An example

Q a smooth manifold of dimension n.

$\pi : T^*Q \to Q$ its cotangent bundle.

On an open subset $U \subset Q$ with a coordinate system $\{(q^1, \cdots, q^n)\}$

an element $p \in \pi^{-1}(U)$ is expressed as

$$p = \sum_i p_i \, dq^i .$$

The function $p \mapsto p_i$ on $\pi^{-1}(U)$ is denoted by p_i,

i.e. $p_i(p) = p\left(\frac{\partial}{\partial q^i}\right)$.

The function $q^i \circ \pi$ on $\pi^{-1}(U)$ is denoted simply by q^i.

Then $\{(q^1, \cdots, q^n, p_1, \cdots, p_n)\}$ is a coordinate system on $\pi^{-1}(U)$.

A symplectic form on $\pi^{-1}(U)$ is given by

$$\omega = \sum_i dq^i \wedge dp_i .$$

This does not depend on the choice of coordinate system, and defines a symplectic form on T^*Q. Thus,

(T^*Q, ω) is a symplectic manifold.

The case of cotangent bundle

$$M = T^*Q \ni p = \sum_i p_i dq^i, \quad \omega = \sum_i dq^i \wedge dp_i$$

$Q \circlearrowright G$ right action

$\rightsquigarrow T^*Q \circlearrowright G$ right action via $pg = g^{-1*}p$.

Let v_ξ be the vector field on Q generated by $\xi \in \mathfrak{g}$. Then the vector field X_ξ on T^*Q generated by ξ is locally expressed as

$$X_\xi = v_\xi^i(q)\frac{\partial}{\partial q^i} - \partial_i v_\xi^j(q)\, p_j \frac{\partial}{\partial p_i}.$$

☺ For $g_t = e^{t\xi}$ and for $p \in T_q^*Q$,

- $q^i(pg_t) = q^i(qg_t)$ hence

$$(X_\xi q^i)(p) = \frac{d}{dt} q^i(pg_t)\Big|_{t=0} = v_\xi^i(q).$$

- $p_i(pg_t) = pg_t\left(\left(\frac{\partial}{\partial q^i}\right)_{qg_t}\right) = g_t^{-1*}p\left(\left(\frac{\partial}{\partial q^i}\right)_{qg_t}\right) = p\left(g_{t*}^{-1}\left(\frac{\partial}{\partial q^i}\right)_{qg_t}\right).$

Since $\frac{d}{dt} g_{t*}^{-1}\left(\frac{\partial}{\partial q^i}\right)_{qg_t}\Big|_{t=0} = \mathcal{L}_{v_\xi}\left(\frac{\partial}{\partial q_i}\right)_q = -\partial_i v_\xi^j(q)\frac{\partial}{\partial q^j}$,

$$X_\xi p_i(p) = \frac{d}{dt} p_i(pg_t)\Big|_{t=0} = p\left(-\partial_i v_\xi^j(q)\frac{\partial}{\partial q^j}\right) = -\partial_i v_\xi^j(q)\, p_j.$$

$\square$

Then

$$i_{X_\xi}\omega = V_\xi^i(q)\,dp_i + dq^i\,\partial_i V_\xi^j(q)\,p_j = d\left(p_i V_\xi^i(q)\right).$$

Thus, $\langle \mu(p), \xi\rangle := -p(V_\xi)$ satisfies $d\langle \mu, \xi\rangle = -i_{X_\xi}\omega$

which is the condition (i) of moment map. Condition (ii)?

I.e. for $p \in T_q^*Q$, $\quad pg(V_\xi(qg)) \overset{?}{=} p(V_{g\xi g^{-1}}(q))$

$$\begin{array}{c} pg(V_\xi(qg)) \\ \| \\ p(g^{-1}_* V_\xi(qg)) \end{array}$$

$V_\xi(qg)$ is tangent to the curve $qge^{t\xi}$ at $t=0$. Thus

$g^{-1}_* V_\xi(qg)$ is tangent to the curve $\underbrace{qge^{t\xi}g^{-1}}_{= qe^{tg\xi g^{-1}}}$ at $t=0$,

hence is equal to $V_{g\xi g^{-1}}(q)$.

i.e. $g^{-1}_* V_\xi(qg) = V_{g\xi g^{-1}}(q)$.

So YES, the condition (ii) is also satisfied.

We conclude:

$$\boxed{\begin{array}{c}\text{A moment map } \mu: T^*Q \to \mathfrak{g}^* \text{ is provided by} \\ \langle \mu(p), \xi\rangle = -p(V_\xi)\end{array}}$$

Suppose Q/G has a structure of a smooth manifold and

$Q \to Q/G$ is a principal G-bundle.

Then, as symplectic manifolds

$$\boxed{\begin{array}{l}\text{the symplectic quotient } \mu^{-1}(0)/G \\ \qquad \cong \text{ the cotangent bundle } T^*(Q/G)\end{array}}$$

$\because$ It is enough to prove this on each $U \subset Q/G$ with a local trivialization $Q|_U \cong U \times G$.

There, $T^*Q|_U \cong T^*U \times T^*G$ as a symplectic manifold.

Let $\underset{p}{\overset{\psi}{}} \longleftrightarrow \underset{(p_U, p_G)}{\overset{\psi}{}}$ be the map.

For $\xi \in \mathfrak{g}$, $v_\xi(q, g) = (0, g\xi)$ and hence

$$\langle \mu(p), \xi \rangle = -(p_U, p_G)(0, g\xi) = -p_G(g\xi).$$

$\therefore \quad \mu(p) = 0 \iff p_G = 0$

$\therefore \quad \mu^{-1}(0)|_U \cong T^*U \times \underset{\substack{\uparrow \\ \text{0-section}}}{G}$ and $\mu^{-1}(0)|_U/G \cong T^*U$. $\square$

An example

Recall that the Yang-Mills theory with gauge group G on d-dimensional spacetime $\mathbb{R}^d = \mathbb{R}^{d-1}_{\mathbf{x}} \times \mathbb{R}_t$ is equivalent to the Hamiltonian system with conjugate variables $A_{ia}(\mathbf{x})$, $E_{jb}(\mathbf{x})$ with $i,j=1,\dots,d-1$, $a=1,\dots,\dim G$ having Poisson bracket

$$\{A_{ia}(\mathbf{x}), E_{jb}(\mathbf{y})\} = \delta_{ij}\delta_{ab}\delta(\mathbf{x}-\mathbf{y}),$$

Hamiltonian

$$H(A,E) = \int d^{d-1}\mathbf{x} \left(\frac{e^2}{2} \sum_i E_i(\mathbf{x})^2 + \frac{1}{2e^2} \sum_{i<j} F_{ij}(\mathbf{x})^2 \right)$$

$$\text{where } F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j]$$

and a constraint

$$\Phi(\mathbf{x}) := \mathbb{D}\cdot\mathbb{E} = \sum_i D_i E_i = \sum_i \left(\partial_i E_i + [A_i, E_i]\right) = 0.$$

The phase space of this theory may be identified with the cotangent bundle $T^*\mathcal{A}$ of the space

$$\mathcal{A} = \{ A \in \Omega^1(\mathbb{R}^{d-1}, \mathfrak{g}) \mid A(\mathbf{x}) \to 0 \text{ as } |\mathbf{x}| \to \infty \}.$$

(We imposed one natural boundary condition at spatial infinity.)

The fiber direction corresponds to the field $\mathbb{E}(\mathbf{x})$.

The group

$$\mathcal{G} = \{ g : \mathbb{R}^{d-1} \to G \mid g(\mathbf{x}) \to 1 \quad \text{as} \quad |\mathbf{x}| \to \infty \}$$

acts on $\mathcal{A}$ (hence on $T^*\mathcal{A}$) by $g : \mathbb{A} \mapsto \mathbb{A}^g = g^{-1}\mathbb{A}g + g^{-1}dg$.

For $\epsilon \in \mathrm{Lie}\,\mathcal{G}$, i.e. $\epsilon : \mathbb{R}^{d-1} \to \mathfrak{g}$ s.t. $\epsilon(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$,

$$\langle \Phi, \epsilon \rangle := \int d^{d-1}\mathbf{x}\, \Phi(\mathbf{x}) \cdot \epsilon(\mathbf{x}) = -\int d^{d-1}\mathbf{x}\, \mathbb{E}(\mathbf{x}) \cdot \mathbb{D}\epsilon(\mathbf{x})$$

$$= -\mathbb{E}(\delta_\epsilon \mathbb{A})$$

Thus, Φ is nothing but the moment map of the action $T^*\mathcal{A} \circlearrowleft \mathcal{G}$, and the reduced phase space is isomorphic to the cotangent bundle of $\mathcal{A}/\mathcal{G}$:

$$\Phi^{-1}(0)/\mathcal{G} \cong T^*(\mathcal{A}/\mathcal{G}).$$

If we regard this as the physical phase space, physical states of the quantum theory are wavefunctionals on $\mathcal{A}/\mathcal{G}$, or equivalently, the wavefunctionals on $\mathcal{A}$ which are invariant under all spatial gauge transformations:

$$\Psi[A^g] = \Psi[A] \quad \forall g \in \mathcal{G}.$$

Alternatively, we may regard the physical phase space as

$$\Phi^{-1}(0)/\mathcal{G}_0 \cong T^*(\mathcal{A}/\mathcal{G}_0)$$

where $\mathcal{G}_0$ is the identity component of $\mathcal{G}$. Then, the physical state condition is

$$\Psi[A^g] = \Psi[A] \quad \forall g \in \mathcal{G}_0.$$

or equivalently

$$\delta_\epsilon \Psi[A] = 0 \qquad \forall\, \mathfrak{g}\text{-valued function } \epsilon(x),$$

that is, the Gauss law constraint,

$$\hat{\Phi}\,\Psi[A] = 0.$$