Chiral anomaly

Consider the case $V_R = V$, $V_L = \{0\}$ for simplicity.

$$Z[A] = \int \mathcal{D}\bar{\psi}_R \mathcal{D}\psi_R \, e^{\int i\bar{\psi}_R \not{D}_A \psi_R \, d^4x}$$

$$= \text{Const}\cdot \int \mathcal{D}\bar{\psi}_R \mathcal{D}\psi_R \mathcal{D}\bar{\psi}_L \mathcal{D}\psi_L \, e^{\int (i\bar{\psi}_R \not{D}_A \psi_R + i\bar{\psi}_L \not{\partial} \psi_L) d^4x}$$

$$= \text{Const} \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \, e^{\int (i\bar{\psi}\not{\partial}\psi + i\bar{\psi}\not{A} P_R \psi) d^4x}$$

where $P_R = \frac{1+\gamma_5}{2}$ projection to R-components.

For the purpose of computation of anomaly, we can consider the Dirac fermion ψ with values in V where A is coupled to $\psi_R = P_R\psi$ only.

$$\delta A \cdot J = i\bar{\psi}_R \not{\delta A} \psi_R = i\bar{\psi} \not{\delta A} P_R \psi.$$

Now let us compute $-\delta_1 \cdots \delta_n \delta_\epsilon W[A]\big|_{A=0}$.

$$-\delta_\epsilon W[A=0] = \left\langle \int d^4x \, D_\mu \epsilon \cdot J^\mu \right\rangle = 0 \text{ by "Lorentz" inv.}$$

$$-\delta\,\delta_\epsilon W[A]\big|_{A=0} = \left\langle \int d^4x\, \partial_\mu \epsilon\, J^\mu \int d^4y\, \delta A\cdot J \right\rangle_{\text{conn}}$$

$$+ \left\langle \int d^4x\, [\delta A_\mu, \epsilon]\cdot J^\mu \right\rangle$$

$\to 0$ by "Lorentz".

$$= \int d^4x\, \partial_\mu \epsilon_a(x) \int d^4y$$

$$i\bar\Psi(x)\,\gamma^\mu e^a P_R\, \Psi(x)\; i\bar\Psi(y)\, \not{\delta A}\, P_R\, \Psi(y)$$

$$= \int d^4x\, \partial_\mu \epsilon_a(x) \int d^4y$$

$$(-1)\, \mathrm{tr}_{V\otimes S}\left(i\gamma^\mu e^a P_R \Psi(x)\bar\Psi(y)\, i\not{\delta A}\, P_R \Psi(y)\bar\Psi(x)\right)$$

$$\delta A = dx^\nu e^b e^{-iqx}$$

$$= \int d^4x\, \partial_\mu \epsilon_a(x) \int d^4y\, e^{-iqy}\, \mathrm{tr}_{V\otimes S}\left(\gamma^\mu e^a P_R \Psi(x)\bar\Psi(y)\,\gamma^\nu e^b P_R \Psi(y)\bar\Psi(x)\right)$$

$$= \int d^4x\, \underline{\partial_\mu \epsilon_a(x)\, e^{-iqx}}\, \mathrm{tr}_V(e^a e^b) \underline{\int \frac{d^4k}{(2\pi)^4}\, \mathrm{tr}_S\left(\gamma^\mu P_R \frac{1}{-\not{k}}\, \gamma^\nu P_R \frac{1}{-(\not{k}-\not{q})}\right)}$$

$\sim i q_\mu \epsilon_a(x) e^{-iqx}$

!!

$\star$

quadratically divergent.

Pauli-Villars regularization

Introduce 3 regulators :

name	0	1	2	3
mass	$\Lambda_0 = 0$	Λ_1	Λ_2	Λ_3
statistics	fermi ($\epsilon_0 = 1$)	bose ($\epsilon_1 = -1$)	fermi ($\epsilon_2 = 1$)	bose ($\epsilon_3 = -1$)
	original	regulators		

This replaces $\star$ by

$$I^{\mu\nu}(q) := \int \frac{d^4k}{(2\pi)^4} \sum_{i=0}^{3} \epsilon_i \underbrace{\mathrm{tr}_S\left(\gamma^\mu P_R \frac{1}{-\not{k} + \Lambda_i} \gamma^\nu P_R \frac{1}{-(\not{k}-\not{q}) + \Lambda_i} \right)}_{= \frac{\mathrm{tr}_S\left(\gamma^\mu P_R (\not{k} + \Lambda_i) \gamma^\nu P_R ((\not{k}-\not{q}) + \Lambda_i)\right)}{(k^2 + \Lambda_i^2)((k-p)^2 + \Lambda_i^2)}}$$

Since γ^μ maps S_R to S_L and S_L to S_R, $P_R \gamma^\mu P_R = 0$.

$\therefore$ The numerator $\mathrm{tr}_S\left(\gamma^\mu P_R (\not{k} + \Lambda_i) \gamma^\nu P_R ((\not{k}-\not{q}) + \Lambda_i)\right)$

$$= \mathrm{tr}_S\left(\gamma^\mu P_R \not{k} \gamma^\nu P_R (\not{k}-\not{q})\right)$$

$P_R (\not{k}-\not{q}) \gamma^\mu = (\not{k}-\not{q}) \gamma^\mu P_R$ & $P_R^2 = P_R$

$$= \mathrm{tr}_S\left(P_R \not{k} \gamma^\nu (\not{k}-\not{q}) \gamma^\mu\right)$$

$$\mathrm{tr}_S P_R \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\lambda = \frac{1}{2} \mathrm{tr}_S \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\lambda + \frac{1}{2} \mathrm{tr}_S \gamma_5 \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\lambda$$

$$= 2\delta^{\mu\nu}\delta^{\rho\lambda} - 2\delta^{\mu\rho}\delta^{\nu\lambda} + 2\delta^{\mu\lambda}\delta^{\nu\rho} - 2\epsilon^{\mu\nu\rho\lambda}$$

$$= 2k^\nu (k-q)^\mu - 2k\cdot(k-q)\,\delta^{\nu\mu} + 2k^\mu (k-q)^\nu \underbrace{- 2\epsilon^{\rho\nu\lambda\mu} k_\rho (k-q)_\lambda}$$

The last term will not contribute as $k_\rho \propto q_\rho$,

$k_\rho k_\lambda \propto \# \delta_{\rho\lambda} + q_\rho q_\lambda$ after the integration (if convergent)

and $\epsilon^{\rho\nu\lambda\mu} q_\rho q_\lambda = \epsilon^{\rho\nu\lambda\mu} \delta_{\rho\lambda} = 0.$

$$\therefore\ I^{\mu\nu}(q) = \int \frac{d^4k}{(2\pi)^4} \sum_{i=0}^{3} \epsilon_i \, 2 \, \frac{(k-q)^\mu k^\nu - \delta^{\mu\nu} k\cdot(k-q) + k^\mu (k-q)^\nu}{(k^2+\Lambda_i^2)((k-q)^2+\Lambda_i^2)}$$

$$\frac{1}{(k^2+\Lambda_i^2)((k-q)^2+\Lambda_i^2)} = \int_0^1 \frac{dx}{[\underbrace{(1-x)(k^2+\Lambda_i^2) + x((k-q)^2+\Lambda_i^2)}_{(\underbrace{k-xq}_{\ell})^2 + \underbrace{x(1-x)q^2 + \Lambda_i^2}_{=:\Delta_i}}]^2}$$

$$= 2\int_0^1 dx \int \frac{d^4\ell}{(2\pi)^4} \sum_{i=0}^{3} \epsilon_i \frac{\star}{[\ell^2+\Delta_i]^2}$$

$$\star = (\ell+xq-q)^\mu (\ell+xq)^\nu + (\ell+xq)^\mu (\ell+xq-q)^\nu - \delta^{\mu\nu} (\ell+xq)\cdot(\ell+xq-q)$$

$$= 2\ell^\mu \ell^\nu - \delta^{\mu\nu}\ell^2 + x(1-x)(\delta^{\mu\nu} q^2 - 2q^\mu q^\nu) + \ell\text{-linear}$$

Exercise: Show that the ℓ-integral is convergent if

- $\sum_{i=0}^{3} \epsilon_i = 0$ (This holds by the set-up)
- $\sum_{i=1}^{3} \epsilon_i \Lambda_i^2 = 0$

From now on, we assume this. Then,

- ℓ-linear term do not contribute to the integral.
- $\ell^\mu \ell^\nu = \frac{1}{4} \delta^{\mu\nu} \ell^2$ after the integration.

Thus, $\star$ can be replaced by

$$\star' = -\frac{1}{2} \delta^{\mu\nu} \ell^2 + x(1-x)\left(\delta^{\mu\nu} q^2 - 2 q^\mu q^\nu\right).$$

The integral may be performed as

$$I^{\mu\nu}(q) = \lim_{\Lambda \to \infty} 2 \int_0^1 dx \int_{\ell^2 \le \Lambda^2} \frac{d^4\ell}{(2\pi)^4} \sum_{i=0}^{3} \epsilon_i \frac{\star'}{[\ell^2 + \Delta_i]^2}$$

$$= \lim_{\Lambda \to \infty} 2 \int_0^1 dx \sum_{i=0}^{3} \epsilon_i \int_{\ell^2 \le \Lambda^2} \frac{d^4\ell}{(2\pi)^4} \frac{\star'}{[\ell^2 + \Delta_i]^2}$$

The bounded ℓ-integral can be done for each i:

$$\int_{\ell^2\le\Lambda^2} \frac{d^4\ell}{(2\pi)^4}\frac{(\ell^2)^n}{(\ell^2+\Delta_i)^2} = \frac{\mathrm{Vol}(S^3)}{2(2\pi)^4}\int_0^{\Lambda^2}\frac{(\ell^2)^{n+1}\,d\ell^2}{(\ell^2+\Delta_i)^2}$$

$$= \frac{\Delta_i^n}{(4\pi)^2}\int_0^{\Lambda^2/\Delta_i}\frac{t^{n+1}\,dt}{(t+1)^2}$$

$$= \frac{1}{(4\pi)^2}\times\begin{cases} -\log\Delta_i + \underline{\log\Lambda^2} + O\left(\frac{\Delta_i}{\Lambda^2}\right) & n=0\\[2ex] 2\Delta_i\log\Delta_i + \underline{\Lambda^2 - 2\Delta_i\log\Lambda^2} + \Delta_i\, O\left(\frac{\Delta_i}{\Lambda^2}\right) & n=1\end{cases}$$

The underlined terms vanish after $\sum_{i=0}^{3}\epsilon_i$, under the assumption $\sum_{i=0}^{3}\epsilon_i = 0$, $\sum_{i=1}^{3}\epsilon_i\Lambda_i^2 = 0$.

Then, we can take the limit $\Lambda\to\infty$ which leaves us with

$$I^{\mu\nu}(q) = \frac{-2}{(4\pi)^2}\int_0^1 dx\sum_{i=0}^{3}\epsilon_i\Big\{\delta^{\mu\nu}\Delta_i\log\Delta_i + x(1-x)\left(\delta^{\mu\nu}q^2 - 2q^\mu q^\nu\right)\log\Delta_i\Big\}$$

For $i=1,2,3$

$$\begin{cases} \log\Delta_i = \log\Lambda_i^2 + O\left(\frac{q^2}{\Lambda_i^2}\right)\\[2ex] \Delta_i\log\Delta_i = \left(x(1-x)q^2 + \Lambda_i^2\right)\log\Lambda_i^2 + x(1-x)q^2 + q^2\, O\left(\frac{q^2}{\Lambda_i^2}\right)\end{cases}$$

$$= \frac{-2}{(4\pi)^2} \int_0^1 dx \Big[x(1-x)\left(2\delta^{\mu\nu} q^2 - 2q^\mu q^\nu\right) \log\left(x(1-x) q^2\right)$$

$$+ \sum_{i=1}^{3} \epsilon_i \Big\{ x(1-x)\left(2\delta^{\mu\nu} q^2 - 2q^\mu q^\nu\right) \log \Lambda_i^2$$

$$+ \delta^{\mu\nu}\left(\Lambda_i^2 \log \Lambda_i^2 + x(1-x) q^2\right) + \underline{q^2 O\left(\frac{q^2}{\Lambda_i^2}\right)} \Big\} \Big]$$

Omit this below

$$\int_0^1 dx\, x(1-x) = \frac{1}{6}$$

$$\int_0^1 dx\, x(1-x) \log x(1-x) = -\frac{5}{18}$$

$$= \frac{-2}{(4\pi)^2} \Big[2\left(\delta^{\mu\nu} q^2 - q^\mu q^\nu\right)\left(\frac{1}{6}\log q^2 - \frac{5}{18}\right)$$

$$+ \sum_{i=1}^{3} \epsilon_i \Big\{ \frac{1}{3}\left(\delta^{\mu\nu} q^2 - q^\mu q^\nu\right) \log \Lambda_i^2 + \delta^{\mu\nu}\left(\Lambda_i^2 \log \Lambda_i^2 + \frac{1}{6} q^2\right) \Big\} \Big].$$

We found

$$I^{\mu\nu}(q) = \frac{-2}{(4\pi)^2} \Big[\left(\delta^{\mu\nu} q^2 - q^\mu q^\nu\right) \frac{1}{3}\left(\log q^2 - \frac{5}{3} + \sum_{i=1}^{3} \epsilon_i \log \Lambda_i^2\right)$$

$$+ \delta^{\mu\nu}\left(\sum_{i=1}^{3} \epsilon_i \Lambda_i^2 \log \Lambda_i^2 - \frac{1}{6} q^2\right) \Big]$$

Thus, we find

$$-\delta\,\delta_\epsilon W[A]\big|_{A=0}$$

$$= i\int d^4x\, \epsilon_a(x)\, e^{-iqx}\, \mathrm{tr}_V(e^a e^b)\, q_\mu \Sigma^{\mu\nu}(q)$$

$$q_\mu \Sigma^{\mu\nu}(q) = \frac{-2}{(4\pi)^2} q^\nu \left(\sum_{i=1}^{3} \epsilon_i \Lambda_i^2 \log \Lambda_i^2 - \frac{1}{6} q^2\right)$$

$$\neq 0.$$

This does not match with $\delta\, \mathcal{Q}_\epsilon[A] = 0$ for the claimed formula.

If fact, this can be cancelled by adding a local counter term to $W[A]$. Let's consider

$$\Delta W[A] = \int d^4x\, \mathrm{tr}_V\left(A_\mu \delta^{\mu\lambda}(C + D\partial^2) A_\lambda\right)$$

$$\delta_\epsilon \Delta W[A] = \int d^4x\, 2\mathrm{tr}_V\left(D_\mu \epsilon\, \delta^{\mu\lambda}(C + D\partial^2) A_\lambda\right)$$

$$\delta\,\delta_\epsilon \Delta W[A]\big|_{A=0} = 2\int d^4x\, \mathrm{tr}_V\left(\partial_\mu \epsilon\, \delta^{\mu\lambda}(C + D\partial^2)\,\delta A_\lambda\right)$$

$$\delta A = dx^\nu e^b e^{-iqx}$$

$$= 2\int d^4x\, \underbrace{\partial^\nu \epsilon_a(x)\, e^{-iqx}}_{\sim\, \epsilon_a(x)\, i q^\nu e^{-iqx}}\, \mathrm{tr}_V(e^a e^b)\left(C + D(-q^2)\right)$$

$$\delta\,\delta_\epsilon\big(W[A] + \Delta W[A]\big) = 0$$

$$\Leftrightarrow \quad \frac{-2}{(4\pi)^2} q^\nu \left(\sum_{i=1}^{3} \epsilon_i \Lambda_i^2 \log \Lambda_i^2 - \frac{1}{6} q^2 \right) = 2\left(C + D(-q^2)\right)$$

i.e. $\quad C = \frac{-1}{(4\pi)^2} \sum_{i=1}^{3} \epsilon_i \Lambda_i^2 \log \Lambda_i^2 \quad$ and $\quad D = -\frac{1}{6(4\pi)^2}$

We may also add

$$\Delta' W[A] = E \int d^4x \, \mathrm{tr}_V \left[A_\mu \left(\delta^{\mu\nu} \partial^2 - \partial^\mu \partial^\nu \right) A_\nu \right]$$

which has $\delta\,\delta_\epsilon \Delta' W[A] \Big|_{A=0} = 0.$

We shall consider $W'[A] = W[A] + \Delta W[A] + \Delta' W[A]$

for the above C and D and for some E.

Now let us compute $\; -\delta_2 \delta_1 \delta_\epsilon W'[A] \Big|_0$

$$= \boxed{ -\int d^4x \, D_\mu \epsilon_a(x) \, \delta_2 \delta_1 \frac{\delta W'[A]}{\delta A_{\mu a}(x)} \Big|_0 } =: ⊛$$

$$- \int d^4x \, [\delta_2 A_\mu, \epsilon]_a(x) \, \boxed{ \delta_1 \frac{\delta W'[A]}{\delta A_{\mu a}(x)} \Big|_0 } + (1 \leftrightarrow 2) \qquad =: ⊛$$

for $\quad \delta_1 A = dx^b e^b e^{-iqx} \quad$ & $\quad \delta_2 A = dx^\rho e^c e^{-ipx}$

$$\textcircled{*} = \underbrace{\delta_1 \frac{\delta W[A]}{\delta A_{\mu a}(x)}\Big|_0}_{①} + \underbrace{\delta_1 \frac{\delta \Delta W[A]}{\delta A_{\mu a}(x)}\Big|_0}_{②} + \underbrace{\delta_1 \frac{\delta \Delta' W[A]}{\delta A_{\mu a}(x)}\Big|_0}_{③}$$

① was essentially computed in

$$\delta_1 \delta_\epsilon W[A]\Big|_0 = \delta_1 \int d^4x\, D_\mu \epsilon_a(x) \frac{\delta W[A]}{\delta A_{\mu a}(x)}\Big|_0$$

$$= \int d^4x\, \partial_\mu \epsilon_a(x)\, \underbrace{\delta_1 \frac{\delta W[A]}{\delta A_{\mu a}(x)}\Big|_0}$$

Recall $-LHS = \int d^4x\, \partial_\mu \epsilon_a(x)\, e^{-iqx}\, \mathrm{tr}_V(e^a e^b)\, I^{\mu\nu}(q)$

$\therefore$ ① $= -e^{-iqx}\, \mathrm{tr}_V(e^a e^b)\, I^{\mu\nu}(q)$

② & ③ are straightforward to obtain.

$$\textcircled{*} = e^{-iqx}\, \mathrm{tr}_V(e^a e^b)\Big\{ -I^{\mu\nu}(q) + 2\delta^{\mu\nu}(C + D(-q^2)) + 2E(\delta^{\mu\nu}(-q^2) + q^\mu q^\nu) \Big\}$$

The above C, D and $E = \hat{E}/3(4\pi)^2$

$$= e^{-iqx}\, \mathrm{tr}_V(e^a e^b)\,(\delta^{\mu\nu} q^2 - q^\mu q^\nu) \times \frac{2}{3(4\pi)^2}\left(\log q^2 - \frac{5}{3} + \sum_{i=1}^{3} \epsilon_i \log \Lambda_i^2 - \hat{E}\right)$$

For ⊛, addition of $\Delta W + \Delta' W$ has no effect.

$$⊛ = \sum_{i=0}^{3} \left\langle \int d^4x \, i \bar{\Psi}_i \not{\partial} \epsilon P_R \Psi_i \int d^4y \, i \bar{\Psi}_i \not{\delta_1 A} P_R \Psi_i \int d^4z \, i \bar{\Psi}_i \not{\delta_2 A} P_R \Psi_i \right\rangle_{\text{conn}}$$

$$\delta_1 A = dx^\nu e^b e^{-iqx}, \quad \delta_2 A = dx^\rho e^c e^{-ipx}$$

$$= \sum_{i=0}^{3} i^3 \int d^4x \, \partial_\mu \epsilon_a(x) \int d^4y \, e^{-iqy} \int d^4z \, e^{-ipz}$$

$$(-\epsilon_i) \, \mathrm{tr}_{V \otimes S} \left(\gamma^\mu e^a P_R \overbrace{\Psi_i(x) \bar{\Psi}_i(y)} \gamma^\nu e^b P_R \overbrace{\Psi_i(y) \bar{\Psi}_i(z)} \gamma^\rho e^c P_R \overbrace{\Psi_i(z) \bar{\Psi}_i(x)} \right)$$

+ exchange

$$= i \int d^4x \, \partial_\mu \epsilon_a(x) \, e^{-i(p+q)x} \int \frac{d^4k}{(2\pi)^4} \sum_{i=0}^{3} \epsilon_i$$

$$\mathrm{tr}_{V \otimes S} \left(\gamma^\mu e^a P_R \frac{1}{-(\not{k}+\not{q}) + \Lambda_i} \gamma^\nu e^b P_R \frac{1}{-\not{k} + \Lambda_i} \gamma^\rho e^c P_R \frac{1}{-(\not{k}-\not{p}) + \Lambda_i} \right)$$

+ exchange

$$\epsilon_a(x) \, i(p+q)_\mu \, e^{-i(p+q)x}$$

$$= -\int d^4x \, \epsilon_a(x) \, e^{-i(p+q)x} \, \mathrm{tr}_V(e^a e^b e^c) \int \frac{d^4k}{(2\pi)^4} \sum_{i=0}^{3} \epsilon_i \, g(k, \Lambda_i)$$

+ exchange

where

$$g(k,m) = \mathrm{tr}_S \left((\not{p}+\not{q}) P_R \frac{1}{-(\not{k}+\not{q}) + m} \gamma^\nu P_R \frac{1}{-\not{k} + m} \gamma^\rho P_R \frac{1}{-(\not{k}-\not{p}) + m} \right)$$

$$(\not{p}+\not{q})\,P_R = (\not{k}+\not{q} - (\not{k}-\not{p}))\,P_R = P_L(\not{k}+\not{q}) - (\not{k}-\not{p})\,P_R$$

$$= (-(\not{k}-\not{p})+m)\,P_R - P_L(-(\not{k}+\not{q})+m) + m(-P_R+P_L)$$

$$g(k,m) = \mathrm{tr}_S\Bigg(P_R \frac{1}{-(\not{k}+\not{q})+m}\gamma^\nu P_R \frac{1}{-\not{k}+m}\gamma^\rho P_R$$

$$- P_L\gamma^\nu P_R \frac{1}{-\not{k}+m}\gamma^\rho P_R \frac{1}{-(\not{k}-\not{p})+m}$$

$$+ m(P_L-P_R)\frac{1}{-(\not{k}+\not{q})+m}\gamma^\nu P_R \frac{1}{-\not{k}+m}\gamma^\rho P_R \frac{1}{-(\not{k}-\not{p})+m}\Bigg)$$

$$= \mathrm{tr}_S\Bigg(\gamma^\nu P_R \frac{1}{-\not{k}+m}\gamma^\rho P_R \frac{1}{-(\not{k}+\not{q})+m}$$

$$- \gamma^\nu P_R \frac{1}{-\not{k}+m}\gamma^\rho P_R \frac{1}{-(\not{k}-\not{p})+m}$$

$$+ m\,\frac{(P_L-P_R)((\not{k}+\not{q})+m)\gamma^\nu P_R(\not{k}+m)\gamma^\rho P_R((\not{k}-\not{p})+m)}{((k+q)^2+m^2)(k^2+m^2)((k-p)^2+m^2)}\Bigg)$$

c

$$c = (-P_R(\not{k}+\not{q}) + mP_L)\,\gamma^\nu(\not{k}+m)\gamma^\rho P_R((\not{k}-\not{p})+m)$$

$$\approx \gamma^\nu(\not{k}+m)\gamma^\rho \underbrace{((\not{k}-\not{p})P_L + mP_R)(-P_R(\not{k}+\not{q}) + mP_L)}_{(\not{k}-\not{p})\,mP_L - mP_R(\not{k}+\not{q})}$$

$\uparrow$ = inside trace

$$= mP_R((\not{k}-\not{p}) - (\not{k}+\not{q})) = -mP_R(\not{p}+\not{q})$$

$$\approx -m P_R(\not{p}+\not{q})\,\gamma^\nu \not{k}\,\gamma^\rho$$

$$\therefore\ g(k,m) = \mathrm{tr}_S\Bigg(\gamma^\nu P_R \frac{1}{-\not{k}+m} \gamma^\rho P_R \frac{1}{-(\not{k}+\not{q})+m}$$

$$- \gamma^\nu P_R \frac{1}{-\not{k}+m} \gamma^\rho P_R \frac{1}{-(\not{k}-\not{p})+m}$$

$$- \frac{m^2 P_R (\not{p}+\not{q}) \gamma^\nu \not{k} \gamma^\rho}{((k+q)^2+m^2)(k^2+m^2)((k-p)^2+m^2)} \Bigg)$$

$$\therefore \int \frac{d^4k}{(2\pi)^4} \sum_{i=0}^{3} \epsilon_i\, g(k,\Lambda_i) = I^{\nu\rho}(-q) - I^{\nu\rho}(p) + J^{\nu\rho}(p,q)$$

where

$$J^{\nu\rho}(p,q) := \int \frac{d^4k}{(2\pi)^4} \sum_{i=1}^{3} \epsilon_i \frac{(-\Lambda_i^2)\, \mathrm{tr}_S(P_R (\not{p}+\not{q}) \gamma^\nu \not{k} \gamma^\rho)}{((k+q)^2+\Lambda_i^2)(k^2+\Lambda_i^2)((k-p)^2+\Lambda_i^2)}$$

$$= \frac{1}{(4\pi)^2} \int_{\triangle} dy\, dz \sum_{i=1}^{3} \epsilon_i \frac{(-\Lambda_i^2)\, \mathrm{tr}_S(P_R (\not{p}+\not{q}) \gamma^\nu (-y\not{q}+z\not{p}) \gamma^\rho)}{y(1-y)q^2 + z(1-z)p^2 + 2yz\, qp + \Lambda_i^2}$$

where $\triangle = \left\{ (y,z) \in \mathbb{R}^2 \;\middle|\; \begin{matrix} y, z \geq 0 \\ y+z \leq 1 \end{matrix} \right\}$

$$\sum_{i=1}^{3} \epsilon_i \frac{-\Lambda_i^2}{\cdots + \Lambda_i^2} = 1 + \text{terms that vanish as } p/\Lambda_i,\ q/\Lambda_i \to 0$$

(omit below)

$$= \frac{1}{(4\pi)^2} \int_{\triangle} dy\, dz\ \mathrm{tr}_S(P_R (\not{p}+\not{q}) \gamma^\nu (-y\not{q}+z\not{p}) \gamma^\rho)$$

$$\int_{\triangle} dy\,dz \cdot y = \int_{\triangle} dy\,dz \cdot z = \frac{1}{6}$$

$$= \frac{1}{(4\pi)^2}\frac{1}{6}\,\mathrm{tr}_S\left(P_R\,\cancel{(p+q)}\,\gamma^\nu(-\not{q}+\not{p})\gamma^\rho\right)$$

$$= \frac{1}{3(4\pi)^2}\Big\{(p+q)^\nu(p-q)^\rho + (p+q)^\rho(p-q)^\nu - \delta^{\nu\rho}\,(p+q)\cdot(p-q) - \epsilon^{\lambda\nu\sigma\rho}(p+q)_\lambda(p-q)_\sigma\Big\}$$

$$= \frac{1}{3(4\pi)^2}\Big\{(\delta^{\nu\rho}q^2 - 2q^\nu q^\rho) - (\delta^{\nu\rho}p^2 - 2p^\nu p^\rho) - 2\epsilon^{\lambda\nu\sigma\rho}q_\lambda p_\sigma\Big\}$$

$$\therefore \int\frac{d^4k}{(2\pi)^4}\sum_{i=0}^{3}\epsilon_i\, g(k,\Lambda_i) = I^{\nu\rho}(-q) - I^{\nu\rho}(p) + J^{\nu\rho}(p,q)$$

$$\begin{aligned}
= \frac{1}{3(4\pi)^2}\Big[& -2(\delta^{\nu\rho}q^2 - q^\nu q^\rho)\Big(\log q^2 - \frac{5}{3} + \sum_{i=1}^{3}\epsilon_i\log\Lambda_i^2\Big) \\
& + \delta^{\nu\rho}\Big(-6\sum_{i=1}^{3}\cancel{\epsilon_i\Lambda_i^2}\log\Lambda_i^2 + q^2\Big) \\
& + 2(\delta^{\nu\rho}p^2 - p^\nu p^\rho)\Big(\log p^2 - \frac{5}{3} + \sum_{i=1}^{3}\epsilon_i\log\Lambda_i^2\Big) \\
& - \delta^{\nu\rho}\Big(-6\sum_{i=1}^{3}\cancel{\epsilon_i\Lambda_i^2}\log\Lambda_i^2 + p^2\Big) \\
& + (\delta^{\nu\rho}q^2 - 2q^\nu q^\rho) - (\delta^{\nu\rho}p^2 - 2p^\nu p^\rho) - 2\epsilon^{\lambda\nu\sigma\rho}q_\lambda p_\sigma\Big]
\end{aligned}$$

$$= \frac{-2}{3(4\pi)^2}\Big[(\delta^{\nu\rho} q^2 - q^\nu q^\rho)\Big(\log q^2 - \frac{8}{3} + \sum_{i=1}^{3} \epsilon_i \log \Lambda_i^2\Big)$$

$$- (\delta^{\nu\rho} p^2 - p^\nu p^\rho)\Big(\log p^2 - \frac{8}{3} + \sum_{i=1}^{3} \epsilon_i \log \Lambda_i^2\Big)$$

$$+ \epsilon^{\lambda\nu\sigma\rho} q_\lambda p_\sigma \Big]$$

$$\therefore \star = -\int d^4x\, \epsilon_a(x)\, e^{-i(p+q)x}\, \mathrm{tr}_V(e^a e^b e^c) \times \text{above}$$

$$+ (\nu, b, q) \leftrightarrow (\rho, c, p)$$

$$= \int d^4x\, \epsilon_a(x)\, e^{-i(p+q)x} \frac{2}{3(4\pi)^2}$$

$$\Big[\mathrm{tr}_V(e^a[e^b, e^c]) \Big\{ (\delta^{\nu\rho} q^2 - q^\nu q^\rho)\Big(\log q^2 - \frac{8}{3} + \sum_{i=1}^{3} \epsilon_i \log \Lambda_i^2\Big)$$

$$- (\delta^{\nu\rho} p^2 - p^\nu p^\rho)\Big(\log p^2 - \frac{8}{3} + \sum_{i=1}^{3} \epsilon_i \log \Lambda_i^2\Big)\Big\}$$

$$+ \mathrm{tr}_V(e^a\{e^b, e^c\})\, \epsilon^{\lambda\nu\sigma\rho} q_\lambda p_\sigma \Big]$$

On the other hand

$$- \int d^4x\, \underbrace{[\delta_2 A_\mu, \epsilon]_a(x)}_{[\delta^\rho_\mu e^c e^{-ipx},\ \epsilon(x)]_a} \boxed{\delta_1 \frac{\delta W'[A]}{\delta A_{\mu a}(x)}\Big|_0}$$

$$\| $$

$$\otimes = e^{-iqx}\, \mathrm{tr}_V(e^a e^b)\, (\delta^{\mu\nu} q^2 - q^\mu q^\nu) \times$$

$$\frac{2}{3(4\pi)^2}\Big(\log q^2 - \frac{5}{3} + \sum_{i=1}^{3} \epsilon_i \log \Lambda_i^2 - \hat{E}\Big)$$

$$= -\int d^4x\, e^{-i(p+q)x}\, \underbrace{\mathrm{tr}_V([e^c, \epsilon(x)]e^b)}_{\mathrm{tr}_V(\epsilon(x)[e^b,e^c])} (\delta^{\mu\nu} q^2 - q^\mu q^\nu) \times$$

$$\frac{2}{3(4\pi)^2} \left(\log q^2 - \frac{5}{3} + \sum_{i=1}^{3} \epsilon_i \log \Lambda_i^2 - \hat{E} \right)$$

If we set $\hat{E} = 1$, this cancels with the $\delta^{\mu\nu} q^2 - q^\mu q^\nu$ terms of ⊛. Similarly for the $(1 \leftrightarrow 2)$ term.

In total, we find

$$-\delta_2 \delta_1 \delta_\epsilon W'[A]\big|_{A=0}$$

$$= \int d^4x\, e^{-i(p+q)x} \frac{2}{3(4\pi)^2} \mathrm{tr}_V(\epsilon(x)\{e^b, e^c\})\, \epsilon^{\lambda\nu\sigma\rho} q_\lambda p_\sigma .$$

These are to be compared with

$$\delta_n \cdots \delta_1 \underbrace{i \int \frac{i}{24\pi^2} \mathrm{tr}_V\left(\epsilon\, d(A dA + \tfrac{1}{2} A^3)\right)}_{X} \Big|_{A=0}$$

$$X[0] = 0$$

$$\delta X[A]\big|_{A=0} = 0$$

$$\delta_2 \delta_1 X[A]\big|_{A=0} = \int \frac{-1}{24\pi^2} \mathrm{tr}_V\left(\epsilon(d\delta_1 A\, d\delta A_2 + d\delta_2 A\, d\delta A_1)\right)$$

$$= \int \frac{-1}{24\pi^2} \operatorname{tr}_V \left(\epsilon \, d(e^b dx^\nu e^{-iqx}) d(e^c dx^\rho e^{-ipx}) + \text{exchange} \right)$$

$$= \int \frac{1}{24\pi^2} e^{-i(p+q)x} \left\{ \operatorname{tr}_V(\epsilon e^b e^c) q_\lambda dx^\lambda dx^\nu p_\sigma dx^\sigma dx^\rho + \text{exchange} \right\}$$

$$= \int \frac{1}{24\pi^2} e^{-i(p+q)x} \operatorname{tr}_V(\epsilon \{e^b, e^c\}) \epsilon^{\lambda\nu\sigma\rho} q_\lambda p_\sigma \, d^4x$$

$$= -\delta_2 \delta_1 \delta_\epsilon W'[A] \big|_{A=0}.$$

Thus if $-\delta_\epsilon W'[A] =: i a_\epsilon[A]$, then we found

$$\boxed{a_\epsilon[A] = \int \frac{i}{24\pi^2} \operatorname{tr}_V \left(\epsilon \, d\left(A dA + \frac{1}{2} A^3\right)\right) \quad \text{modulo } O(A^3) \text{ terms}}$$