# Axial anomaly in a gauge and gravitational background

Dirac fermion can be defined in a general gravitational background $g = g_{\mu\nu} dx^\mu dx^\nu$ (and a gauge background $A = A_\mu dx^\mu$) in which the $\gamma$-matrices obey $\{\gamma^\mu, \gamma^\nu\} = -2g^{\mu\nu}$.

The action is given by

$$S = \int \sqrt{g}\, d^d x \, (-i)\, \bar{\psi} \not{D}_{g,A} \psi ,$$

$$\not{D}_{g,A} = \gamma^\mu D_\mu \; ; \quad D_\mu = \partial_\mu + \omega_\mu + A_\mu ,$$

Here, $\omega_\mu$ is the Levi-Civita connection on spinors

$$\omega_\mu = -\frac{1}{4} \omega_\mu{}^a{}_b \, \delta_{ac} \gamma^{cb}$$

where the connection form $\omega_\mu{}^a{}_b$ and the rotation generator $\gamma^{cb} = \frac{1}{2}[\gamma^c, \gamma^b]$ are with respect to an orthonormal frame $\{e_a\}$, $g(e_a, e_b) = \delta_{ab}$.

The chirality operator in even $d = 2n$ is given by

$$\gamma_{d+1} := \sqrt{g}\, i^{\frac{d(d-1)}{2}} \gamma^1 \cdots \gamma^d .$$

Check that it satisfies $(\gamma_{d+1})^2 = 1$, $\gamma_{d+1}\gamma^\mu = -\gamma^\mu \gamma_{d+1}$.

Axial anomaly:

$$\mathcal{D}_{g,A}(\bar{\psi} e^{-i\epsilon\gamma_{d+1}})\mathcal{D}_{g,A}(e^{-i\epsilon\gamma_{d+1}}\psi)$$

$$= \mathcal{D}_{g,A}\bar{\psi}\,\mathcal{D}_{g,A}\psi \cdot e^{i a_\epsilon^{d+1}[g,A]} \quad ;$$

$$a_\epsilon^{d+1}[g,A] = 2\,\mathrm{Tr}\left[\epsilon\gamma_{d+1} e^{-\not{D}_{g,A}^2/\Lambda^2}\right]$$

Similar computation (*)

$$= 2\int_{\mathbb{R}^d} \epsilon\, \mathrm{tr}_V\left(e^{\frac{i}{2\pi}F_A}\right) \det\left(\frac{\frac{i}{4\pi}R}{\sinh\left(\frac{i}{4\pi}R\right)}\right)^{\frac{1}{2}}$$

where $R$ = $d\times d$ matrix whose $(\mu,\nu)$ entry is the 2-form

$$R^\mu{}_\nu = \frac{1}{2}R_{\rho\lambda}{}^\mu{}_\nu\, dx^\rho \wedge dx^\lambda.$$

(*): As in the case of just $A$, the discussion of survival under $\mathrm{tr}_S(\gamma_{d+1} -)$ and the $\Lambda\to\infty$ limit simplifies the computation considerably.

In addition, there are two main ingredients:

① Lichnerowicz/Weitzenböck formula

$$\not{D}_{g,A}^2 = D_{g,A}^\dagger D_{g,A} + \frac{1}{4} R + \frac{1}{2} \gamma^{\mu\nu} F_{\mu\nu}$$

② "Schwinger's result" Phys.Rev. 82 (1951) 664

For the particle in $\mathbb{R}^2$ with constant magnetic field,

say $H = \frac{1}{2} p_x^2 + \frac{1}{2} (p_y - Bx)^2$,

$$\langle x, y | e^{-\beta H} | x, y \rangle = \frac{B/2}{2\pi \sinh(\beta B/2)}.$$

This is a simple exercise in quantum mechanics.
(Do it!)

For details, see the book by Fujikawa and Suzuki, "Path integrals and quantum anomalies" (there is also a Japanese version),

or the original papers:


Fujikawa, Phys.Rev.D 21 (1980) 2848

Endo and Takao, Prog.Theor.Phys. 703 (1985) 803

Fujikawa, Ojima and Yajima, Phys.Rev.D 34 (1986) 3223

The result is applicable to the Dirac fermion on a general Riemannian manifold $(X, g)$ with a spin structure with values in a vector bundle $E$ with a connection $A$:

$$a_{\epsilon}^{d+1}[g, A] = 2\int_X \epsilon\, ch(E)\, \hat{A}(TX)$$

with

$$ch(E) = tr_E\left(e^{\frac{i}{2\pi}F_A}\right),$$

$$\hat{A}(TX) = \det_{TX}\left(\frac{\frac{i}{4\pi}R}{\sinh\left(\frac{i}{4\pi}R\right)}\right)^{\frac{1}{2}}.$$

$ch(E)$ and $\hat{A}(TX)$ are closed forms on $X$ and their cohomology classes do not depend on the choice of $A$ and $g$. The classes are called Chern character of $E$ and A-roof genus of $TX$. The latter can be expressed in terms of the Pontrjagin classes

$$\hat{A} = 1 - \frac{1}{24}p_1 + \frac{1}{5760}(-4p_2 + p_1^2) + \cdots$$