon\gamma_{d+1}}\Psi, & \bar{\Psi} & \to & \bar{\Psi} e^{i\epsilon\gamma_{d+1}} \\ \| & & \| & \| & & \| \\ \begin{pmatrix}\psi_R \\ \psi_L\end{pmatrix} & & \begin{pmatrix}e^{i\epsilon}\psi_R \\ e^{-i\epsilon}\psi_L\end{pmatrix} & (\bar{\psi}_L, \bar{\psi}_R) & & (\bar{\psi}_L e^{i\epsilon}, \bar{\psi}_R e^{-i\epsilon}) \end{array}$$

with constant ϵ, the modes transform as

$$\begin{pmatrix} b_{\dots}^R \\ b_{\dots}^L \end{pmatrix} \to \begin{pmatrix} e^{i\epsilon} b_{\dots}^R \\ e^{-i\epsilon} b_{\dots}^L \end{pmatrix}, \quad (\bar{b}_{\dots}^L, \bar{b}_{\dots}^R) \to (\bar{b}_{\dots}^L e^{i\epsilon}, \bar{b}_{\dots}^R e^{-i\epsilon}).$$

Thus, the transformation of the measure is :

$$\mathcal{D}(\bar{\Psi} e^{-i\epsilon\gamma_{d+1}})\mathcal{D}(e^{-i\epsilon\gamma_{d+1}}\Psi)$$

$$= \prod_{i=1}^{n_R} d(e^{-i\epsilon}\bar{b}_{0i}^L)\, d(e^{-i\epsilon} b_{0i}^R) \prod_{j=1}^{n_L} d(e^{i\epsilon}\bar{b}_{0j}^R)\, d(e^{i\epsilon} b_{0j}^L)$$

$$\times \prod_{n=1}^{\infty} d(e^{-i\epsilon}\bar{b}_n^L)\, d(e^{i\epsilon}\bar{b}_n^R)\, d(e^{-i\epsilon} b_n^R)\, d(e^{i\epsilon} b_n^L)$$

$$= \prod_{i=1}^{n_R} e^{2i\epsilon}\, d\bar{b}_{0i}^L\, d b_{0i}^R \prod_{j=1}^{n_L} e^{-2i\epsilon}\, d\bar{b}_{0j}^R\, d b_{0j}^L$$

$$\times \prod_{n=1}^{\infty} d\bar{b}_n^L\, d\bar{b}_n^R\, d b_n^R\, d b_n^L$$

$$= \underbrace{(e^{2i\epsilon})^{n_R}(e^{-2i\epsilon})^{n_L}}_{e^{2i\epsilon(n_R - n_L)}} \cdot \mathcal{D}\bar{\Psi}\,\mathcal{D}\Psi.$$

Since $n_R - n_L = \dim S_0^R(E) - \dim S_0^L(E)$

$$= \operatorname{index}(\not{D} : S^R(E) \to S^L(E)),$$

this reproduces

$$a_\epsilon^{d+1}[g, A] = 2\epsilon \cdot \operatorname{index}(\not{D} : S^R(E) \to S^L(E)).$$

for a constant ϵ.

(An extension of) Fujikawa's method tells us

$$a_\epsilon^{d+1}[g, A] = 2\int_X \epsilon \, \mathrm{ch}(E)\, \hat{A}(TX)$$

$$\overset{\substack{\epsilon \text{ constant} \\ \downarrow}}{=} 2\epsilon \int_X \mathrm{ch}(E)\, \hat{A}(TX)$$

which means

$$\mathrm{index}\left(\not{D} : S^R(E) \to S^L(E) \right) = \int_X \mathrm{ch}(E)\, \hat{A}(TX).$$

This is nothing but the formula for the index that follows from Atiyah-Singer index theorem.

Thus, Fujikawa's computation of axial anomaly rederives the Atiyah-Singer formula for the index of the Dirac operator.

Remark

$\mathrm{Tr}_{S(E)}\left(\gamma_{d+1}\, e^{-\not{D}^2/\Lambda^2}\right)$ is an example of <u>Witten index</u>.

We may consider a quantum mechanics with the space of states $\mathcal{H}$ and Hamiltonian H given by

$$\mathcal{H} = S(E),$$
$$H = \not{D}^2.$$

This system also has additional self adjoint operators

$$(-1)^F = \gamma_{d+1} \quad \text{with eigenvalue } \pm 1$$
$$Q = \not{D}.$$

The three operators $H, (-1)^F, Q$ obey

$$Q^2 = H, \quad (-1)^F Q = -Q(-1)^F \qquad \text{—— } \star$$

and hence $[H, Q] = [H, (-1)^F] = 0$ (Q & $(-1)^F$ are conserved).

Such a system is called a supersymmetric quantum mechanics (SQM). Q is called the supercharge and $\star$ is called the supersymmetry algebra.

What we have seen on the eigenvalues λ and eigenvectors of $\not{D}^2$, $R \xleftrightarrow{1:1} L$ correspondence for $\lambda > 0$,

$$\rightsquigarrow \quad \mathrm{Tr}_{S(E)}\left(\gamma_{d+1}\, e^{-\not{D}^2/\Lambda^2}\right) = \dim S_0^R(E) - \dim S_0^L(E) \quad \text{etc}$$

all follow just from the relation ✪ (I have intentionally presented them so that this is clear), and hold in any SQM.

In particular,

$$\mathrm{Tr}_{\mathcal{H}}\left((-1)^F e^{-\beta H}\right) = \dim \mathcal{H}_0^B - \dim \mathcal{H}_0^F$$

$$= \mathrm{index}\left(Q : \mathcal{H}^B \to \mathcal{H}^F\right)$$

where $\mathcal{H} = \mathcal{H}^B \oplus \mathcal{H}^F$ according to $(-1)^F = +1$ or -1.

$$\mathcal{H}_0^B = \mathcal{H}^B \cap \mathrm{Ker}\, Q, \quad \mathcal{H}_0^F = \mathcal{H}^F \cap \mathrm{Ker}\, Q.$$

This quantity is the Witten index of the SQM.

It is **very much** recommended to read the original paper

E.Witten, “Constraints on supersymmetric breaking”, Nuclear Physics B202 (1982) 253

If the system has a Lagrangian description, with a set of variables X and Euclidean Lagrangian $L(X, \dot{X})$, then the Witten index has a path-integral expression:

$$\mathrm{Tr}_{\mathcal{H}} (-1)^F e^{-\beta H} = \int_{X(\tau+\beta)=X(\tau)} \mathcal{D}X \; e^{-\int_0^\beta d\tau \, L(X(\tau), \dot{X}(\tau))}$$

Our system $(\mathcal{H}, Q, (-1)^F) = (S(E), \not{D}, \gamma_{d+1})$ does have a Lagrangian description, and the path-integral for the Witten index can be computed exactly.

(Done e.g. in Alvarez-Gaume, Commun.Math.Phys. 90 (1983) 161
Friedan and Windey, Physica 15D (1985) 71)

The answer is

$$\int_X \mathrm{ch}(E) \hat{A}(TX).$$

Thus, the Atiyah-Singer formula is also rederived from physics. Historically, this is earlier than Fujikawa's method (in the general case).