§ ADHM Construction

after Atiyah - Drinfeld - Hitchin - Manin 1978.

Data: X^1, X^2, X^3, X^4 : $k\times k$ Hermitian matrices

$H_\uparrow, H_\downarrow$: $k \times n$ complex matrices

Satisfying some conditions to be determined.

For $x = (x^1, x^2, x^3, x^4) \in \mathbb{R}^4$, introduce a $2k\times(2k+n)$ matrix

$$D(x) = 2k\{ (\overbrace{x - X}^{2k}, \overbrace{H}^{n})$$

where
$$x = (x^4 \mathbb{1}_2 + i \sigma\cdot \mathbf{x}) \otimes \mathbb{1}_k \quad : 2k\times 2k$$

$$X = \mathbb{1}_2 \otimes X^4 + i \underbrace{\sigma \otimes \mathbf{X}}_{= \sum_{i=1}^{3} \sigma_i \otimes X^i} \quad : 2k \times 2k$$

$$H = \begin{pmatrix} H_\uparrow \\ H_\downarrow \end{pmatrix} \quad : 2k\times n$$

Condition ① $\forall x \in \mathbb{R}^4$, $D(x)$ has rank $2k$

($\Rightarrow$ $2k\times 2k$ matrix $D(x)D(x)^\dagger$ is invertible)

wpi IPMU INSTITUTE FOR THE PHYSICS AND MATHEMATICS OF THE UNIVERSE

Then, let $u(x)$ be $(2k+n)\times n$ matrix satisfying

$$\boxed{\begin{array}{l} \cdot\ D(x)\,u(x) = 0\,, \\ \cdot\ u(x)^\dagger u(x) = \mathbb{1}_n\,. \end{array}}$$

I.e. $u(x) = (u_1(x), \cdots, u_n(x))$, then $\{u_i(x)\}_{i=1}^n \subset \mathbb{C}^{2k+n}$ form an orthonormal basis of $\mathrm{Ker}\,D(x) \subset \mathbb{C}^{2k+n}$.

Note $u(x)$ is unique up to orthogonal basis change of $\mathrm{Ker}\,D(x)$,

i.e. $u(x) \to u(x)\,g(x)$ for a $U(n)$ valued fcn $g(x)$ of x.

Note $P(x) = u(x)\,u(x)^\dagger$ is the projection to $\mathrm{Ker}\,D(x) \subset \mathbb{C}^{2k+n}$.

$Q(x) = D(x)^\dagger \left(D(x)\,D^\dagger(x)\right)^{-1} D(x)$ is the projection to

$$\mathrm{Span}\left(D^\dagger(x)\right) = \left(\mathrm{Ker}\,D(x)\right)^\perp \subset \mathbb{C}^{2k+n}.$$

So $P(x) + Q(x) = \mathbb{1}_{2k+n}$

$P(x)^2 = P(x)$, $Q(x)^2 = Q(x)$, $P(x)\,Q(x) = Q(x)\,P(x) = 0$.

Put $\boxed{A = u^\dagger du}$

$u^\dagger u = 1_n \Rightarrow A^\dagger = du^\dagger u = -u^\dagger du = -A$

$\therefore$ A is a gauge potential for gauge group $U(n)$.

東京大学 THE UNIVERSITY OF TOKYO

$$F_A = dA + A\wedge A = du^\dagger \wedge du + u^\dagger du \wedge u^\dagger du$$

<u>Note</u> $u^\dagger dP\wedge dP\, u = u^\dagger(du\, u^\dagger + u\, du^\dagger)\wedge(du\, u^\dagger + u\, du^\dagger)u$

$$= (u^\dagger du\, u^\dagger + du^\dagger)\wedge(du + u\, du^\dagger u)$$

$$= u^\dagger du \wedge u^\dagger du + du^\dagger \wedge du + \underbrace{u^\dagger du\, du^\dagger u + du^\dagger u\, du^\dagger u}_{0}$$

$$= F_A$$

$$\therefore F_A = u^\dagger dP\wedge dP\, u \overset{P+Q=1}{=} u^\dagger dQ \wedge dQ\, u \qquad Du=0,\ u^\dagger D^\dagger = 0$$

$$= u^\dagger\left(dD^\dagger (DD^\dagger)^{-1} D \cancel{+ D^\dagger d((DD^\dagger)^{-1}D)}\right)\wedge\left(\cancel{d(D^\dagger (DD^\dagger)^{-1})D} + D^\dagger (DD^\dagger)^{-1} dD\right)u$$

$$= u^\dagger dD^\dagger (DD^\dagger)^{-1} DD^\dagger (DD^\dagger)^{-1} dD\, u = u\, dD^\dagger (DD^\dagger)^{-1} dD\, u$$

$$= u^\dagger \begin{pmatrix} dx^\dagger \\ 0 \end{pmatrix} (DD^\dagger)^{-1} \begin{pmatrix} dx & 0 \end{pmatrix} u .$$

If $(DD^\dagger)^{-1}$ commutes with $\sigma_1 \otimes \mathbb{1}_k, \sigma_2 \otimes \mathbb{1}_k, \sigma_3 \otimes \mathbb{1}_k$, then

$$F_A = u^\dagger \begin{pmatrix} dx^\dagger \wedge dx \\ 0 \end{pmatrix} (DD^\dagger)^{-1} \begin{pmatrix} 1 & 0 \end{pmatrix} u$$

ASD !

i.e. $\underline{F_A + * F_A = 0 \quad !}$ anti-instanton !!

wpi iPMU INSTITUTE FOR THE PHYSICS AND MATHEMATICS OF THE UNIVERSE

Condition ② $D(x)D(x)^\dagger$ commutes with $\sigma_i \otimes \mathbb{1}_k$ $i=1,2,3.$

$$D(x)D(x)^\dagger = (x-X,\ H)\begin{pmatrix} x^\dagger - X^\dagger \\ H^\dagger \end{pmatrix} = xx^\dagger - xX^\dagger - Xx^\dagger + XX^\dagger + HH^\dagger$$

- $xx^\dagger = \mathbb{1}_{2k}\cdot |x|^2$ Obviously commutes with $\sigma_i \otimes \mathbb{1}_k$ no condition

- $$xX^\dagger + Xx^\dagger = (x^4\mathbb{1}_2 + i\sigma_i x^i)\otimes\mathbb{1}_k \cdot (\mathbb{1}_2\otimes X^4 - i\sigma_j\otimes X^j)$$
$$+ (\mathbb{1}_2\otimes X^4 + i\sigma_j \otimes X^j)\cdot(x^4\mathbb{1}_2 - i\sigma_i x^i)\otimes\mathbb{1}_k$$
$$= 2x^4\mathbb{1}_2\otimes X^4 + x^i\underbrace{(\sigma_i\sigma_j + \sigma_j\sigma_i)}_{2\delta_{ij}\cdot\mathbb{1}_2}\otimes X^j$$
$$= 2\,\mathbb{1}_2\otimes\sum_{\mu=1}^{4} x^\mu X^\mu$$ also commutes with $\sigma_i\otimes\mathbb{1}_k$
no condition.

- $$XX^\dagger = (\mathbb{1}_2\otimes X^4 + i\sigma_i\otimes X^i)\cdot(\mathbb{1}_2\otimes X^4 - i\sigma_j\otimes X^j)$$
$$= \mathbb{1}_2\otimes(X^4)^2 + \underbrace{\sigma_i\sigma_j}_{\underbrace{\frac{1}{2}(\sigma_i\sigma_j+\sigma_j\sigma_i)}_{\delta_{ij}} + \underbrace{\frac{1}{2}(\sigma_i\sigma_j-\sigma_j\sigma_i)}_{i\epsilon_{ijk}\sigma_k}}\otimes X^iX^j + \underbrace{i\sigma_i\otimes X^iX^4 - i\sigma_j\otimes X^4X^j}_{i\sigma_i\otimes[X^i,X^4]}$$
$$\longrightarrow \frac{i}{2}\epsilon_{ijk}\sigma_k\otimes[X^i,X^j]$$
$$= \mathbb{1}_2\otimes\sum_{\mu=1}^{4}(X^\mu)^2 + i\sigma_3\otimes\left([X^1,X^2]+[X^3,X^4]\right)$$
$$+ i\sigma_1\otimes\left([X^2,X^3]+[X^1,X^4]\right) + i\sigma_2\otimes\left([X^3,X^1]+[X^2,X^4]\right).$$

東京大学 THE UNIVERSITY OF TOKYO

$$H = \begin{pmatrix} H_\uparrow \\ H_\downarrow \end{pmatrix} \Rightarrow$$

$$HH^\dagger = \begin{pmatrix} H_\uparrow H_\uparrow^\dagger & H_\uparrow H_\downarrow^\dagger \\ H_\downarrow H_\uparrow^\dagger & H_\downarrow H_\downarrow^\dagger \end{pmatrix} = \mathbb{1}_2 \otimes \frac{H_\uparrow H_\uparrow^\dagger + H_\downarrow H_\downarrow}{2} + \sigma_3 \otimes \frac{H_\uparrow H_\uparrow^\dagger - H_\downarrow H_\downarrow^\dagger}{2}$$

$$+ \frac{\sigma_1 + i\sigma_2}{2} \otimes H_\uparrow H_\downarrow^\dagger + \frac{\sigma_1 - i\sigma_2}{2} \otimes H_\downarrow H_\uparrow^\dagger$$

Condition ② $\Longleftrightarrow$

- $i[X^1, X^2] + i[X^3, X^4] + \dfrac{H_\uparrow H_\uparrow^\dagger - H_\downarrow H_\downarrow^\dagger}{2} = 0$
- $i[X^2, X^3] + i[X^1, X^4] + \dfrac{H_\uparrow H_\downarrow^\dagger + H_\downarrow H_\uparrow^\dagger}{2} = 0$
- $i[X^3, X^1] + i[X^2, X^4] + \dfrac{iH_\uparrow H_\downarrow^\dagger - iH_\downarrow H_\uparrow^\dagger}{2} = 0$

Or equivalently, for

$k \times k$ $\mathbb{C}$

$Z := iX^1 + X^2$ $\quad Q := H_\uparrow$ $\quad k \times n$ $\mathbb{C}$

$\tilde{Z} := iX^3 + X^4$ $\quad \tilde{Q} := H_\downarrow^\dagger$ $\quad n \times k$ $\mathbb{C}$,

- $[Z, Z^\dagger] - [\tilde{Z}^\dagger, \tilde{Z}] + QQ^\dagger - \tilde{Q}^\dagger \tilde{Q} = 0$
- $[Z, \tilde{Z}] + Q\tilde{Q} = 0$

What is $\nu[A] = ?$

A "Quick" (mathematical) answer:

$$E = \bigcup_{x \in \mathbb{R}^4} \operatorname{Ker} D_{(x)}, \qquad F = \bigcup_{x \in \mathbb{R}^4} \operatorname{Span} D^\dagger_{(x)}$$

extend to vector bundles over S^4

s.t. $E \oplus F \cong S^4 \times \mathbb{C}^{2k+n}$ (trivial bdle)

Chern-character

$$ch(E) = \left[tr_E \left(e^{\frac{i}{2\pi} F_A} \right) \right] = \sum_{n \geq 0} \underbrace{\frac{1}{n!} \left(\frac{i}{2\pi} \right)^n \left[tr_E (F_A^{\,n}) \right]}_{ch_n(E)}$$

Note

$$\nu[A] = -\frac{1}{8\pi^2} \int_{\mathbb{R}^4 \to S^4} \underset{\substack{\| \\ tr(F_A^2) \\ SU(n)}}{Tr(F_A^2)} = \frac{1}{2} \left(\frac{i}{2\pi} \right)^2 \int_{S^4} tr(F_A^2)$$

$$= \int_{S^4} ch_2(E).$$

$$\underset{\substack{\| \\ ch(E) + ch(F)}}{ch(E \oplus F)} = ch(S^4 \times \mathbb{C}^{2k+n}) = 2k+n$$

$$\nu[A] = \int_{S^4} ch_2(E) = -\underbrace{\int_{S^4} ch_2(F)}_{\substack{\| \\ k \text{ (see below)}}} = -k$$

wpi iPMU INSTITUTE FOR THE PHYSICS AND MATHEMATICS OF THE UNIVERSE

<u>A down-to-earth answer</u> :

$V(x)$: orthonormal basis of $\mathrm{Span}(D(x)^\dagger)$ $\quad \left(D(x)^\dagger \xrightarrow{\text{Gram-Schmidt}} V(x) \right)$

$$B := V^\dagger dV \qquad (A = u^\dagger du)$$

$$F_A = u^\dagger dP \wedge dP\, u \quad , \quad F_B = V^\dagger dQ \wedge dQ\, V$$

$$\mathrm{tr}(F_A^2) + \mathrm{tr}(F_B^2) = \mathrm{tr}(u^\dagger dP dP\, \overset{P}{u u^\dagger}\, dP dP\, u) + \mathrm{tr}(V^\dagger dQ dQ\, \overset{Q}{V V^\dagger}\, dQ dQ\, V)$$

$$= \mathrm{tr}(P dP dP\, P dP dP) + \mathrm{tr}(Q \underbrace{dQ dQ}_{Q\, dP dP}\, Q \underbrace{dQ dQ}_{dP dP})$$

$P+Q=1$

$$\underbrace{Q\,dP}_{= -dQ\,P\, =\, dP\,P}\,dP$$

$$\mathrm{tr}(dP P\, dP dP P\, dP) = -\mathrm{tr}(P dP dP\, P dP dP)$$

$$= 0$$

$$\therefore \nu[A] = -\nu[B].$$

$$D(x) = (x - X, H) \xrightarrow{|x| \to \infty} (x, 0)$$

$$\therefore V(x) \xrightarrow{|x|\to\infty} \frac{1}{|x|}\begin{pmatrix} x^\dagger \\ 0 \end{pmatrix} , \quad B = V^\dagger dV \to \frac{x}{|x|}\, d\Big(\overbrace{\frac{x^\dagger}{|x|}}^{g_1(x)}\Big) \otimes \mathbb{1}_k$$

$$\nu[B] = \underbrace{\frac{1}{24\pi^2}\int \mathrm{tr}_2 (g^{-1} dg)^3}_{=1} \cdot \underbrace{\mathrm{tr}_k \mathbb{1}_k}_{=k} = k .$$

$$\therefore \underline{\nu[A] = -k} .$$

東京大学 THE UNIVERSITY OF TOKYO

Summary

Given X^1, X^2, X^3, X^4 $k\times k$ Hermitian
$H_\uparrow, H_\downarrow$ $k\times n$ complex } obeying ① & ②

(such (X, H) is called an ADHM data),

take $(2k+n)\times n$ $u(x)$ satisfying

$$(x-X, H)\, u(x) = 0 \quad \& \quad u(x)^\dagger u(x) = \mathbb{1}_n,$$

and put $A = u^\dagger du$. Then

- $F_A + *F_A = 0$
- $\nu[A] = -k$

FACT

Any $U(n)$ gauge potential A such that $F_A + *F_A = 0$ and $\nu[A] = -k$ is of this form for some ADHM data.

Remark 1. $A = \underset{\substack{\uparrow \\ \text{traceless}}}{A'} + a\,\mathbb{1}_n \Rightarrow F_A = F_{A'} + da\,\mathbb{1}_n$

F_A ASD $\Rightarrow$ $F_{A'}$ ASD & da ASD

$$\Rightarrow -\int da \wedge * da = \int da \wedge da \underset{\substack{\uparrow \\ a \sim g^{-1}dg \text{ at } \infty}}{=} 0 \qquad \therefore da \equiv 0.$$

$\therefore\ a = i\,d\theta$ for some $\theta : \mathbb{R}^4 \to \mathbb{R}$

$\quad = g_{(1)}^{-1} dg_{(1)}$ for $g_{(1)} = e^{i\theta} : \mathbb{R}^4 \to U(1)$.

$$g := \begin{pmatrix} g_{(1)} & & \\ & \ddots & \\ & & g_{(1)} \end{pmatrix} \quad n\times n$$

$A = A' + g^{-1}dg = (A')^g$ or $A' = A^{g^{-1}}$

Replacing $u(x)$ by $u(x)\,g(x)^{-1}$, if necessary,

we may assume A is an SU(n) gauge potential

Remark 2 $\quad X^\mu \to h X^\mu h^{-1}$, $\quad H_\alpha \to h H_\alpha g^{-1}$ $\quad$ for $h \in U(k)$, $g \in SU(n)$,

$$(x - X, H) \to (\mathbb{1}_2 \otimes h)(x - X, H)\left(\begin{array}{c|c} \mathbb{1}_2 \otimes h^{-1} & 0 \\ \hline 0 & g^{-1} \end{array}\right)$$

$$\rightsquigarrow \quad U(x) \to \left(\begin{array}{c|c} \mathbb{1}_2 \otimes h & \\ \hline & g \end{array}\right) U(x) \quad : \quad \text{A unchanged}$$

$\boxed{\text{Thus, the ADHM data } (X, H) \text{ is equivalent to } (hXh^{-1}, hHg^{-1}),\ (h, g) \in U(k) \times SU(n).}$

Remark 2′ $\quad D(x) = (x - X, H) \sim (x, 0)$ as $|x| \to \infty$.

$D(x)U(x) = 0$, $U^\dagger U = 1_n$ has a solution of the form

$$U(x) \xrightarrow[|x| \to \infty]{} \begin{pmatrix} 0 \\ g_\infty \end{pmatrix} \quad ; \quad g_\infty \in U(n).$$

As the boundary condition, we may require $g_\infty = 1$

i.e. $\quad U(x) \longrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ as $|x| \to \infty$.

Then the above $g \in SU(n)$ transformation must be acompanied with $U(x) \to U(x) g^{-1}$ that does $A \to A^{g^{-1}}$: gauge transformation that does not become 1 at ∞.

Remark 3 fix $g_{-k} : S^3_\infty \to SU(n)$ $n[g_{-k}] = -k$.

$$\widetilde{\mathcal{M}}_k(SU(n)) := \dfrac{\left\{ A : SU(n) \text{ potential} \;\middle|\; \begin{array}{l} F_A + *F_A = 0 \\ \nu[A] = -k \\ A \to g_{-k}^{-1} dg_{-k} \text{ at } \infty \end{array} \right\}}{\left\{ g : \mathbb{R}^4 \to G \;\middle|\; g \to 1 \text{ at } \infty \right\}}$$

The moduli space of framed $SU(n)$ instantons with $\nu = -k$

$$\cong \left\{ (X, H) \;\middle|\; \text{ADHM data} \right\} / U(k),$$

by summary, $\boxed{\text{FACT}}$, and above remarks.

- It is a space of $\dim_{\mathbb{R}}$ $4k^2 + 2 \cdot 2kn - 3k^2 - k^2 = 4kn$

- The two spaces have natural Riemannian metrics and $\cong$ is an isomorphism of Riemannian manifolds :

LHS comes from $\|\delta A\|^2 = -\int_{\mathbb{R}^4} \mathrm{Tr}(\delta A \wedge * \delta A)$ $(\times \text{const})$.

RHS comes from $ds^2 = \sum_{\mu=1}^4 |dX^\mu|^2 + \sum_{\alpha = \tau, \nu} |dH_\alpha|^2$.

wpi IPMU INSTITUTE FOR THE PHYSICS AND MATHEMATICS OF THE UNIVERSE

e.g. $k=1,\ n=2$ (1 anti-instanton for $SU(2)$).

X^μ : 1×1 write $X^\mu = x_0^\mu$.

Q : 1×2, $\tilde{Q}$: 2×1

Eqns are
$$\left.\begin{aligned} QQ^\dagger - \tilde{Q}^\dagger\tilde{Q} &= 0 \\ Q\tilde{Q} &= 0 \end{aligned}\right\}$$
solved by
$$Q = (\rho, 0)$$
$$\tilde{Q}^\dagger = (0, \rho)$$

up to $U(1)\times SU(2)$ rotation.

$$(x - X, H) = (x - x_0, \rho \mathbb{1}_2)$$

Assume for simplicity $x_0 = 0$. As $u(x)$ we may take

$$u(x) = \begin{pmatrix} -\rho \mathbb{1}_2 \\ x \end{pmatrix} \Big/ \sqrt{\rho^2 + |x|^2}$$

$$A = u^\dagger du = \frac{(-\rho, x^\dagger)}{\sqrt{|x|^2+\rho^2}}\, d\left(\frac{1}{\sqrt{|x|^2+\rho^2}}\begin{pmatrix} -\rho \\ x \end{pmatrix}\right) = \frac{1}{\sqrt{|x|^2+\rho^2}}\begin{pmatrix} 0 \\ dx \end{pmatrix} - \frac{1}{2}\frac{d|x|^2}{\sqrt{}^{\,3}}\begin{pmatrix} -\rho \\ x \end{pmatrix}$$

$$= \frac{1}{|x|^2+\rho^2}\, x^\dagger dx - \frac{1}{2}\,\frac{d|x|^2}{|x|^2+\rho^2}$$

This is precisely the form of anti-instanton we had found!

東京大学 THE UNIVERSITY OF TOKYO

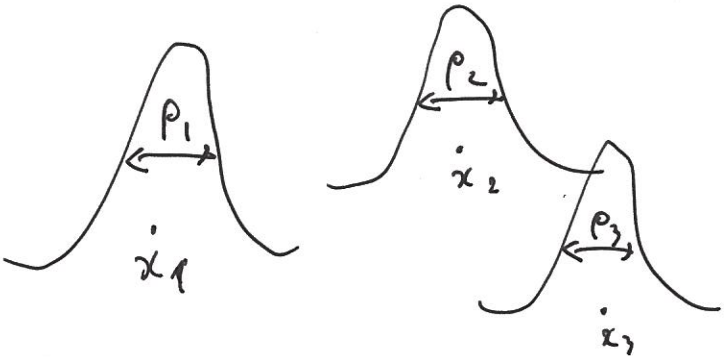
wpi IPMU INSTITUTE FOR THE PHYSICS AND MATHEMATICS OF THE UNIVERSE

e.g. $n=2,\ k>1$

A particular solution : $X^\mu = \begin{pmatrix} x_1^\mu & & \\ & \ddots & \\ & & x_k^\mu \end{pmatrix}$

$$Q = \begin{pmatrix} \rho_1 & 0 \\ \vdots & \vdots \\ \rho_k & 0 \end{pmatrix}, \quad \tilde{Q}^\dagger = \begin{pmatrix} 0 & \rho_1 \\ \vdots & \vdots \\ 0 & \rho_k \end{pmatrix}$$

→ "Superposition" of 1 antiinstantons at x_i, size ρ_i



But that's not general.

e.g. $n>2,\ k=1$ $\qquad Q,\ \tilde{Q}^\dagger \ :\ 1\times n$

$$\left.\begin{aligned} QQ^\dagger - \tilde{Q}^\dagger\tilde{Q} &= 0 \\ Q\tilde{Q} &= 0 \end{aligned}\right\} \text{ solved by } \quad \begin{aligned} Q &= (\rho, 0, \overbrace{0\cdots 0}^{n-2}) \\ \tilde{Q}^\dagger &= (0, \rho, 0\cdots 0) \end{aligned}$$

up to $U(1)\times SU(n)$ rotation.

$$(x-X,\ H) = (x-x_0,\ \rho\mathbb{1}_2,\ 0_{2\times(n-2)})$$

$$U(x) = \left(\begin{array}{c|c} -\rho\mathbb{1}_2 & \\ x-x_0 & 0_{4\times(n-2)} \\ \hline 0_{(n-2)\times 2} & \mathbb{1}_{n-2} \end{array}\right), \quad A = \left(\begin{array}{c|c} A_{SU(2)} & 0 \\ \hline 0 & 0_{n-2} \end{array}\right) \quad \text{up to } SU(n) \text{ rotation.}$$

$\frac{1}{\sqrt{|x-x_0|^2+\rho^2}}\times$

This IS general.

東京大学 THE UNIVERSITY OF TOKYO

wpi iPMU INSTITUTE FOR THE PHYSICS AND MATHEMATICS OF THE UNIVERSE

$G = SO(n)$

$$J_k := \begin{pmatrix} 0 & -1_k \\ 1_k & 0 \end{pmatrix}^{\swarrow 2k\times 2k}, \quad \epsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad 2\times 2$$

ADHM data :

X^μ $2k\times 2k$ ~~antisymmetric~~ hermitian s.t. $J_k^T (X^\mu)^* J_k = X^\mu$

H $4k\times n$ complex s.t. $(\epsilon^T \otimes J_k^T) H^* = H$.

obeying ① & ②.

$$D(x)\, U(x) = 0 \Rightarrow (x^* - X^*, H^*)\, U^*(x) = 0$$

$$\Rightarrow \underbrace{(\epsilon^T \otimes J_k^T)(x^* - X^*, H^*)\begin{pmatrix} \epsilon\otimes J_k & \\ & 1 \end{pmatrix}}_{= (x - X, H)} \begin{pmatrix} \epsilon^{-1}\otimes J_k^{-1} & \\ & 1 \end{pmatrix} U^*(x) = 0$$

$U \mapsto \begin{pmatrix} \epsilon^{-1}\otimes J_k^{-1} & 0 \\ 0 & 1 \end{pmatrix} U^*$ defines an orthogonal structure.

$$\widetilde{\mathcal{M}}_k(SO(n)) \cong \{ (X, H) \mid \text{ADHM} \} / Sp(k)$$

$$X^\mu \to h X^\mu h^{-1} \qquad h \in Sp(k)$$

$$H_\alpha \to h H_\alpha g^{-1} \qquad g \in SO(n)$$

東京大学 THE UNIVERSITY OF TOKYO

$G = Sp(n)$

$$J_n := \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix} \quad 2n \times 2n \;, \qquad \epsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad 2 \times 2$$

ADHM data

- X^μ $\quad k \times k$ symmetric, hermitian
- H $\quad 2k \times 2n$ $\quad \epsilon^T H^* J_n = H$
- Obeying ① and ②

$$D(x)\,u(x) = 0 \;\Rightarrow\; (x^* - X^*, H^*)\, u^*(x) = 0$$

$$\Rightarrow \underbrace{\epsilon^T (x^* - X^*, H^*) \begin{pmatrix} \epsilon & 0 \\ 0 & J_n \end{pmatrix}}_{(x - X, H)} \begin{pmatrix} \epsilon^{-1} & 0 \\ 0 & J_n^{-1} \end{pmatrix} u^*(x) = 0$$

$u \mapsto \begin{pmatrix} \epsilon^{-1} & 0 \\ 0 & J_n^{-1} \end{pmatrix} u^*$ defines a symplectic structure

$$\widetilde{M}_k(Sp(n)) = \{ (X, H) \mid \text{ADHM} \} / O(k)$$

$$X^\mu \to h X^\mu h^{-1} \qquad h \in O(k)$$

$$H_\alpha \to h H_\alpha g^{-1} \qquad g \in Sp(n)$$

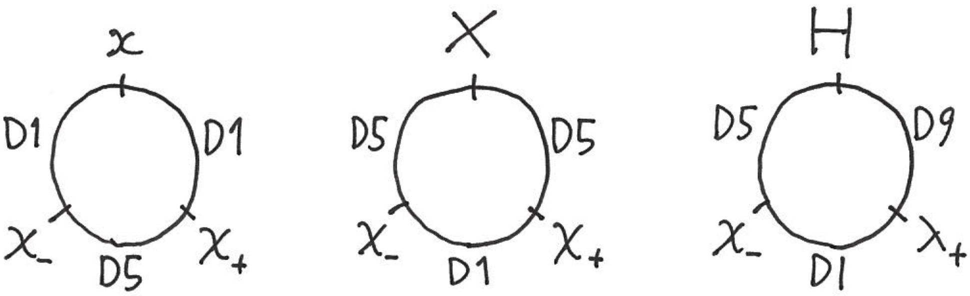
String theory construction (M. Douglas 1996)

$$\left.\begin{array}{ll} n & \text{D9-branes} \\ k & \text{D5-branes} \end{array}\right\} \Rightarrow \begin{array}{lll} X & \text{from} & \text{5-5 string} \\ H & \text{from} & \text{5-9} \end{array}$$

$$\text{D1-brane probe} \Rightarrow \begin{array}{ll} \chi_+, \chi_- & 1-5 \\ \lambda_+ & 1-9 \\ x & 1-1 \end{array}$$

$$L^{D1}_{eff} \supset \bar{\chi}_- \partial_+ \chi_- + \bar{\chi}_+ \partial_- \chi_+ + \bar{\lambda}_+ \partial_- \lambda_+ \qquad \partial_\pm := \tfrac{1}{2}(\partial_t \pm \partial_\sigma)$$

$$+ \chi_- \underline{(x - X, H)} \begin{pmatrix} \chi_+ \\ \lambda_+ \end{pmatrix} + c.c.$$



massless modes : $\begin{pmatrix} \chi_+ \\ \lambda_+ \end{pmatrix} = \sum_{a=1}^{n} U_a(x(t,\sigma))\, \lambda_+^a(t,\sigma)$

$$\rightsquigarrow L^{D1}_{eff} \supset \sum_{a,b} \bar{\lambda}^a_+(t,\sigma) \left[\partial_- \delta^a_b + \partial_- x^\mu(t,\sigma) \left(U^\dagger \partial_\mu U\right)^a_b (x(t,\sigma)) \right] \lambda^b_+(t,\sigma)$$

D1 probing n D9's with background $A = U^\dagger dU$.

<u>The theory on D5's</u> : a 6-dimensional $\mathcal{N}=(1,0)$ supersymmetric gauge theory with gauge group $U(k)$ and "hypermultiplets":

- one adjoint $(X = Z, \tilde{Z})$
- N fundamentals $(H = Q, \tilde{Q})$

The vacuum eqns are precisely the condition ② (ADHM eqn):

$$[Z, Z^\dagger] - [\tilde{Z}^\dagger, \tilde{Z}] + QQ^\dagger - \tilde{Q}^\dagger \tilde{Q} = 0 \qquad (D)$$

$$[Z, \tilde{Z}] + Q\tilde{Q} = 0 \qquad (F)$$

I.e. the vacuum manifold is precisely $\widetilde{\mathcal{M}}_k(SU(n))$.

The dimensional reduction to 4d is $\mathcal{N}=2$ supersymmetric theory.

In $\mathcal{N}=1$ language, it is $U(k)$ gauge theory with chiral multiplets:

3 adjoints $\Phi, Z, \tilde{Z}$

N fundamentals Q and N antifundamentals $\tilde{Q}$

and superpotential

$$W = \tilde{Q}\Phi Q + \mathrm{Tr}(\tilde{Z}[\Phi, Z])$$

(D) and (F) are precisely the D-term and F-term eqns at $\Phi = 0$. $\widetilde{\mathcal{M}}_k(SU(n))$ is the "Higgs branch" of the moduli space of vacua.

The above is in "Type IIB superstring theory".

$\rightsquigarrow$ $U(n)$ on n D9-branes

$U(k)$ on k D5-branes

Relevant for $SO(n)$ and $Sp(n)$ instantons are "Orientifolds" of Type IIB. There are two types:

- $O9^-$ $\rightsquigarrow$ $O(n)$ on n D9-branes

  $Sp(k)$ on $2k$ D5-branes

- $O9^+$ $\rightsquigarrow$ $Sp(n)$ on $2n$ D9-branes

  $O(k)$ on k D5-branes

The conditions $J_k^T (X^M)^* J_k = X^M$, $(\epsilon^T \otimes J_k^T) H^* = H$ $(SO(n))$

or $(X^M)^T = X^M$, $\epsilon^T H^* J_n = H$ $(Sp(n))$

are "Orientifold projection conditions" for the 5-5 and 5-9 open string states.