Descent and its derivation

Let $I(\xi)$ be an adjoint invariant polynomial function of $\xi \in \mathfrak{g}$ of degree $n+1$.

For a G-gauge potential A, $I(F_A)$ is a $2n+2$ form which is

- gauge invariant $\delta_\epsilon I(F_A) = 0$ and
- closed $dI(F_A) = 0$.

Moreover, there are sequence of differential forms $I^0_{2n+1}[A]$, $I^1_{2n}[\epsilon, A]$, $I^2_{2n-1}[\epsilon_1, \epsilon_2, A]$, $\cdots$ obeying

$$I(F_A) = dI^0_{2n+1}[A],$$

$$\delta_\epsilon I^0_{2n+1}[A] = dI^1_{2n}[\epsilon, A],$$

$$\delta_{\epsilon_1} I^1_{2n}[\epsilon_2, A] - \delta_{\epsilon_2} I^1_{2n}[\epsilon_1, A] - I^1_{2n}[[\epsilon_1, \epsilon_2], A] = dI^2_{2n-1}[\epsilon_1, \epsilon_2, A],$$

$$\vdots$$

Called the **descent equations**.

For $q = 0, 1, \cdots, 2n+1$, $I^{q}_{2n-q+1}[\epsilon_1, \cdots, \epsilon_q, A]$ is a $2n-q+1$ form depending linearly on infinitesimal gauge transformations $\epsilon_1, \cdots, \epsilon_q$. The descent equation for a general q is

$$\sum_{i=1}^{q+1} (-1)^{i-1} \delta_{\epsilon_i} I^{q}_{2n-q+1}[\epsilon_1, \cdots \cancel{\epsilon_i}, \cdots \epsilon_{q+1}, A]$$
$$+ \sum_{i<j} (-1)^{i+j} I^{q}_{2n-q+1}[[\epsilon_i, \epsilon_j], \cdots \cancel{\epsilon_i}, \cdots \cancel{\epsilon_j}, \cdots, A]$$
$$= d\, I^{q+1}_{2n-q}[\epsilon_1, \cdots, \epsilon_{q+1}, A], \qquad 0 \leq q \leq 2n+1.$$

The form $I^{0}_{2n+1}[A]$ satisfying $I(F_A) = d\, I^{0}_{2n+1}[A]$ can be found in the same way as we found Chern-Simons form from the Chern character:

Write $I(\xi)$ using an adjoint invariant multilinear function $I(\xi_1, \cdots, \xi_{n+1})$ as $I(\xi) = I(\xi, \cdots, \xi)$. By the adjoint invariance

$$d\, I(F_A) = \sum_{i=1}^{n+1} I(F_A, \cdots, \underbrace{dF_A + [A, F_A]}_{D_A F_A = 0 \text{ by Bianchi}}, \cdots, F_A) = 0.$$

Also, for an arbitrary variation $A \to A + \delta A$,

$$\delta I(F_A) = \sum_{i=1}^{n+1} I(F_A, \cdots, \underbrace{\delta F_A}_{D_A \delta A}, \cdots, F_A) = \underbrace{d \sum_{i=1}^{n+1} I(F_A, \cdots, \overset{i}{\delta A}, \cdots, F_A)}_{=: J(F_A, \delta A)}$$

For a family $A_t = t \cdot A$, $t \in [0,1]$,

$$I(F_A) = I(F_{A_1}) - I(F_{A_0}) = \int_0^1 dt \, \frac{\partial}{\partial t} I(F_{A_t}) \qquad dJ(F_{A_t}, A)$$

$$= d \underbrace{\int_0^1 dt \, J(F_{A_t}, A)}_{} =: I^0_{2n+1}[A]$$

In what follows, we shall find I^q_{2n-q+1} for $q \geq 1$.

Notation: For a manifold X

$\mathcal{A}(X)$ = the space of G-gauge potentials on X

$\cong \Omega^1(X, \mathfrak{g})$ $\mathfrak{g}$-valued 1-forms on X,

$\mathcal{G}(X)$ = the space of G-gauge transformations on X

$\cong \mathrm{Map}(X, G)$ G-valued functions on X.

We shall consider gauge potentials & gauge transformations on

$$\hat{X} := X \times \mathcal{G}(X).$$

Define <u>projection</u> & <u>evaluatation</u> maps

$$\begin{array}{ccc} & \xrightarrow{p_X} & X \\ \hat{X} & & \\ & \xrightarrow{ev} & G \end{array} \quad ; \quad \begin{array}{ccc} & \mapsto & x \\ (x, g) & & \\ & \mapsto & g(x) \end{array}$$

- Any $A \in \mathcal{A}(X)$ determines a gauge potential on $\hat{X}$

$$\hat{A} = P_X^* A \in \mathcal{A}(\hat{X}).$$

- e_V can be regarded as a gauge transformation on $\hat{X}$

$$e_V \in \mathcal{G}(\hat{X}).$$

A differential form on $\hat{X}$ is said to be a $\binom{q}{p}$-form

when it is a section of $\Lambda^p T^* X \otimes \Lambda^q T^* \mathcal{G}(X)$

$$\Omega^j(\hat{X}) = \bigoplus_{p+q=j} \underbrace{\Omega_p^q(\hat{X})}_{\text{the space of } \binom{q}{p}\text{-forms on } \hat{X}.}$$

The exterior derivative on $\hat{X}$ is denoted by

$$\hat{d} = d + \delta \ ; \quad d : \Omega_p^q \to \Omega_{p+1}^q, \quad \delta : \Omega_p^q \to \Omega_p^{q+1}.$$

Note: $\hat{A} = P_X^* A \in \Omega_1^0(\hat{X}; \mathfrak{g}) \subset \Omega^1(\hat{X}; \mathfrak{g})$

while $\hat{A}^{e_V} = \underbrace{e_V^{-1} \hat{A} e_V + e_V^{-1} d e_V}_{\text{in } \Omega_1^0} + \underbrace{e_V^{-1} \delta e_V}_{\text{in } \Omega_0^1}$

$I^{0}_{2n+1}[\hat{A}^{ev}] \in \Omega^{2n+1}(\hat{X})$ has all components

$$=: \underbrace{\hat{I}^{0}_{2n+1}[A]}_{\text{in } \Omega^{0}_{2n+1}} + \underbrace{\hat{I}^{1}_{2n}[A]}_{\text{in } \Omega^{1}_{2n}} + \underbrace{\hat{I}^{2}_{2n-1}[A]}_{\text{in } \Omega^{2}_{2n-1}} + \cdots + \underbrace{\hat{I}^{2n+1}_{0}[A]}_{\text{in } \Omega^{2n+1}_{0}}$$

As $d I^{0}_{2n+1}[A] = I(F_A)$ holds on any space, we have

$$\hat{d}\, I^{0}_{2n+1}[\hat{A}^{ev}] = I(F_{\hat{A}^{ev}}).$$

$$\mathrm{LHS} = (d+\delta)\left(\hat{I}^{0}_{2n+1}[A] + \hat{I}^{1}_{2n}[A] + \hat{I}^{2}_{2n-1}[A] + \cdots + \hat{I}^{2n+1}_{0}[A]\right)$$

$$= \underbrace{d\hat{I}^{0}_{2n+1}[A]}_{\Omega^{0}_{2n+2}} + \underbrace{\delta\hat{I}^{0}_{2n+1}[A] + d\hat{I}^{1}_{2n}[A]}_{\Omega^{1}_{2n+1}}$$

$$+ \underbrace{\delta\hat{I}^{1}_{2n}[A] + d\hat{I}^{2}_{2n-1}[A]}_{\Omega^{2}_{2n}} + \cdots \cdots + \underbrace{\delta\hat{I}^{2n+1}_{0}[A]}_{\Omega^{2n+2}_{0}}.$$

$$\mathrm{RHS} = I(F_{\hat{A}}) = I(p_X^* F_A) = p_X^* I(F_A) \in \Omega^{0}_{2n+2}.$$

The eqn splits to

$$d\,\hat{I}^{0}_{2n+1}[A] = p_X^* I(F_A) \qquad \hat{0}$$

$$\delta \hat{I}^{0}_{2n+1}[A] + d\,\hat{I}^{1}_{2n}[A] = 0 \qquad \hat{1}$$

$$\delta \hat{I}^{1}_{2n}[A] + d\,\hat{I}^{2}_{2n-1}[A] = 0 \qquad \hat{2}$$

$$\vdots \qquad\qquad \vdots$$

$$\delta \hat{I}^{2n+1}_{0}[A] = 0 \qquad \widehat{2n+2}$$

- For a differential form $\hat{\omega}$ on $\hat{X}$, we denote by $\hat{\omega}|_1$ the pullback of $\hat{\omega}$ by $x \in X \mapsto (x,1) \in \hat{X}$.

- $\mathcal{G}(X)$ acts on $\hat{X}$ from the right, $g : (x, g') \mapsto (x, g'g)$.

  $\rightsquigarrow$ Each $\epsilon \in \mathrm{Map}(X, \mathfrak{g}) = \mathrm{Lie}\,\mathcal{G}(X)$ induces a vector field $\hat{\epsilon}$ on $\hat{X}$.

Put $I^{1}_{2n}[\epsilon, A] := \hat{I}^{1}_{2n}[A](\hat{\epsilon})\Big|_1$

$\quad I^{2}_{2n-1}[\epsilon_1, \epsilon_2, A] := -\hat{I}^{2}_{2n-1}[A](\hat{\epsilon}_1, \hat{\epsilon}_2)\Big|_1$

$\vdots$

Then, $\widehat{⓪}|_1$, $\widehat{①}(\hat{\epsilon})|_1$, $\widehat{②}(\hat{\epsilon}_1, \hat{\epsilon}_2)|_1$, ⋯ are nothing but the descent equations

$$d I^0_{2n+1}[A] = I(F_A), \qquad ⓪$$

$$\delta_\epsilon I^0_{2n+1}[A] = d I^1_{2n}[\epsilon, A], \qquad ①$$

$$\delta_{\epsilon_1} I^1_{2n}[\epsilon_2, A] - \delta_{\epsilon_2} I^1_{2n}[\epsilon_1, A] - I^1_{2n}[[\epsilon_1, \epsilon_2], A]$$
$$= d I^2_{2n-1}[\epsilon_1, \epsilon_2, A], \qquad ②$$

$$\vdots \qquad \vdots$$

This can be shown as follows.

Obviously, $\delta(\text{any})|_1 = 0$.

Also, $e_v^{-1} d e_v|_1 = 0$ since $e_v(x, 1) \equiv 1$.

Thus, $\hat{A}^{e_v}|_1 = \hat{A}|_1 = A$ and hence $d\hat{A}^{e_v}|_1 = dA$.

$$\therefore \quad \hat{I}^0_{2n+1}[A]|_1 = I^0_{2n+1}[\hat{A}^{e_v}]|_1 = I^0_{2n+1}[A]. \qquad \text{— (a)}$$

By this, $\widehat{⓪}|_1$ is what we already know:

$$d I^0_{2n+1}[A] = P_x^* I(F_A)|_1 = I(F_A), \text{ which is } ⓪$$

To show the rest, some preparation is needed.

A useful tool is the left action by $g \in \mathcal{G}(X)$ on $\hat{X}$,

$$L_g : (x, g') \in \hat{X} \longmapsto (x, gg') \in \hat{X}.$$

- It satisfies

$$p_X \circ L_g = p_X \quad : \hat{X} \to X \qquad \text{and}$$

$$L_g^* e_v = p_X^* g \cdot e_v \qquad \text{in } \mathcal{G}(\hat{X}).$$

Using these, we find

$$L_g^* \hat{A}^{e_v} = (L_g^* \hat{A})^{L_g^* e_v} = (\underbrace{L_g^* p_X^* A}_{(p_X \circ L_g)^* A \,=\, p_X^* A})^{p_X^* g \cdot e_v}$$

$$= (p_X^* A)^{p_X^* g \cdot e_v} = (p_X^* A^{p_X^* g})^{e_v} = (p_X^* A^g)^{e_v}$$

$$= \widehat{A^g}^{\,e_v}.$$

$$\therefore \ L_g^* I^0_{2n+1}[\hat{A}^{e_v}] = I^0_{2n+1}[\widehat{A^g}^{\,e_v}]$$

$$\text{i.e.} \quad L_g^* \hat{I}^q_{2n-q+1}[A] = \hat{I}^q_{2n-q+1}[A^g] \qquad \text{—— (b)}$$

- For a function f on $\widehat{X}$,

$$(\hat{\epsilon} f)(x,1) = \frac{d}{dt} f(x, \overbrace{1\cdot e^{t\epsilon}}^{= e^{t\epsilon}\cdot 1})\Big|_{t=0} = \frac{d}{dt}\left(L_{e^{t\epsilon}}^{*} f\right)(x,1)\Big|_{t=0}$$

$$\therefore \quad \hat{\epsilon} f\big|_1 = \frac{d}{dt} L_{e^{t\epsilon}}^{*} f\Big|_{1,\,t=0}. \qquad \text{——(c)}$$

For $\alpha \in \Omega_0^1(\hat{X})$,

$$\hat{\epsilon}\,\alpha(\hat{\epsilon}')(x,1) = \frac{d}{dt}\,\alpha\left(\hat{\epsilon}'(x, e^{t\epsilon})\right)\Big|_{t=0}$$

$$\hat{\epsilon}'(x,g) = L_{g*}\hat{\epsilon}'(x,1)$$

$$= \frac{d}{dt}\,\underbrace{\alpha\left(L_{e^{t\epsilon}*}\hat{\epsilon}'(x,1)\right)}_{\left(L_{e^{t\epsilon}}^{*}\alpha\right)\left(\hat{\epsilon}'(x,1)\right)}\Big|_{t=0}$$

$$\therefore \quad \hat{\epsilon}\,\alpha(\hat{\epsilon}')\big|_1 = \frac{d}{dt}\left(L_{e^{t\epsilon}}^{*}\alpha\right)(\hat{\epsilon}')\Big|_{1,\,t=0}.$$

Similarly, for $\alpha \in \Omega_p^q(\hat{X})$,

$$\hat{\epsilon}\,\alpha(\hat{\epsilon}_1, \cdots, \hat{\epsilon}_q)\big|_1 = \frac{d}{dt}\left(L_{e^{t\epsilon}}^{*}\alpha\right)(\hat{\epsilon}_1, \cdots, \hat{\epsilon}_q)\Big|_{1,\,t=0}.$$

$$\text{——(d)}$$

• Also, we shall use

$$d \circ i_{\hat{\epsilon}} + i_{\hat{\epsilon}} \circ d = 0 \quad \text{on } \Omega_p^q(\hat{X}). \qquad \text{—— (e)}$$

This is becase $\hat{\epsilon}(x,g)$ is the tangent vector of the curve $(x, ge^{t\epsilon})$ at $t=0$ and is invariant under variation of x.

• <u>A side remark</u>

If $\check{\epsilon}$ is the vector field on $\hat{X}$ induced from $\epsilon \in Map(X, \mathfrak{g})$ via the <u>left</u> action of $\mathcal{G}(X)$ on $\hat{X}$, we also have

$$d \circ i_{\check{\epsilon}} + i_{\check{\epsilon}} \circ d = 0.$$

Thus

$$\delta \circ i_{\check{\epsilon}} + i_{\check{\epsilon}} \circ \delta = \mathcal{L}_{\check{\epsilon}} \ (\text{the Lie derivative by } \check{\epsilon}).$$

Together with the commutativity of the left and the right actions (which implies $[\check{\epsilon}, \hat{\epsilon}'] = 0$) and $\check{\epsilon} = \hat{\epsilon}$ at $(x, 1)$, this can be used to give an alternative and mechanical derivation of the above formula (d).

Now, we are ready.

- ① $(\hat{\epsilon})|_1$ is $\delta \hat{I}^0_{2n+1}[A](\hat{\epsilon})|_1 + d\hat{I}^1_{2n}[A](\hat{\epsilon})|_1 = 0$.

The first term $\delta \hat{I}^0_{2n+1}[A](\hat{\epsilon})|_1$

$$= \hat{\epsilon}\, \hat{I}^0_{2n+1}[A]|_1$$

$$\overset{(c)}{=} \frac{d}{dt} L^*_{e^{t\epsilon}} \hat{I}^0_{2n+1}[A]\Big|_{1,t=0}$$

$$\overset{(b)}{=} \frac{d}{dt} \hat{I}^0_{2n+1}[A^{e^{t\epsilon}}]\Big|_{1,t=0} \overset{(a)}{=} \frac{d}{dt} I^0_{2n+1}[A^{e^{t\epsilon}}]\Big|_{t=0}$$

$$= \delta_\epsilon I^0_{2n+1}[A].$$

The second term $d\hat{I}^1_{2n}[A](\hat{\epsilon})|_1$

$$= i_{\hat{\epsilon}}\, d\hat{I}^1_{2n}[A]|_1$$

$$\overset{(e)}{=} -d\, i_{\hat{\epsilon}} \hat{I}^1_{2n}[A]|_1 = -d\left(i_{\hat{\epsilon}} \hat{I}^1_{2n}[A]|_1\right)$$

$$\overset{\text{def}}{=} -d\, I^1_{2n}[\epsilon, A].$$

$\therefore$ ① $(\hat{\epsilon})|_1$ reads $\delta_\epsilon I^0_{2n+1}[A] - d\, I^1_{2n}[\epsilon, A] = 0$, which is ①.

• ② $(\hat{\epsilon}_1,\hat{\epsilon}_2)|_1$ is $\delta \hat{I}^1_{2n}[A](\hat{\epsilon}_1,\hat{\epsilon}_2)\big|_1 + d\,\hat{I}^2_{2n-2}[A](\hat{\epsilon}_1,\hat{\epsilon}_2)\big|_1 = 0$

Note that

$$\delta \hat{I}^1_{2n}[A](\hat{\epsilon}_1,\hat{\epsilon}_2) = \hat{\epsilon}_1 \hat{I}^1_{2n}[A](\hat{\epsilon}_2) - \hat{\epsilon}_2 \hat{I}^1_{2n}[A](\hat{\epsilon}_1) - \hat{I}^1_{2n}[A]\big(\underbrace{[\hat{\epsilon}_1,\hat{\epsilon}_2]}_{=\,\widehat{[\epsilon_1,\epsilon_2]}\ \leftarrow \text{right action}}\big)$$

and

$$\begin{aligned}
\hat{\epsilon}_1 \hat{I}^1_{2n}[A](\hat{\epsilon}_2)\big|_1 &\overset{(d)}{=} \frac{d}{dt}\left(L^*_{e^{t\epsilon_1}} \hat{I}^1_{2n}[A]\right)(\hat{\epsilon}_2)\Big|_{1,t=0} \\
&\overset{(b)}{=} \frac{d}{dt} \hat{I}^1_{2n}[A^{e^{t\epsilon_1}}](\hat{\epsilon}_2)\Big|_{1,t=0} \\
&\overset{\text{def}}{=} \frac{d}{dt} I^1_{2n}[\epsilon_2, A^{e^{t\epsilon_1}}]\Big|_{t=0} \\
&= \delta_{\epsilon_1} I^1_{2n}[\epsilon_2, A].
\end{aligned}$$

Thus, the first term $\delta \hat{I}^1_{2n}[A](\hat{\epsilon}_1,\hat{\epsilon}_2)\big|_1$

$$= \delta_{\epsilon_1} I^1_{2n}[\epsilon_2, A] - \delta_{\epsilon_2} I^1_{2n}[\epsilon_1, A] - I^1_{2n}[[\epsilon_1,\epsilon_2], A].$$

The second term $d\,\hat{I}^2_{2n-1}[A](\hat{\epsilon}_1,\hat{\epsilon}_2)\big|_1$

$$= \iota_{\hat{\epsilon}_2} \iota_{\hat{\epsilon}_1} d \hat{I}^2_{2n-1}[A] \Big|_1 \overset{(e)}{=} d\, \iota_{\hat{\epsilon}_2} \iota_{\hat{\epsilon}_1} \hat{I}^2_{2n-1}[A] \Big|_1$$

$$= d \left(\hat{I}^2_{2n-1}[A](\epsilon_1, \epsilon_2) \Big|_1 \right)$$

$$\overset{\text{def}}{=} d\, I^2_{2n-1}[\epsilon_1, \epsilon_2, A],$$

$\therefore$ (2) $(\hat{\epsilon}_1, \hat{\epsilon}_2)\Big|_1$ reads

$$\delta_{\epsilon_1} I^1_{2n}[\epsilon_2, A] - \delta_{\epsilon_2} I^1_{2n}[\epsilon_1, A] - I^1_{2n}[[\epsilon_1, \epsilon_2], A]$$

$$= d\, I^2_{2n-1}[\epsilon_1, \epsilon_2, A], \quad \text{which is (2)}.$$

- For $0 \leq q \leq 2n+1$, if we put

$$I^q_{2n-q+1}[\epsilon_1, \cdots, \epsilon_q, A] := (-1)^{\frac{q(q-1)}{2}} \hat{I}^q_{2n-q+1}[A](\hat{\epsilon}_1, \cdots, \hat{\epsilon}_q)\Big|_1,$$

$\widehat{(q+1)}$ $(\hat{\epsilon}_1, \cdots, \hat{\epsilon}_{q+1})\Big|_1$ reads

$$\sum_{i=1}^{q+1} (-1)^{i-1} \delta_{\epsilon_i} I^q_{2n-q+1}[\epsilon_1, \cdots \cancel{\epsilon_i}, \cdots \epsilon_{q+1}, A]$$

$$+ \sum_{i<j} (-1)^{i+j} I^q_{2n-q+1}[[\epsilon_i, \epsilon_j], \cdots \cancel{\epsilon_i}, \cdots \cancel{\epsilon_j}, \cdots, A]$$

$$= d\, I^{q+1}_{2n-q}[\epsilon_1, \cdots, \epsilon_{q+1}, A].$$

$\square$