More on integrality

Suppose $A \to g^{-1}dg$ as $|x| \to \infty$, so that $F_A \to 0$ at ∞

and $S_{YM}[A]$ is finite.

(For $d > 2$, $F_A = 0$ near ∞ implies $A \to g^{-1}dg$.) —$*$ proved below

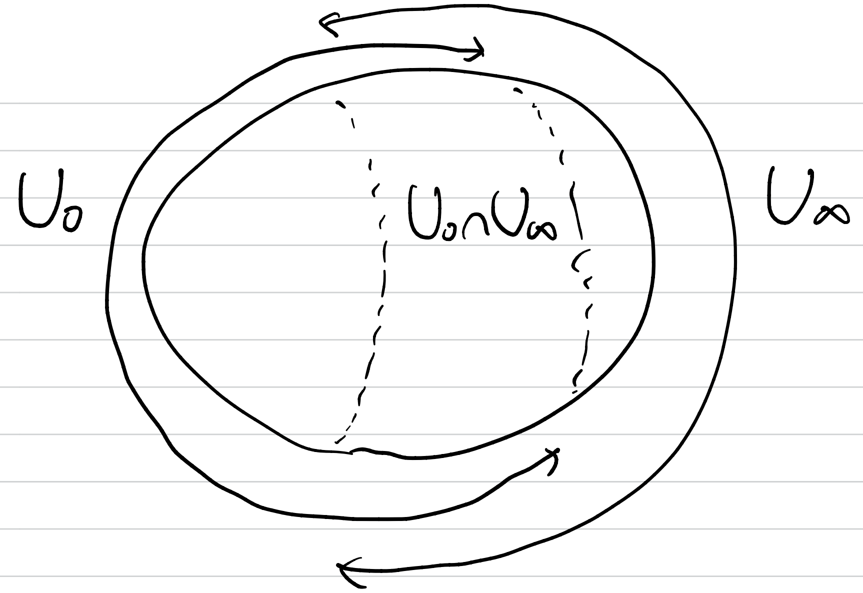
In Lecture 12, it was shown that there is some K_V s.t.

$$\int_{\mathbb{R}^d} ch_{d,V}[A] \in K_V \mathbb{Z}.$$

In fact, it can be shown that

$$\int_{\mathbb{R}^d} ch_{d,V}[A] \in \mathbb{Z}.$$

$A \to g^{-1}dg$ as $|x| \to \infty$ means that the gauge potential defines a connection on a principal G-bundle on S^d. Indeed, a principal G bundle P is given by an open cover $\{U_0, U_\infty\}$ of S^d with a map $g_{\infty 0} : U_\infty \cap U_0 \to G$, and a connection on P is given by gauge potentials A_0 on U_0 and A_∞ on U_∞ related on the overlap $U_\infty \cap U_0$ by the gauge transformation by $g_{\infty 0}$, $A_0 = A_\infty^{g_{\infty 0}}$.



In the present case, $S^d = \mathbb{R}^d \cup \{\infty\}$,

$$\left.\begin{aligned} & U_0 = \mathbb{R}^d, \quad A_0 = A \\ & U_\infty = \text{a neighborhood of } \infty, \quad A_\infty = 0 \\ & g_{\infty 0} = g \end{aligned}\right\} (\#)$$

defines a G-bundle with a connection on S^d.

It also defines a vector bundle $E = P \times_G V$ with fiber V.

Then $$\mathrm{ch}(E)\big|_{U_0} = \mathrm{ch}_V[A],$$

$$\mathrm{ch}(E)\big|_{U_\infty} = 0,$$

and

$$\int_{S^d} \mathrm{ch}(E) = \int_{\mathbb{R}^d} \mathrm{ch}_V[A] = \int_{\mathbb{R}^d} \mathrm{ch}_{d,V}[A].$$

At this point, we use the Atiyah-Singer formula for the index of Dirac operator (which can be derived by Fujikawa's method or SQM path-integral):

$$\mathrm{index}(\not{D} : S^R(E) \to S^L(E)) = \int_{S^d} \mathrm{ch}(E)\,\hat{A}(TS^d).$$

<u>Claim</u> $\hat{A}(TS^d) = 1$

<u>proof</u> If we realize S^d as the unit sphere in $\mathbb{R}^{d+1}$, the tangent bundle of $\mathbb{R}^{d+1}$ restricted on S^d has decomposition

$$T\mathbb{R}^{d+1}|_{S^d} \cong TS^d \oplus N$$

where N is the normal bundle of S^d in $\mathbb{R}^{d+1}$.

Note: $T\mathbb{R}^{d+1}|_{S^d}$ and N are both topologically trivial.

(The former is trivial as $T\mathbb{R}^{d+1}$ is, and the unit normal vector field provides a trivialization of the latter.)

On the other hand, $\hat{A}$ can be expressed in terms of Pontrjagin classes as

$$\hat{A} = 1 - \frac{1}{24}P_1 + \frac{1}{5760}(-4P_2 + P_1^2) + \cdots$$

The total Pontryagin class $p = 1 + p_1 + p_2 + \cdots$ satisfies

$$p(\xi \oplus \eta) = p(\xi)\, p(\eta).$$

Applying this to $\xi = TS^d$, $\eta = N$ and using triviality of $TS^d \oplus N$ ($\cong T\mathbb{R}^{d+1}|_{S^d}$) and N, we find

$$\underbrace{p(T\mathbb{R}^{d+1}|_{S^d})}_{1} = p(TS^d)\, \underbrace{p(N)}_{1},$$

i.e. $p(TS^d) = 1$, that is $p_j(TS^d) = 0 \quad \forall j \geq 1$.

This proves $\hat{A}(TS^d) = 1$. //

Thus the index formula says

$$\operatorname{index}(\not{D} : S^R(E) \to S^L(E)) = \int_{S^d} ch(E).$$

Since the index is an integer,

$$\int_{S^d} ch(E) \underset{\substack{\uparrow \\ (\#)}}{=} \int_{\mathbb{R}^d} ch_{d,V}[A] \quad \text{is an integer.}$$

<u>Remark</u> In Lecture 12, it was shown

$$\int_{\mathbb{R}^d} ch_{d,V}[A] = \lim_{R\to\infty} \int_{S_R^{d-1}} \omega_{d-1,V}[g^{-1}dg].$$

This is for the configuration (#) in which $A_\infty \equiv 0$ is assumed in particular. The same can be derived without such an assumption:

To construct a general G-bundle on S^d we can take U_0 & U_∞ to be neighborhoods of balls D_0^d & D_∞^d which divides S^d along a $(d-1)$-sphere, $D_0^d \cap D_\infty^d \cong S^{d-1}$.

$$\int_{S^d} ch(E) = \int_{D_0^d} \underbrace{ch_{d,V}[A_0]}_{= d\omega_{d-1,V}[A_0]} + \int_{D_\infty^d} \underbrace{ch_{d,V}[A_\infty]}_{= d\omega_{d-1,V}[A_\infty]}$$

$$= \int_{\partial D_0^d = S^{d-1}} \omega_{d-1,V}[A_0] + \int_{\partial D_\infty^d = -\partial D_0^d = -S^{d-1}} \omega_{d-1,V}[A_\infty]$$

$$= \int_{S^{d-1}} \left(\omega_{d-1,V}[A_0] - \omega_{d-1,V}[A_\infty]\right)$$

Here we recall $A_0 = A_\infty^{g_{\infty 0}}$ and the change of CS form under gauge transformations:

$$\omega_{d-1,V}[A^g] - \omega_{d-1,V}[A]$$
$$= \omega_{d-1,V}[g^{-1}dg] + d\alpha_{d-2,V}[g,A].$$

Then we find that the right hand side is

$$\int_{S^{d-1}} \left(\omega_{d-1,V}[g_{\infty 0}^{-1} dg_{\infty 0}] + \cancel{d\alpha_{d-2,V}[g_{\infty 0}, A_\infty]} \right) \to 0$$

as $\partial S^{d-1} = \phi$

Thus,

$$\int_{S^d} ch(E) = \int_{S^{d-1}} \omega_{d-1,V}[g_{\infty 0}^{-1} dg_{\infty 0}].$$

(However, the assumption (#) does not lose any generality: We can acheive that by taking the limit where D^d_∞ and U_∞ are vanishingly small neighborhoods of $\infty \in S^d$.)

Proof of $*$: $F_A = 0$ near ∞ implies $A \to g^{-1}dg$ if $d>2$.

Suppose $F_A = 0$ near ∞, i.e. on $O_R = \{ x \in \mathbb{R}^d \mid |x| > R \}$ for some large R. Pick & fix any point $x_0 \in O_R$.

For $\forall x \in O_R$, draw any path γ_t from $\gamma_0 = x_0$ to $\gamma_1 = x$, and solve ODE $g_t^{-1} \frac{dg_t}{dt} = \dot{\gamma}_t^\mu A_\mu(\gamma_t)$ with the initial condition $g_0 = 1$.

(e.g. If G is a matrix group like $SU(n)$, $SO(n)$, $USp(n)$, the solution is

$$g_t = \mathrm{Pexp}\left(- \int_0^t dt' \, \dot{\gamma}^\mu_{t'} A_\mu(\gamma_{t'}) \right)^{-1}$$

where Pexp is the path-ordered exponential.)

Since $O_R \cong S^{d-1} \times \mathbb{R}$ is simply connected if $d>2$, as $F_A \equiv 0$ on O_R, g_1 does not depend on the choice of path γ_t. And we can define $g(x) := g_1$. This defines a map $g : O_R \to G$ s.t. $g^{-1}dg = A|_{O_R}$. //