Chiral anomaly by Fujikawa's method

$V_R = V,\ V_L = \{0\}$

For the purpose of computing the anomaly, we may consider a Dirac fermion with values in V

$$S = \int d^4x\,(-i)\,\bar{\Psi}\,\not{D}_{A,R}\,\Psi \quad ; \quad \not{D}_{A,R} = \not{\partial} + \not{A}\,P_R$$

Chiral rotation:

$$\Psi^g = (g P_R + P_L)^{-1}\Psi, \quad \bar{\Psi}^g = \bar{\Psi}\,(P_R + g P_L)$$

$$\mathcal{D}\bar{\Psi}^g\,\mathcal{D}\Psi^g = \mathrm{Det}(P_R + g P_L)^{-1} \cdot \mathrm{Det}(g P_R + P_L)\,\mathcal{D}\bar{\Psi}\,\mathcal{D}\Psi$$

$$= \mathrm{Det}\left[(P_R + g^{-1} P_L)(g P_R + P_L)\right]\mathcal{D}\bar{\Psi}\,\mathcal{D}\Psi$$

$$= \mathrm{Det}(g P_R + g^{-1} P_L)\,\mathcal{D}\bar{\Psi}\,\mathcal{D}\Psi$$

$$\therefore\ i a^R_\epsilon = \mathrm{Tr}(\epsilon P_R - \epsilon P_L) = \mathrm{Tr}(\epsilon \gamma_5) \ \cdots \text{divergent.}$$

A regularization:

$$i a^R_\epsilon[A] = \mathrm{Tr}\left(\epsilon\,\gamma_5\, e^{-\not{D}_{A,R}^2/\Lambda^2}\right)$$

Note: $\not{D}_{A,R} = \not{\partial} + \not{A}\,P_R = P_R\,\not{\partial}\,P_L + P_L\,\not{D}_A\,P_R$

$$\therefore\ \not{D}_{A,R}^2 = P_R\,\not{\partial}\,\not{D}_A\,P_R + P_L\,\not{D}_A\,\not{\partial}\,P_L$$

$$i\,a^R_\epsilon[A] = \mathrm{Tr}\left(P_R\,\epsilon\, e^{-\partial\!\!\!/ \, D\!\!\!\!/_A/\Lambda^2} - P_L\,\epsilon\, e^{-D\!\!\!\!/_A \partial\!\!\!/ /\Lambda^2}\right)$$

$$= \frac{1}{2}\mathrm{Tr}\left(\epsilon\left(e^{-\partial\!\!\!/ \, D\!\!\!\!/_A/\Lambda^2} - e^{-D\!\!\!\!/_A \partial\!\!\!/ /\Lambda^2}\right)\right) \leftarrow ①$$

$$+ \frac{1}{2}\mathrm{Tr}\left(\gamma_5\,\epsilon\left(e^{-\partial\!\!\!/ \, D\!\!\!\!/_A/\Lambda^2} + e^{-D\!\!\!\!/_A \partial\!\!\!/ /\Lambda^2}\right)\right) \leftarrow ②$$

$$② = \frac{1}{2}\int d^4x \int \frac{d^4k}{(2\pi)^4}\, e^{-ikx}\, \mathrm{tr}_{V\otimes S}\left[\epsilon\gamma_5\left(e^{-\partial\!\!\!/ \, D\!\!\!\!/_A/\Lambda^2} + e^{-D\!\!\!\!/_A \partial\!\!\!/ /\Lambda^2}\right)\right] e^{ikx}$$

long computation

$$= \int d^4x\, \frac{-1}{24\pi^2}\,\epsilon^{\mu\nu\rho\lambda}\,\mathrm{tr}_V\left[\epsilon\,\partial_\mu\left(A_\nu\partial_\rho A_\lambda + \frac{1}{2}A_\nu A_\rho A_\lambda\right)\right]$$

$$① = \frac{1}{2}\int d^4x \int \frac{d^4k}{(2\pi)^4}\, e^{-ikx}\, \mathrm{tr}_{V\otimes S}\left[\epsilon\left(e^{-\partial\!\!\!/ \, D\!\!\!\!/_A/\Lambda^2} - e^{-D\!\!\!\!/_A \partial\!\!\!/ /\Lambda^2}\right)\right] e^{ikx}$$

long computation

$$= \int d^4x\, \frac{1}{48\pi^2}\,\mathrm{tr}_V\Big[\epsilon\Big(6\Lambda^2\partial\cdot A + \partial^2\partial\cdot A + 2A_\mu\partial^\mu\partial\cdot A - 2\partial^\mu\partial\cdot A\, A_\mu$$
$$- A^\mu\partial^2 A_\mu + \partial^2 A_\mu A^\mu + \partial^\mu(A_\mu A\cdot A - A^\nu A_\mu A_\nu + A\cdot A A_\mu)\Big)\Big]$$

$$= \delta_\epsilon \int d^4x\, \frac{1}{48\pi^2}\,\mathrm{tr}\Big(-3\Lambda^2 A\cdot A + A^\mu\partial^2 A_\mu - \frac{3}{2}A_\mu\partial^\mu\partial\cdot A - A\cdot A\,\partial\cdot A$$
$$- 2A^\mu\partial^\nu A_\mu A_\nu - \frac{1}{4}A_\mu A_\nu A^\mu A^\nu\Big)$$

Thus, we find

$$i\,a^R_\epsilon[A] = \int \frac{-1}{24\pi^2}\,\mathrm{tr}_V\left[\epsilon\, d\left(AdA + \frac{1}{2}A^3\right)\right] + \delta_\epsilon\, \mathrm{loc}[A].$$