Remarks on axial anomaly in d=2n

For $d=2n$ Dirac fermion ψ with values in a rep V of G,

$$\mathcal{D}_A(\bar\psi e^{-i\epsilon\gamma_{d+1}})\mathcal{D}_A(e^{-i\epsilon\gamma_{d+1}}\psi)$$

$$= \mathcal{D}_A\bar\psi\,\mathcal{D}_A\psi\,\exp\left[2i\int_{\mathbb{R}^d}\epsilon\,\mathrm{tr}_V\left(\frac{1}{n!}\left(\frac{i}{2\pi}F_A\right)^n\right)\right]$$

① In the theory where G is gauged, as a part of the action, we may consider the Theta term

$$-S_{V,\theta_V}[A] = i\int_{\mathbb{R}^d}\theta_V\,\mathrm{tr}_V\left(\frac{1}{n!}\left(\frac{i}{2\pi}F_A\right)^n\right).$$

Then, the axial anomaly formula says

$$\boxed{\begin{array}{l}\text{The axial rotation } \psi\to e^{-i\epsilon\gamma_{d+1}}\psi,\ \bar\psi\to\bar\psi e^{-i\epsilon\gamma_{d+1}} \\ \text{shifts } \theta_V\to\theta_V+2\epsilon.\end{array}}$$

- In $d=4$ & G simple, there is a canonically normalized Theta term

$$-S_\theta[A] = i\int_{\mathbb{R}^4}\theta\,\mathrm{tr}\left(\frac{1}{2}\left(\frac{i}{2\pi}F_A\right)^2\right)$$

using "the standard trace" defined by

$$\mathrm{tr}(XY\cdots) := \frac{1}{2h^\vee}\mathrm{tr}_{\mathfrak{g}}(XY\cdots) \qquad \forall X, Y, \cdots \in \mathfrak{g}.$$

Note that $\mathrm{tr}(XY) = -\frac{1}{2}X\cdot Y$.

Then, the shift is

$$\theta \longrightarrow \theta + 4T_V\epsilon.$$

- We may consider a Dirac fermion with a complex mass

$$\mathcal{L}_\psi = -i\overline{\psi}_R \not{D}_A \psi_R - i\overline{\psi}_L \not{D}_A \psi_L + \overline{\psi}_L m \psi_R + \overline{\psi}_R m^* \psi_L$$

The phase of mass, $m = |m| e^{i\arg(m)}$, can be absorbed

by $\psi' = e^{\frac{i}{2}\arg(m)\gamma_{d+1}}\psi$, $\overline{\psi}' = \overline{\psi}\, e^{\frac{i}{2}\arg(m)\gamma_{d+1}}$,

But this induces the shift $\theta_V \to \theta_V + \arg(m)$

($\theta \to \theta + 2T_V \arg(m)$ in 4d, G simple).

$$\boxed{\begin{array}{l} \text{System with complex } m\text{, theta parameter } \theta_V \text{ (resp. } \theta) \\ \equiv \text{System with } m=|m|\text{, theta parameter } \theta_V + \arg(m) \\ \qquad\qquad\qquad\qquad \text{(resp. } \theta + 2T_V \arg(m)) \end{array}}$$

② Chern-Simons form

$\frac{1}{2}$ the density of axial anomaly is the $2n$-form part of the Chern character $ch_V[A] = tr_V(e^{\frac{i}{2\pi}F_A})$, and hence is denoted by $ch_{2n,V}[A] = tr_V\left[\frac{1}{n!}\left(\frac{i}{2\pi}F_A\right)^n\right]$.

It is closed and gauge invariant,

$$d\, ch_{2n,V}[A] = 0, \quad ch_{2n,V}[A^g] = ch_{2n,V}[A].$$

In fact, it is exact, i.e. written as

$$ch_{2n,V}[A] = d\omega_{2n-1,V}[A]$$

for a $(2n-1)$-form $\omega_{2n-1,V}[A]$ called the Chern-Simons form.

The expression can be found as follows:

For any variation $A \to A + \delta A$,

$$\delta\, tr_V(F_A^n) = n\, tr_V(\delta F_A F_A^{n-1}) = n\, tr_V(D_A(\delta A) F_A^{n-1})$$
$$= n\, d\, tr_V(\delta A F_A^{n-1})$$

where we used $\delta F_A = D_A \delta A$, $d\, tr_V(\mathcal{O}) = tr_V(D_A \mathcal{O})$, and Bianchi identity $D_A F_A = 0$.

For a one-parameter family of gauge potentials $A_t = t\cdot A$,

$$\mathrm{tr}_V F_A^n = \mathrm{tr}_V F_{A_1}^n - \mathrm{tr}_V F_{A_0}^n = \int_0^1 dt \frac{\partial}{\partial t} \mathrm{tr}_V F_{A_t}^n \underset{\uparrow\ \text{above}}{=} n\, d\, \mathrm{tr}_V\left(\underbrace{\frac{\partial A_t}{\partial t}}_{A} F_{A_t}^{n-1} \right)$$

$$= d \int_0^1 dt\, n\, \mathrm{tr}_V (A F_{A_t}^{n-1})$$

$$\therefore\ \omega_{2n-1,V}[A] = \frac{1}{n!}\left(\frac{i}{2\pi}\right)^n n \int_0^1 dt\, \mathrm{tr}_V (A F_{A_t}^{n-1})$$

$$;\ F_{A_t} = dA_t + A_t^2 = t\,dA + t^2 A^2$$

More explicitly

$$\omega_{1,V}[A] = \frac{i}{2\pi}\int_0^1 dt\, \mathrm{tr}_V A = \frac{i}{2\pi}\mathrm{tr}_V A$$

$$\omega_{3,V}[A] = \left(\frac{i}{2\pi}\right)^2 \int_0^1 dt\, \mathrm{tr}_V \left(A (t\,dA + t^2 A^2) \right)$$

$$= \left(\frac{i}{2\pi}\right)^2 \mathrm{tr}_V \left(\frac{1}{2} A\,dA + \frac{1}{3} A^3 \right)$$

$$\omega_{5,V}[A] = \frac{1}{2}\left(\frac{i}{2\pi}\right)^3 \int_0^1 dt\, \mathrm{tr}_V \Big(\underbrace{A (t\,dA + t^2 A^2)^2}_{t^2 A(dA)^2 + t^3 \underbrace{A(dA\,A^2 + A^2 dA)}_{2A^3 dA} + t^4 A^5} \Big)$$

$$= \frac{1}{2}\left(\frac{i}{2\pi}\right)^3 \mathrm{tr}_V \left(\frac{1}{3} A(dA)^2 + \frac{1}{2} A^3 dA + \frac{1}{5} A^5 \right)$$

$\vdots$

Chen-Simons form is <u>NOT</u> gauge invariant:

$$\omega_{2n-1,V}[A^g] = \omega_{2n-1,V}[A] + \omega_{2n-1,V}[g^{-1}dg] + d\alpha_{2n-2,V}[g,A]$$

for some $(2n-2)$-form $\alpha_{2n-2,V}[g,A]$.

The expression for $\alpha_{2n-2,V}$ can be found by extending the method to find $\omega_{2n-1,V}$ from $ch_{2n,V}$.

[see Zumino's Les Houches Lecture]

For low n's, they are

$$\alpha_{0,V}[g,A] = 0$$

$$\alpha_{2,V}[g,A] = -\frac{1}{2}\left(\frac{i}{2\pi}\right)^2 \mathrm{tr}_V\left(dg\,g^{-1}A\right)$$

$$\alpha_{4,V}[g,A] = -\frac{1}{2\cdot 3!}\left(\frac{i}{2\pi}\right)^3 \mathrm{tr}_V\Big[dg\,g^{-1}(AdA + dA\,A + A^3) + \frac{1}{2}(A\,dg\,g^{-1})^2 + A(dg\,g^{-1})^3\Big]$$

$$\vdots$$

③ Integrality.

Suppose $A \to g^{-1}dg$ as $|x| \to \infty$, so that $F_A \to 0$ at ∞

and $S_{YM}[A]$ is finite.

(For $d>2$, $F_A = 0$ near ∞ implies $A \to g^{-1}dg$)

Then

$$\int_{\mathbb{R}^d} ch_{d,V}[A] \in K_V \mathbb{Z}$$

for some K_V. This means that $e^{-S_{V,\theta_V}[A]}$ is invariant under $\theta_V \to \theta_V + 2\pi/K_V$.

For $d=4$ & G simple and simply connected,

$$\int_{\mathbb{R}^4} tr\left(\frac{1}{2!}\left(\frac{i}{2\pi}F_A\right)^2\right) \in \mathbb{Z}$$

and $e^{-S_\theta[A]}$ is invariant under $\theta \to \theta + 2\pi$.

For this reason, θ_V or θ is called Theta **angle**.

∵

$$\int_{\mathbb{R}^d} ch_{d,V}[A] = \lim_{R\to\infty} \int_{D^d_R} ch_{d,V}[A]$$

$ch_{d,V}[A] = d\omega_{d-1,V}[A]$

$D^d_R = \{|x| \leq R\}$

$g^{-1}dg$ on S_R^{d-1} at large R

$$= \lim_{R\to\infty} \int_{\partial D_R^d = S_R^{d-1}} \omega_{d-1,V}[A]$$

$$= \lim_{R\to\infty} \int_{S_R^{d-1}} \omega_{d-1,V}[g^{-1}dg]$$

Note that

$$\omega_{d-1,V}[g^{-1}dg] = g^* \omega_{d-1,V}(G)$$

where $\omega_{d-1,V}(G)$ is a closed $d-1$ form on G whose cohomology class is a multiple of an integral class

$$[\omega_{d-1,V}(G)] \in K_V H^{d-1}(G,\mathbb{Z}).$$

Then

$$\int_{S_R^{d-1}} \omega_{d-1,V}[g^{-1}dg] = \int_{g_*[S_R^{d-1}]} \omega_{d-1,V}(G) \in K_V \mathbb{Z}.$$

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Examples

- $\omega_{1,V}[g^{-1}dg] = \frac{i}{2\pi} \mathrm{tr}_V(g^{-1}dg)$

  For $G = U(1) \cong S^1$, $V =$ charge 1 representation $\mathbb{C}(1)$

  $$\int_{S^1} \omega_{1,\mathbb{C}(1)}[g^{-1}dg] = \int_{S^1} \frac{i}{2\pi} g^{-1}dg$$

  $= (-1) \times$ the winding number of the map $g : S^1 \to U(1) \cong S^1$.

- $\omega_{3,V}[g^{-1}dg] = \frac{1}{24\pi^2} \mathrm{tr}_V(g^{-1}dg)^3$

  For $G = SU(2) \cong S^3$, $V =$ fundamental rep $\mathbb{C}^2$,

  $$\int_{S^3} \omega_{3,\mathbb{C}^2}[g^{-1}dg] = \int_{S^3} \frac{1}{24\pi^2} \mathrm{tr}_{\mathbb{C}^2}(g^{-1}dg)^3$$

  $=$ the winding number of the map $g : S^3 \to SU(2) \cong S^3$

For other (G, V, d), $\int_{S^{d-1}} \omega_{d-1,V}[g^{-1}dg]$ can be interpreted as "winding number" in the same way, or such an interpretation is not straight or even absent.

e.g.

· $d=2$, G simple.

$\mathrm{tr}_V X = 0 \quad \forall X \in \mathfrak{g}$, $\forall$ rep V.

$\therefore$ $ch_{2,V}[A] = 0$, $\omega_{1,V}[g^{-1}dg] = 0$ for $\forall$ rep V

Thus axial anomaly is absent.

However, we may have a non-trivial map $S^1 \to G$ when G is not simply connected ($\pi_1 G$ is a finite group).

· $d=4$, G simple, simply connected (e.g. $G = SU(n), USp(n), Spin(n)$)

We may use the standard "tr". Then

$\int_{S^3} \frac{1}{24\pi^2} \mathrm{tr}(g^{-1}dg)^3$ measures the winding number of $g: S^3 \to G$ and defines $H^3(G,\mathbb{Z}) \cong \pi_3(G) \cong \mathbb{Z}$.

This means $\int_{\mathbb{R}^4} \mathrm{tr}\left(\frac{1}{2}\left(\frac{i}{2\pi}F_A\right)^2\right)$ can take all possible integer values. Then the periodicity of θ is strictly 2π: $\theta \sim \theta + 2\pi$.

Anomaly descent

(In what follows, we omit writing "V" to simplify expressions.)

- $ch_{2n+2}[A]$ is gauge invariant, closed and exact

$$\delta_\epsilon \, ch_{2n+2}[A] = 0, \quad d\, ch_{2n+2}[A] = 0,$$

$$ch_{2n+2}[A] = d\,\omega_{2n+1}[A].$$

- $\omega_{2n+1}[A]$ is not gauge invariant, but its infinitesimal gauge transformation is exact

$$\delta_\epsilon \, \omega_{2n+1}[A] = d\,\omega_{2n}[\epsilon, A].$$

- $\omega_{2n}[\epsilon, A]$ satisfies

$$\delta_{\epsilon_1} \omega_{2n}[\epsilon_2, A] - \delta_{\epsilon_2} \omega_{2n}[\epsilon_1, A] - \omega_{2n}[[\epsilon_1, \epsilon_2], A] = d\,\omega_{2n-1}[\epsilon_1, \epsilon_2, A]$$

$\vdots$

This is called the anomaly descent.

The derivation can be found in Zumino's Les Houches Lecture.
It may also be posted as an additional note.

Note: $\int_{\mathbb{R}^d} \omega_d[\epsilon, A]$ satisfies the Wess-Zumino consistency condition and can be a candidate for anomaly (up to a constant multiplication).

Indeed, $\omega_4[\epsilon, A] = \frac{d}{dt}\alpha_4[e^{t\epsilon}, A]\Big|_{t=0}$

$$= -\frac{1}{2\cdot 3!}\left(\frac{i}{2\pi}\right)^3 \mathrm{tr}\left[d\epsilon\left(AdA + dAA + A^3\right)\right]$$

$$= \frac{1}{2\cdot 3!}\left(\frac{i}{2\pi}\right)^3 \mathrm{tr}\left[\epsilon\, d\left(AdA + dAA + A^3\right)\right] + d(-)$$

$$= \frac{-1}{2\pi}\cdot\underbrace{\frac{i}{24\pi^2}\mathrm{tr}\left[\epsilon\, d\left(AdA + \frac{1}{2}A^3\right)\right]} + d(-)$$

This is nothing but the 4d chiral anomaly up to the factor of $\pm 1/2\pi$.

Thus, $d=6$ axial anomaly seems to be related to $d=4$ chiral anomaly.

Why? (We'll come back to this in a moment.)

A practical use

Consider a 4-dimensional theory with

R-handed fermion in a representation V_R of G

L-handed fermion in a representation V_L of G.

The condition of G-anomaly to be absent is

$$\mathrm{ch}_{V_R,6}[A] - \mathrm{ch}_{V_L,6}[A] = 0.$$

Exercise Show that the standard model has no gauge anomaly. In this case,

$$G = SU(3) \times SU(2) \times U(1)$$

$$V_R = \left[(\mathbb{1}, \mathbb{1}, -1) \oplus (\mathbb{3}, \mathbb{1}, \tfrac{2}{3}) \oplus (\mathbb{3}, \mathbb{1}, -\tfrac{1}{3}) \right]^{\oplus 3}$$

$$V_L = \left[(\mathbb{1}, \mathbb{2}, -\tfrac{1}{2}) \oplus (\mathbb{3}, \mathbb{2}, \tfrac{1}{6}) \right]^{\oplus 3}$$

d=2n+2 axial anomaly vs d=2n chiral anomaly

$ch_{2n+2,V}[A]$ ... $\frac{1}{2}$ density of axial anomaly in $d=2n+2$ Dirac fermion in rep. V of G

descent $\Big\} \Downarrow$

$\omega_{2n,V}[\epsilon, A]$ ... $\frac{-1}{2\pi}$ density of chiral anomaly in $d=2n$ R-handed fermion in rep. V of G.

Why?

→ ① Anomaly inflow [Callan-Harvey 1985]

② Index theory [Alvarez-Gaume - Ginsparg 1984 after Atiyah-Singer 1984]

γ^μ : Gamma matrices in $d=2n$ ⇒

$$\left.\begin{aligned}
\Gamma^\mu &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \gamma^\mu \\
\Gamma^{2n+1} &= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \otimes \mathbb{1}_{2^n} \\
\Gamma^{2n+2} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \mathbb{1}_{2^n}
\end{aligned}\right\} \text{Gamma matrices in } D=2n+2$$

$$\Gamma_{D+1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \gamma_{d+1}$$

$$\not{D}^{(D)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \not{D}^{(d)} + 2i \begin{pmatrix} 0 & \partial_z \\ \partial_{\bar{z}} & 0 \end{pmatrix} \otimes \mathbb{1}_{2^n} \quad ; \; z = x^{2n+1} + i\, x^{2n+2}$$

$D=2n+2$ Dirac fermion with complex mass

$$\mathcal{L} = -i\bar{\Psi}_R \not{\partial}^{(D)} \Psi_R - i\bar{\Psi}_L \not{\partial}^{(D)} \Psi_L + m\bar{\Psi}_L \Psi_R + m^* \bar{\Psi}_R \Psi_L$$

Let us consider a position dependent mass

$$m = 2M^2 (x^{2n+1} - i x^{2n+2}) = 2M^2 \bar{z}$$

For $\Psi_R = \begin{pmatrix} a_R \\ a_L \end{pmatrix}$, $\Psi_L = \begin{pmatrix} b_L \\ b_R \end{pmatrix}$ $\quad$ a_R, b_R : R in $d=2n$
a_L, b_L : L in $d=2n$

Dirac equation:

$$\begin{pmatrix} -i\not{\partial}^{(d)} a_R \\ +i\not{\partial}^{(d)} a_L \end{pmatrix} + 2\left[\begin{pmatrix} 0 & \partial_z \\ \partial_{\bar{z}} & 0 \end{pmatrix}\begin{pmatrix} a_R \\ a_L \end{pmatrix} + M^2 z \begin{pmatrix} b_L \\ b_R \end{pmatrix} \right] = 0$$

$$\begin{pmatrix} -i\not{\partial}^{(d)} b_L \\ +i\not{\partial}^{(d)} b_R \end{pmatrix} + 2\left[\begin{pmatrix} 0 & \partial_z \\ \partial_{\bar{z}} & 0 \end{pmatrix}\begin{pmatrix} b_L \\ b_R \end{pmatrix} + M^2 \bar{z} \begin{pmatrix} a_R \\ a_L \end{pmatrix} \right] = 0$$

$$[\quad] = 0 \iff a_R = b_R = e^{-M^2|z|^2} \psi_R(x^1, \cdots, x^d)$$

$$a_L = b_L = 0$$

Get a single R-handed massless fermion supported at $z=0$.

Now consider $D = 2n+2$ Dirac fermion in a rep. V of G

with $m = 2M^2 \bar{z}$.

- There is a R-handed fermion in rep V supp at $z=0$

$\rightsquigarrow$ Effective action $W^{R}_{2n,V}[A|_0]$.

- The phase of mass $\arg(m) = \arg(\bar{z}) = -\arg(z)$

$\overset{\text{axial anomaly}}{\rightsquigarrow}$ Theta term $S_{\Delta\theta}[A]$ with $\Delta\theta = -\arg(z)$

$$-S_{\Delta\theta}[A] = i \int_{\mathbb{R}^d \times \mathbb{C}} \Delta\theta \, \underbrace{ch_{2n+2,V}[A]}_{d\omega_{2n+1,V}[A]}$$

$$= i \int_{\mathbb{R}^d \times \mathbb{C}} d\arg(z)\, \omega_{2n+1,V}[A]$$

$\mathbb{C} \rightarrow \lim_{\epsilon \to 0} \mathbb{C} \setminus D^2_\epsilon$ to be precise

Its gauge variation is

$$-\delta_\epsilon S_{\Delta\theta}[A] = \lim_{\epsilon \to 0} i \int_{\mathbb{R}^d \times (\mathbb{C} \setminus D^2_\epsilon)} d\arg(z)\, \underbrace{\delta_\epsilon \omega_{2n+1,V}[A]}_{d\omega_{2n,V}[\epsilon, A]}$$

$$= \lim_{\epsilon \to 0} i \int_{\mathbb{R}^2 \times \partial D^2_\epsilon} d\arg(z)\, \omega_{2n,V}[\epsilon, A]$$

$$= 2\pi i \int_{\mathbb{R}^d} \omega_{2n,V}[\epsilon, A]\Big|_0$$

<u>The $D=2n+2$ system (Dirac fermion) has no gauge anomaly:</u>

$$0 = \delta_\epsilon\left(-W^R_{2n,V}[A|_0] - S_{\Delta 0}[A]\right)$$

$$= -\delta_\epsilon W^R_{2n,V}[A|_0] + 2\pi i \int_{\mathbb{R}^d} \omega_{2n,V}[\epsilon, A]\Big|_0$$

$\therefore$ chiral anomaly in $d=2n$ R-handed fermion in rep. V of G indeed has density $-2\pi\, \omega_{2n,V}[\epsilon, A]$ obtained via descent.