Finding Instantons for $G = SU(2)$

Recall $S_E[A] \geq \frac{8\pi^2}{g^2} |\nu[A]|$

"=" $\iff$ $F_A \pm * F_A = 0$ if $\nu[A] \lessgtr 0$.

Ansatz $A = f(|x|^2)\, g^{-1} dg$

where $g(x) = \frac{1}{|x|} \begin{pmatrix} x^4 + ix^3 & ix^1 + x^2 \\ ix^1 - x^2 & x^4 - ix^3 \end{pmatrix}$. [$g(x) \in SU(2)$ if $x \neq 0$].

Note: $n_{S^3_R}[g] = -1$ for $S^3_R = \{x \mid |x| = R\}$

with the natural orientation from $S^3_R = \partial\{x \mid |x| \leq R\}$

where the orientation of $\{x \mid |x| \leq R\}$ is such that

$dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 > 0$. [The main exercise].

Thus we look for $f(|x|^2)$ s.t. $F_A + *F_A = 0$,

anti-instanton (a connection with an anti-self-dual (ASD) curvature).

$\sigma = (\sigma_1, \sigma_2, \sigma_3)$ Pauli matrices

write $$x = \begin{pmatrix} x^4 + i x^3 & i x^1 + x^2 \\ i x^1 - x^2 & x^4 - i x^3 \end{pmatrix} = x^4 \mathbb{1}_2 + i \mathbf{x}\cdot\sigma$$

$$x^\dagger = x^4 \mathbb{1}_2 - i \mathbf{x}\cdot\sigma$$

· $x x^\dagger = x^\dagger x = |x|^2 \mathbb{1}_2$

· $dx \wedge dx^\dagger$
$dx^\dagger \wedge dx$
$$= (dx^4 \pm i\, dx^i \sigma_i) \wedge (dx^4 \mp i\, dx^j \sigma_j)$$

$$= \pm 2i\, \sigma_i\, dx^i \wedge dx^4 + \underbrace{\sigma_i \sigma_j}_{= i\epsilon_{ijk}\sigma_k} dx^i \wedge dx^j$$

$$\left.\begin{aligned} &= 2i\sigma_3 (\pm dx^3 \wedge dx^4 + dx^1 \wedge dx^2) \\ &+ 2i\sigma_1 (\pm dx^1 \wedge dx^4 + dx^2 \wedge dx^3) \\ &+ 2i\sigma_2 (\pm dx^2 \wedge dx^4 + dx^3 \wedge dx^1) \end{aligned}\right\}$$

Self-dual
anti-self-dual.

$$A = f(|x|^2)\, \frac{x^\dagger}{|x|}\, d\!\left(\frac{x}{|x|}\right) = \boxed{\frac{f(|x|^2)}{|x|^2}} \left(x^\dagger dx - \frac{1}{2} d|x|^2 \right)$$

$$\boxed{\frac{f(|x|^2)}{|x|^2}} =: F(|x|^2)$$

$$dA = F' \, d|x|^2 \wedge \left(x^\dagger dx - \cancel{\tfrac{1}{2} d|x|^2} \right) + F\, dx^\dagger \wedge dx$$

$$A^2 = F^2 \left(x^\dagger dx - \tfrac{1}{2} d|x|^2 \right) \wedge \left(x^\dagger dx - \tfrac{1}{2} d|x|^2 \right) = F^2\, \underbrace{x^\dagger dx}_{= d|x|^2 - dx^\dagger x} \wedge\, x^\dagger dx$$

$$F_A = dA + A^2$$

$$= (F' + F^2)\,\underbrace{d|x|^2 \wedge x^\dagger dx}_{\displaystyle \| \atop \displaystyle \frac{1}{2}\Big(\underbrace{|x|^2 dx^\dagger \wedge dx}_{\text{ASD}} - x^\dagger \underbrace{dx \wedge dx^\dagger}_{\text{SD}}\, x\Big)} + (F - |x|^2 F^2)\,\underbrace{dx^\dagger \wedge dx}_{\text{ASD}}$$

F_A ASD $\Rightarrow$ $F' + F^2 = 0$

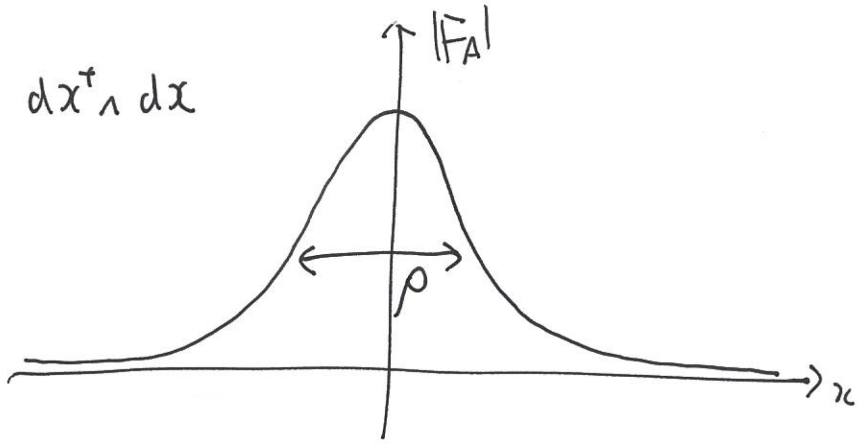
i.e. $-\dfrac{dF}{F^2} = d|x|^2 \quad \therefore \quad \dfrac{1}{F} = |x|^2 + \underset{\uparrow}{\rho^2}$

integration constant.

$$F = \frac{1}{|x|^2 + \rho^2}$$

$$\therefore \quad A = \frac{1}{|x|^2+\rho^2}\,\frac{1}{2}\left(x^\dagger dx - dx^\dagger x\right)$$

$$F_A = \frac{\rho^2}{(|x|^2+\rho^2)^2}\, dx^\dagger \wedge dx$$



Indeed, we have found a solution localized at $x = 0$.

The constant ρ can be regarded as the size of the (anti) instanton.

Unlike in the 1-d problem (double well), the size is a free parameter.

Other solutions?

One way to find new solutions: Use Symmetry.

$$S_E[A] = -\frac{1}{2g^2}\int_{\mathbb{R}^4}\sum_{\mu,\nu}\mathrm{Tr}\left(F_{\mu\nu}(x)F_{\mu\nu}(x)\right)d^4x$$

is invariant under

- Isometry of $\mathbb{R}^4$ $\Big\langle$ translations ($\mathbb{R}^4$), rotations ($SO(4)$)
- dilatation $A_\mu(x) \to \lambda A_\mu(\lambda x)$.

- Translations do give new solutions: $A(x) \to A(x - x_0)$

  x_0: the "center" of instanton
  "location"

- Rotations are equivalent to gauge transformations:

$$SO(4) \cong (SU(2)\times SU(2))/\{(\pm 1_2, \pm 1_2)\}$$

$$[x^\mu \mapsto M^\mu{}_\nu x^\nu] \leftrightarrow [\, x = x^4 + i\vec{x}\cdot\vec{\sigma} \mapsto g_L x g_R^{-1}\,]$$

$$\Rightarrow A \to g_R A g_R^{-1}$$

- Dilatation changes ρ to ρ/λ.

[Dilatation symmetry is the reason why we have a free parameter corresponding to the size.]

Thus, we have identified a 5-parameter family of anti-instantons, up to gauge transformations. $\left(\begin{array}{l} 4 \cdots \text{location} \\ 1 \cdots \text{size} \end{array}\right)$.

Remark: Isometry + dilatation are parts of Conformal symmetries which form a group $SO(5,1)$. It's $SO(5)$ subgroup preserves the solution (up to gauge transf.).

$\dim\left(SO(5,1)/SO(5)\right) = \frac{6\cdot 5}{2} - \frac{5\cdot 4}{2} = 15-10 = 5$ matches the # we have identified. Thus, nothing new comes out of the more general conformal symmetry.

$\boxed{\text{FACT}}$: That's all!

i.e. there is no other solution.

Remark Constant gauge transformations (or equivalently, gauge tranformations that do not go to 1 at $x=\infty$) can be regarded as a global symmetry, not a gauge symmetry. Note that any $g = \text{const}$ does not fix our solution other than $g = \pm 1$. Thus, including these transformations, we have $5+3=8$ parameter family of anti-instantons.

The energy density of θ-vacua

$$\mathcal{E}(\theta) = -2\underbrace{K}_{\text{fluctuation determinant.}} e^{-\frac{8\pi^2}{g^2}} \cos\theta$$

- 8-parameter family of (anti) instantons.

  $\longrightarrow \left(\frac{1}{\sqrt{2\pi g^2}}\right)^8$ from the zero mode measure.

- Integration over the location x_0 is taken care of:

  — the "VT" factor (which is dropped here).

- Integration over constant gauge transformations just gives a numerical factor.

- $\mathcal{E}(\theta)$ must have dimension of mass^4.

$\longrightarrow$

$$\mathcal{E}(\theta) = -\cos\theta \, e^{-\frac{8\pi^2}{g^2}} g^{-8} \int_0^\infty \frac{d\rho}{\rho^5} \underbrace{f(\rho M)}_{\text{some function of } \rho M}.$$

M is a mass scale needed to regularize the theory (cut-off).

Renormalization group

Consider a theory with gauge group G,

a Dirac fermion in a representation R_f, and

a complex scalar in a representation R_b.

For simple G, $\mathrm{tr}_R(XY)$ in any R is proportional to $\mathrm{Tr}(XY)$.

Let's define #s $h^\vee$, T_R by

$$\mathrm{tr}_{adj}(XY) = 2h^\vee \,\mathrm{Tr}(XY)$$

$h^\vee$ … dual Coxeter number of G.

$$\mathrm{tr}_R(XY) = 2T_R \,\mathrm{Tr}(XY)$$

The cut-off dependence of the gauge coupling is governed by

$$\boxed{b_1 := \frac{11}{3}h^\vee - \frac{4}{3}T_{R_f} - \frac{1}{3}T_{R_b}}$$

at one-loop:

$$\beta_{1\text{-loop}}(g) = M\frac{d}{dM}g = -\frac{g^3}{16\pi^2}b_1 .$$

i.e. $M\frac{d}{dM}\left(\frac{8\pi^2}{g^2}\right) = b_1 + O(g^2)$

$$\frac{8\pi^2}{g^2(M_1)} = \frac{8\pi^2}{g^2(M_2)} + b_1 \log\left(\frac{M_1}{M_2}\right) + O(g^2)$$

or $$e^{-\frac{8\pi^2}{g^2(M_1)}} = e^{-\frac{8\pi^2}{g^2(M_2)}}\left(\frac{M_2}{M_1}\right)^{b_1}$$

SU(2) Yang-Mills:
$b_1 = \frac{22}{3}$.

Any physical observable should not depend on M

as long as the coupling g changes this way $\longrightarrow$

$$\mathcal{E}(\theta) = -A\cos\theta\, e^{-\frac{8\pi^2}{g^2(M)}} g(M)^{-8} \int_0^\infty \frac{d\rho}{\rho^5} (\rho M)^{\frac{22}{3}} \left(1+O(g^2)\right)$$

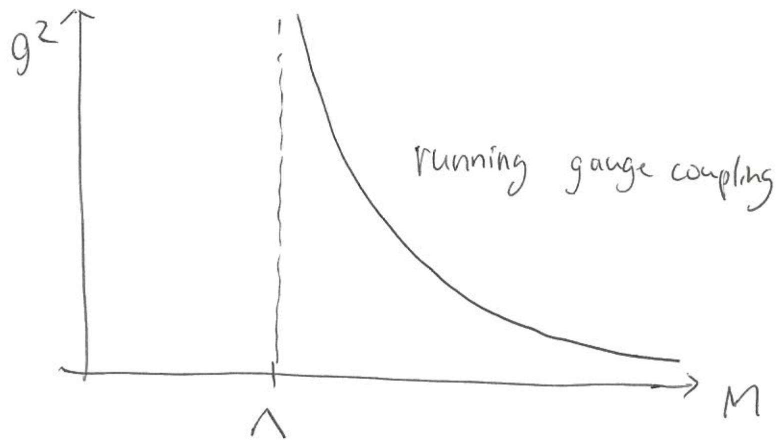
The ρ-integral diverges at $\rho\to\infty$.

- $\rho \to \infty$: dilute gas approx. breaks down.
- But even before that; long distances $\longleftrightarrow$ low energies (small M)

$$\frac{8\pi^2}{g^2(M_1)} = \frac{8\pi^2}{g^2(M_2)} + b_1 \log\left(\frac{M_1}{M_2}\right) \Rightarrow$$

$$\frac{1}{g^2(M)} = \frac{b_1}{8\pi^2} \log\left(\frac{M}{\Lambda}\right)$$

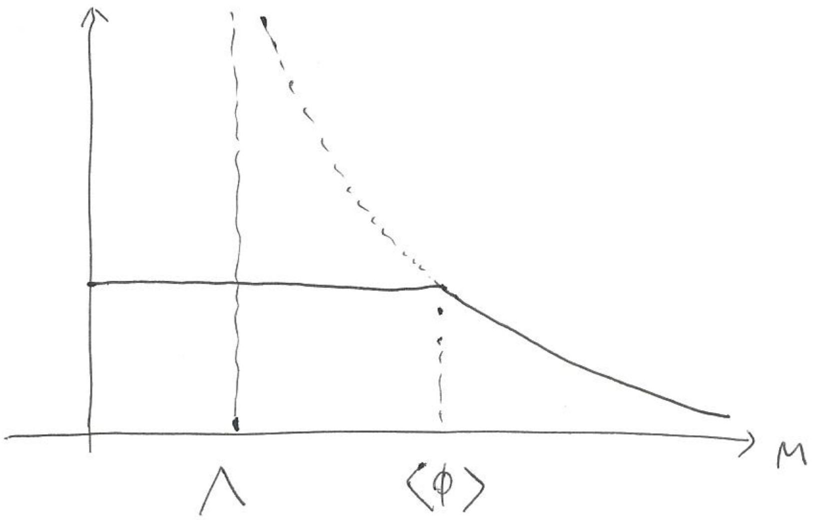
(Λ: the scale parameter of the theory ("dynamical scale"))



At long enough distance scales, the theory is not weakly coupled.

Semiclassical approximation is totally invalid!

If the gauge symmetry is broken by Higgs mechanism, the running stops at the symmetry breaking scale.



If $\langle\phi\rangle/\Lambda$ is large enough so that $g^2(\langle\phi\rangle) \ll 1$, instanton computation should be valid even for $\rho \to \infty$.

We will (hopefully) see such examples later.