Chiral anomaly (Outline: see the additional note for details)

Consider the case $V_R = V$, $V_L = \{0\}$ for simplicity.

$$Z[A] = \int \mathcal{D}\bar{\psi}_R \mathcal{D}\psi_R \, e^{\int i\bar{\psi}_R \not{D}_A \psi_R \, d^4x}$$

$$= \text{Const} \cdot \int \mathcal{D}\bar{\psi}_R \mathcal{D}\psi_R \mathcal{D}\bar{\psi}_L \mathcal{D}\psi_L \, e^{\int (i\bar{\psi}_R \not{D}_A \psi_R + i\bar{\psi}_L \not{\partial} \psi_L) d^4x}$$

$$= \text{Const} \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \, e^{\int (i\bar{\psi} \not{\partial} \psi + i\bar{\psi} \not{A} P_R \psi) d^4x}$$

where $P_R = \frac{1+\gamma_5}{2}$ projection to R-components.

For the purpose of computation of anomaly, we can consider the Dirac fermion ψ with values in V where A is coupled to $\psi_R = P_R \psi$ only:

$$\delta A \cdot J = i\bar{\psi}_R \not{\delta A} \psi_R = i\bar{\psi} \not{\delta A} P_R \psi.$$

Now let us compute $-\delta_1 \cdots \delta_n \delta_\epsilon W[A]\big|_{A=0}$.

$$-\delta_\epsilon W[A=0] = \left\langle \int d^4x \, D_\mu \epsilon \cdot J^\mu \right\rangle = 0 \text{ by "Lorentz" inv.}$$

$$-\delta\,\delta_\epsilon W[A]\Big|_{A=0} = \left\langle \int d^4x\, \partial_\mu \epsilon\, J^\mu \int d^4y\, \delta A\cdot J \right\rangle_{conn}$$

$$+ \left\langle \int d^4x\, [\delta A_\mu, \epsilon]\cdot J^\mu \right\rangle = 0 \text{ by "Lorentz".}$$

$$\delta A = dx^\nu e^b e^{-iqx}$$

$$= \int d^4x\, \partial_\mu \epsilon_a(x) \int d^4y\, e^{-iqy} (-1)\, \mathrm{tr}_{V\otimes S}\left(i\gamma^\mu e^a P_R \psi(x)\bar\psi(y)\, i\gamma^\nu e^b P_R \psi(y)\bar\psi(x)\right)$$

$$= \int d^4x\, \underbrace{\partial_\mu \epsilon_a(x) e^{-iqx}}_{\sim\, i q_\mu \epsilon_a(x) e^{-iqx}}\, \mathrm{tr}_V(e^a e^b) \underbrace{\int \frac{d^4k}{(2\pi)^4} \mathrm{tr}_S\left(\gamma^\mu P_R \frac{1}{-\not k}\gamma^\nu P_R \frac{1}{-(\not k - \not q)}\right)}_{\star}$$

quadratically divergent.

Pauli-Villars regularization

Introduce 3 regulators :

name	0	1	2	3
mass	$\Lambda_0 = 0$	Λ_1	Λ_2	Λ_3
statistics	fermi ($\epsilon_0 = 1$)	bose ($\epsilon_1 = -1$)	fermi ($\epsilon_2 = 1$)	bose ($\epsilon_3 = -1$)
	original	regulators		

This replaces $\star$ by

$$I^{\mu\nu}(q) := \int \frac{d^4k}{(2\pi)^4} \sum_{i=0}^{3} \epsilon_i\, \mathrm{tr}_S\left(\gamma^\mu P_R \frac{1}{-\not k + \Lambda_i}\gamma^\nu P_R \frac{1}{-(\not k - \not q) + \Lambda_i}\right)$$

It turns out that the integral is convergent if $\sum_{i=1}^{3} \epsilon_i \Lambda_i^2 = 0$.

Then, after some computation, we find

$$I^{\mu\nu}(q) = \frac{-2}{(4\pi)^2}\Big[(\delta^{\mu\nu} q^2 - q^\mu q^\nu)\frac{1}{3}\Big(\log q^2 - \frac{5}{3} + \sum_{i=1}^{3} \epsilon_i \log \Lambda_i^2\Big)$$
$$+ \delta^{\mu\nu}\Big(\sum_{i=1}^{3} \epsilon_i \Lambda_i^2 \log \Lambda_i^2 - \frac{1}{6} q^2\Big)\Big]$$

and hence

$$-\delta\delta_\epsilon W[A]\Big|_{A=0}$$
$$= i\int d^4x\, \epsilon_a(x)\, e^{-iqx}\, \mathrm{tr}_V(e^a e^b)\, q_\mu I^{\mu\nu}(q)$$
$$q_\mu I^{\mu\nu}(q) = \frac{-2}{(4\pi)^2} q^\nu \Big(\sum_{i=1}^{3} \epsilon_i \Lambda_i^2 \log \Lambda_i^2 - \frac{1}{6} q^2\Big)$$
$$\neq 0.$$

This does not match with $\delta\, \mathcal{a}_\epsilon[A] = 0$ for the claimed formula.

If fact, this can be cancelled by adding a local counter term $\Delta W[A]$ to $W[A]$. In fact

$$\Delta W[A] = \int d^4x\, \mathrm{tr}_V\big(A_\mu \delta^{\mu\lambda}(C + D\partial^2) A_\lambda\big)$$

with $C = \frac{-1}{(4\pi)^2}\sum_{i=1}^{3} \epsilon_i \Lambda_i^2 \log \Lambda_i^2$ & $D = -\frac{1}{6(4\pi)^2}$

does the job.

We may also add

$$\Delta' W[A] = E \int d^4x \, \mathrm{tr}_V \left[A_\mu \left(\delta^{\mu\nu} \partial^2 - \partial^\mu \partial^\nu \right) A_\nu \right]$$

which has $\delta \delta_\epsilon \Delta' W[A] \big|_{A=0} = 0.$

We shall consider $W'[A] = W[A] + \Delta W[A] + \Delta' W[A]$

for the above C and D and for some E.

Next, let us compute $-\delta_2 \delta_1 \delta_\epsilon W'[A] \big|_{A=0}$

$$= \boxed{-\int d^4x \, D_\mu \epsilon_a(x) \, \delta_2 \delta_1 \frac{\delta W'[A]}{\delta A_{\mu a}(x)} \Big|_0} =: \text{☆}$$

$$\boxed{-\int d^4x \, [\delta_2 A_\mu, \epsilon]_a(x) \, \delta_1 \frac{\delta W'[A]}{\delta A_{\mu a}(x)} \Big|_0} + (1 \leftrightarrow 2) =: \circledast$$

for $\delta_1 A = dx^\nu e^b e^{-iqx}$ & $\delta_2 A = dx^\rho e^c e^{-ipx}$

$\circledast$ is basically computed in $-\delta_1 \delta_\epsilon W[A] \big|_0$ and is

$$= -\int d^4x \, e^{-i(p+q)x} \, \mathrm{tr}_V(\epsilon(x)[e^b, e^c]) \left(\delta^{\mu\nu} q^2 - q^\mu q^\nu \right) \times$$

$$\left[\frac{2}{3(4\pi)^2} \left(\log q^2 - \frac{5}{3} + \sum_{i=1}^{3} \epsilon_i \log \Lambda_i^2 \right) - 2E \right]$$

$$\star = \sum_{i=0}^{3} \left\langle \int d^4x \; i\bar{\psi}_i \not{\partial}\epsilon P_R \psi_i \int d^4y \; i\bar{\psi}_i \not{\delta}_1\!A \, P_R \psi_i \int d^4z \; i\bar{\psi}_i \not{\delta}_2\!A \, P_R \psi_i \right\rangle_{\text{conn}}$$

$$= -\int d^4x \, \epsilon_a(x) \, e^{-i(p+q)x} \, \mathrm{tr}_V(e^a e^b e^c) \int \frac{d^4k}{(2\pi)^4} \sum_{i=0}^{3} \epsilon_i \, g(k,\Lambda_i)$$

+ exchange ;

$$g(k,m) = \mathrm{tr}_S\left((\not{p}+\not{q}) P_R \frac{1}{-(\not{k}+\not{q})+m} \gamma^\nu P_R \frac{1}{-\not{k}+m} \gamma^\rho P_R \frac{1}{-(\not{k}-\not{p})+m} \right)$$

$$= \mathrm{tr}_S\Bigg(\gamma^\nu P_R \frac{1}{-\not{k}+m} \gamma^\rho P_R \frac{1}{-(\not{k}+\not{q})+m}$$

$$- \gamma^\nu P_R \frac{1}{-\not{k}+m} \gamma^\rho P_R \frac{1}{-(\not{k}-\not{p})+m}$$

$$- \frac{m^2 P_R (\not{p}+\not{q}) \gamma^\nu \not{k} \gamma^\rho}{((k+q)^2+m^2)(k^2+m^2)((k-p)^2+m^2)} \Bigg) .$$

$$\int \frac{d^4k}{(2\pi)^4} \sum_{i=0}^{3} \epsilon_i \, g(k,\Lambda_i) = I^{\nu\rho}(-q) - I^{\nu\rho}(p) + J^{\nu\rho}(p,q) \; ;$$

$$J^{\nu\rho}(p,q) := \int \frac{d^4k}{(2\pi)^4} \sum_{i=1}^{3} \epsilon_i \frac{(-\Lambda_i^2)\, \mathrm{tr}_S\left(P_R (\not{p}+\not{q}) \gamma^\nu \not{k} \gamma^\rho\right)}{((k+q)^2+\Lambda_i^2)(k^2+\Lambda_i^2)((k-p)^2+\Lambda_i^2)}$$

$$= \frac{1}{3(4\pi)^2} \left\{ (\delta^{\nu\rho} q^2 - 2q^\nu q^\rho) - (\delta^{\nu\rho} p^2 - 2p^\nu p^\rho) - 2\epsilon^{\lambda\nu\rho\sigma} q_\lambda p_\sigma \right\} .$$

+ terms that vanish as $p/\Lambda_i \to 0$, $q/\Lambda_i \to 0$.

$$\star = -\int d^4x\, \epsilon_a(x)\, e^{-i(p+q)x}\, \mathrm{tr}_V(e^a e^b e^c)\left[I^{\nu\rho}(-q) - I^{\nu\rho}(p) + J^{\nu\rho}(p,q) \right]$$
$$+ \ (\nu, b, q) \leftrightarrow (\rho, c, p)$$

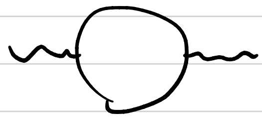
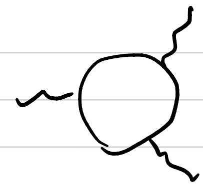
$$= \int d^4x\, \epsilon_a(x)\, e^{-i(p+q)x} \frac{2}{3(4\pi)^2}\Big[\mathrm{tr}_V(e^a\{e^b, e^c\})\, \epsilon^{\lambda\nu\sigma\rho} q_\lambda p_\sigma$$
$$+ \mathrm{tr}_V(e^a[e^b, e^c])\Big\{ (\delta^{\nu\rho} q^2 - q^\nu q^\rho)\Big(\log q^2 - \frac{8}{3} + \sum_{i=1}^{3} \epsilon_i \log \Lambda_i^2\Big)$$
$$- (\delta^{\nu\rho} p^2 - p^\nu p^\rho)\Big(\log p^2 - \frac{8}{3} + \sum_{i=1}^{3} \epsilon_i \log \Lambda_i^2\Big)\Big\}\Big]$$

If we set $E = 1/3(4\pi)^2$, the $\delta^{\nu\rho} q^2 - q^\nu q^\rho$ & $\delta^{\nu\rho} p^2 - p^\nu p^\rho$ terms cancel in $\star + * + *_{1\leftrightarrow 2}$, and we find

$$-\delta_2 \delta_1 \delta_\epsilon W'[A]\Big|_{A=0}$$
$$= \int d^4x\, e^{-i(p+q)x} \frac{2}{3(4\pi)^2} \mathrm{tr}_V(\epsilon(x)\{e^b, e^c\})\, \epsilon^{\lambda\nu\sigma\rho} q_\lambda p_\sigma .$$
$$= \delta_2 \delta_1\, i \int \frac{i}{24\pi^2} \mathrm{tr}_V(\epsilon\, dA \wedge dA).$$

$$\therefore \quad a_\epsilon[A] = \int \frac{i}{24\pi^2} \mathrm{tr}_V\Big(\epsilon\, d\big(A dA + \frac{1}{2} A^3\big)\Big)$$

modulo $O(A^3)$ terms

<u>Summary</u> By computing  & ,

we have seen

$$a^5_\epsilon[A] = \int \frac{-1}{4\pi^2}\, \epsilon\, \mathrm{tr}_V(dA\,dA) + O(A^3)$$

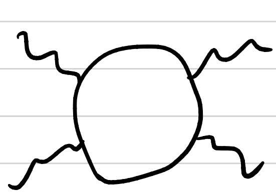
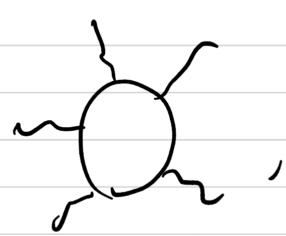
$$a^R_\epsilon[A] = \int \frac{i}{24\pi^2}\, \mathrm{tr}_{V_R}(\epsilon\, dA\,dA) + O(A^3)$$

modulo $\delta^R_\epsilon\, \mathrm{local}[A]$.

This is consistent with

$$a^5_\epsilon[A] = \int \frac{-1}{4\pi^2}\, \epsilon\, \mathrm{tr}_V(F_A^2)$$

$$a^R_\epsilon[A] = \int \frac{i}{24\pi^2}\, \mathrm{tr}_{V_R}\left(\epsilon\, d\left(A\,dA + \tfrac{1}{2}A^3\right)\right).$$

We may compute , , ...

to fix the higher order terms $O(A^3)$.

But that is not necessary if we use the general structure of anomalies.

The general structure of anomalies

① The anomaly $a_\epsilon[A,\Phi]$ is local, i.e.

$\int d^dx$ polynomial of derivatives of (ϵ, A, ϕ),

because it comes from regularization procedure, which is non-trivial only for divergent diagrams $\leftrightarrow$ local.

② (Already discussed) There is a freedom to modify the action by local counter terms. Thus the anomaly $a_\epsilon[A,\Phi]$ is defined only modulo δ_ϵ local functional of $[A,\Phi]$.

③ $$\delta_{\epsilon_1}\delta_{\epsilon_2}\mathcal{D}_A\phi - \delta_{\epsilon_2}\delta_{\epsilon_1}\mathcal{D}_A\phi = \delta_{[\epsilon_1,\epsilon_2]}\mathcal{D}_A\phi$$

(we've been considering right action). Thus

$$\boxed{\delta_{\epsilon_1} a_{\epsilon_2}[A,\Phi] - \delta_{\epsilon_2} a_{\epsilon_1}[A,\Phi] = a_{[\epsilon_1,\epsilon_2]}[A,\Phi]}$$

Wess-Zumino consistency condition

Let us consider the case of chiral anomaly with $V_L = \{0\}$.

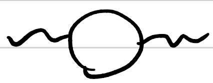
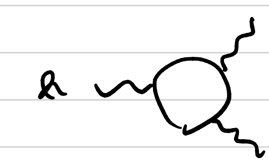
(i) $a^R_\epsilon[A] = c \int \mathrm{tr}_{V_R}\left(\epsilon\, d\left(AdA + \frac{1}{2}A^3\right)\right)$

$\underset{\text{constant}}{\uparrow}$

Satisfies the WZ consistency condition. (Exercise)

(ii) It is also the unique solution to the WZ condition (of course modulo δ_ϵ local$[A]$) with the "initial condition"

$$a^R_\epsilon[A] = c\int \mathrm{tr}_{V_R}(\epsilon\, dA dA) + O(A^3).$$

Thus, our computation from  &  is enough to prove

$$a^R_\epsilon[A] = \int \frac{i}{24\pi^2}\, \mathrm{tr}_{V_R}\left(\epsilon\, d\left(AdA + \frac{1}{2}A^3\right)\right).$$

In view of the relation between $a^5_\epsilon[A]$ & $a^{tot}_{(\epsilon,0)}[0,A]$ for $U(1)_5 \times G$, this also confirms

$$a^5_\epsilon[A] = \int \frac{-1}{4\pi^2}\, \epsilon\, \mathrm{tr}_V(F_A^2).$$

Fujikawa's method

... an alternative, direct method to compute anomalies.

Axial anomaly (G ungauged)

ψ : massless Dirac fermion with values in a unitary representation V of a compact Lie group G.

$$S = \int d^4x \, (-i) \overline{\psi} \not{D}_A \psi .$$

Axial anomaly [Note: right action $\psi^\epsilon = (e^{i\epsilon\gamma_5})^{-1}\psi$]

$$\mathcal{D}(\overline{\psi} e^{-i\epsilon\gamma_5}) \, \mathcal{D}(e^{-i\epsilon\gamma_5}\psi) = (\mathrm{Det}\, e^{i\epsilon\gamma_5})^2 \, \mathcal{D}\overline{\psi}\, \mathcal{D}\psi$$

$$\therefore \; i a^5_\epsilon = \frac{d}{dt} \left(\mathrm{Det}\, e^{it\epsilon\gamma_5}\right)^2 \Big|_{t=0} = 2\, \mathrm{Tr}(i\epsilon\gamma_5)$$

.... divergent.

A regularization:

$$a^5_\epsilon [A] = 2\, \mathrm{Tr}\left(\epsilon \gamma_5 \, e^{-\not{D}_A^2/\Lambda^2} \right).$$

Note: $\not{D}_A^\dagger = \not{D}_A$ since $\gamma^{\mu\dagger} = -\gamma^\mu$ and $D_\mu^\dagger = -D_\mu$ (V unitary).

Thus $\not{D}_A$ has real eigenvalues and hence $e^{-\not{D}_A^2/\Lambda^2}$ can provide a regularization.

Use the plane wave basis $\varphi_{k,i,\alpha}(x) = e^{ikx}\, e_i \otimes e_\alpha$

of the space of V-valued spinors

($\{e_i\} \subset V$, $\{e_\alpha\} \subset S$ basis)

to evaluate the trace:

$$G^5_\epsilon[A] = 2\,\mathrm{Tr}\left(\epsilon\, \gamma_5\, e^{-\not{D}_A^2/\Lambda^2}\right)$$

$$= 2\int d^4x \int \frac{d^4k}{(2\pi)^4} \sum_{i,\alpha} \left(\varphi_{k,i,\alpha}(x),\ \epsilon(x)\,\gamma_5\, e^{-\not{D}_A^2/\Lambda^2}\varphi_{k,i,\alpha}(x)\right)$$

$$= 2\int d^4x\, \epsilon(x) \int \frac{d^4k}{(2\pi)^4}\, \mathrm{tr}_{V\otimes S}\left(\gamma_5\, e^{-ikx}\, e^{-\not{D}_A^2/\Lambda^2}\, e^{ikx}\right)$$

$$e^{-ikx}\not{D}_A\, e^{ikx} = \gamma^\mu(ik_\mu + \partial_\mu + A_\mu),$$

$$e^{-ikx}\not{D}_A^2\, e^{ikx} = \gamma^\mu\gamma^\nu(ik_\mu + \partial_\mu + A_\mu)(ik_\nu + \partial_\nu + A_\nu)$$

- $\gamma^\mu\gamma^\nu = \frac{1}{2}\{\gamma^\mu,\gamma^\nu\} + \frac{1}{2}[\gamma^\mu,\gamma^\nu] = -\delta^{\mu\nu} + \gamma^{\mu\nu}$
- $(ik_\mu + \partial_\mu + A_\mu)(ik_\nu + \partial_\nu + A_\nu) - (\mu \leftrightarrow \nu) = F_{\mu\nu}$

$$= -\delta^{\mu\nu}(ik_\mu + \partial_\mu + A_\mu)(ik_\nu + \partial_\nu + A_\nu) + \frac{1}{2}\gamma^{\mu\nu}F_{\mu\nu}$$

$$= k^2 \underbrace{-2ik^\mu(\partial_\mu + A_\mu) + (\partial^\mu + A^\mu)(\partial_\mu + A_\mu)}_{=:X} + \frac{1}{2}\gamma^{\mu\nu}F_{\mu\nu}.$$

$$e^{-ikx} e^{-\not{D}_A^2/\Lambda^2} e^{ikx} = \exp\left(-e^{-ikx}\not{D}_A^2/\Lambda^2 e^{ikx}\right)\cdot 1$$

$$= e^{-k^2/\Lambda^2 - (X + \frac{1}{2}\gamma^{\mu\nu}F_{\mu\nu})/\Lambda^2}\cdot 1$$

$$= e^{-k^2/\Lambda^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left[(X + \frac{1}{2}\gamma^{\mu\nu}F_{\mu\nu})/\Lambda^2\right]^n\cdot 1$$

- As $\mathrm{tr}_S(\gamma_5\gamma^{\mu_1}\cdots\gamma^{\mu_s}) = 0$ if $s \le 3$, at least two powers of $\frac{1}{2}\gamma^{\mu\nu}F_{\mu\nu}/\Lambda^2$ is necessary to survive $\mathrm{tr}_S(\gamma_5 —)$.
- $\displaystyle\int\frac{d^4k}{(2\pi)^4} e^{-k^2/\Lambda^2}(X/\Lambda^2)^\ell(\gamma^{\mu\nu}F_{\mu\nu}/\Lambda^2)^{m+2}\cdot 1$ or its reordering

$$\sim \Lambda^4\frac{\Lambda^{p\le\ell}}{(\Lambda^2)^{\ell+m+2}} \le \Lambda^{-\ell-2m}$$

$\therefore$ Only the terms with $\ell = m = 0$ survives the limit $\Lambda \to \infty$.

$$a_\epsilon^5[A] = 2\int d^4x\, \epsilon(x)\int\frac{d^4k}{(2\pi)^4} e^{-k^2/\Lambda^2}\frac{1}{2}\mathrm{tr}_{V\otimes S}\left[\gamma_5\left(\frac{1}{2}\gamma^{\mu\nu}F_{\mu\nu}/\Lambda^2\right)^2\right]$$

Use $\displaystyle\int\frac{d^4k}{(2\pi)^4} e^{-k^2/\Lambda^2} = \frac{\Lambda^4}{(4\pi)^2}$

$$= \int d^4x\, \epsilon(x) \frac{1}{(4\pi)^2} \mathrm{tr}_V (F_{\mu\nu} F_{\rho\lambda}) \underbrace{\frac{1}{4} \mathrm{tr}_S (\gamma_5 \gamma^{\mu\nu} \gamma^{\rho\lambda})}_{-\epsilon^{\mu\nu\rho\lambda}}$$

$$= \int d^4x\, \epsilon(x) \frac{-1}{16\pi^2} \epsilon^{\mu\nu\rho\lambda} \mathrm{tr}_V (F_{\mu\nu} F_{\rho\lambda})$$

$$= \int \frac{-1}{4\pi^2} \epsilon\, \mathrm{tr}_V (F_A \wedge F_A).$$

Note: The axial rotation group is Abeian

$$e^{i\epsilon_1 \gamma_5} \circ e^{i\epsilon_2 \gamma_5} = e^{i(\epsilon_1 + \epsilon_2)\gamma_5}$$

and the infinitesimal formula $a^5_\epsilon [A] = \int \frac{-1}{4\pi^2} \epsilon\, \mathrm{tr}_V F_A^2$ is additive

$$a^5_{\epsilon_1}[A] + a^5_{\epsilon_2}[A] = a^5_{\epsilon_1 + \epsilon_2}[A].$$

Thus, it **integrates** to the anomaly formula for finite axial rotations,

$$\mathcal{D}_A (\bar{\psi} e^{-i\epsilon\gamma_5}) \mathcal{D}_A (e^{-i\epsilon\gamma_5} \psi) = \mathcal{D}_A \bar{\psi} \mathcal{D}_A \psi\, e^{i a^5_\epsilon [A]}$$

$$= \mathcal{D}_A \bar{\psi} \mathcal{D}_A \psi \exp\left[i \int \frac{-1}{4\pi^2} \epsilon\, \mathrm{tr}_V (F_A^2) \right]$$

Chiral anomaly

$V_R = V$, $V_L = \{0\}$

For the purpose of computing the anomaly, we may consider a Dirac fermion with values in V

$$S = \int d^4x\, (-i)\, \bar{\Psi} \not{D}_{A,R} \Psi \quad ; \quad \not{D}_{A,R} = \not{\partial} + \not{A} P_R$$

Chiral rotation:

$$\Psi^g = (g P_R + P_L)^{-1} \Psi, \quad \bar{\Psi}^g = \bar{\Psi}(P_R + g P_L)$$

$$\mathcal{D}\bar{\Psi}^g \mathcal{D}\Psi^g = \mathrm{Det}(P_R + g P_L)^{-1} \cdot \mathrm{Det}(g P_R + P_L)\, \mathcal{D}\bar{\Psi}\mathcal{D}\Psi$$

$$= \mathrm{Det}\left[(P_R + g^{-1} P_L)(g P_R + P_L)\right] \mathcal{D}\bar{\Psi}\mathcal{D}\Psi$$

$$= \mathrm{Det}(g P_R + g^{-1} P_L)\, \mathcal{D}\bar{\Psi}\mathcal{D}\Psi$$

$$\therefore\ i a^R_\epsilon = \mathrm{Tr}(\epsilon P_R - \epsilon P_L) = \mathrm{Tr}(\epsilon \gamma_5) \quad \cdots \text{divergent.}$$

A regularization:

$$i a^R_\epsilon [A] = \mathrm{Tr}\left(\epsilon \gamma_5\, e^{-\not{D}_{A,R}^2/\Lambda^2}\right)$$

⋮ similar computation
⋮

$$= \int \frac{-1}{24\pi^2} \mathrm{tr}_V\left[\epsilon\, d\left(A dA + \tfrac{1}{2} A^3\right)\right] + \delta_\epsilon \mathrm{loc}[A].$$

Axial anomaly in a general even dimension $d = 2n$

$$\{\gamma^\mu, \gamma^\nu\} = -2\delta^{\mu\nu} \qquad \mu,\nu = 1,\dots,d \quad \text{represented on } S = \mathbb{C}^{2^n}$$

$$(\gamma^1 \cdots \gamma^d)^2 = (-1)^{\frac{d(d-1)}{2}}$$

$$\gamma_{d+1} := i^{\frac{d(d-1)}{2}} \gamma^1 \cdots \gamma^d \ ; \quad \gamma_{d+1}^2 = 1, \quad \gamma_{d+1}\gamma^\mu = -\gamma^\mu \gamma_{d+1}$$

ψ massless Dirac fermion on $\mathbb{R}^d$ with values in a unitary rep V of a compact G.

$$S = \int d^dx \, (-i) \, \overline{\psi} \not{D}_A \psi$$

Axial anomaly

$$\mathcal{D}_A(\overline{\psi} e^{-i\epsilon\gamma_{d+1}}) \mathcal{D}_A(e^{-i\epsilon\gamma_{d+1}}\psi) = \mathcal{D}_A\overline{\psi}\mathcal{D}_A\psi \cdot e^{i a_\epsilon^{d+1}[A]} \ ;$$

$$a_\epsilon^{d+1}[A] = 2 \operatorname{Tr}\left[\epsilon \gamma_{d+1} e^{-\not{D}_A^2/\Lambda^2}\right]$$

$$= 2 \int d^dx \, \epsilon(x) \int \frac{d^dk}{(2\pi)^d} \operatorname{tr}_{V\otimes S}\left(\gamma_{d+1} e^{-ikx} e^{-\not{D}_A^2/\Lambda^2} e^{ikx}\right)$$

$\vdots$ same computation (Exercise)

$$= 2 \int_{\mathbb{R}^d} \epsilon \, \frac{1}{n!} \operatorname{tr}_V \left(\frac{i}{2\pi} F_A\right)^n$$