More about diagrams

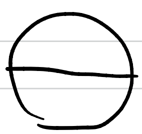
- Connected or not ✓

- Loops

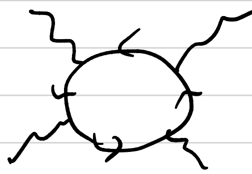
E = # external lines

P = # propagators

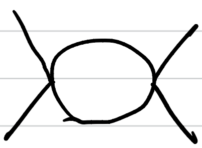
V = # vertices



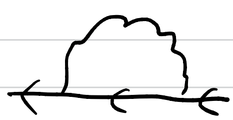
$E=0$
$P=3$
$V=2$



$E=4$
$P=8$
$V=4$



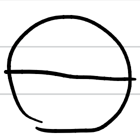
$E=4$
$P=6$
$V=2$



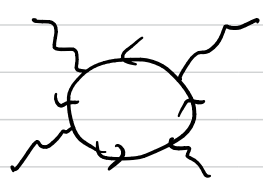
$E=2$
$P=4$
$V=2$

Then # internal lines $I = P-E$

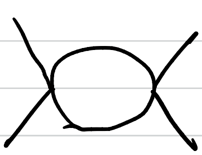
and # loops $L = I - V + 1 = P - E - V + 1$ if connected



$L = 3-2+1 = 2$



$L = 4-4+1 = 1$



$L = 2-2+1 = 1$

$L = 2-2+1 = 1$

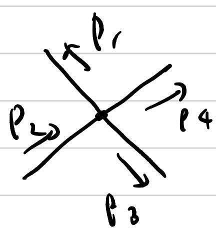
$$\int \prod_{v\in V} d^4y_v \int \prod_{e\in E} d^4p_e\, e^{-ip_e(x_e-y_{v(e)})} \int \prod_{i\in I} d^4p_i\, e^{-ip_i(y_{t(i)}-y_{s(i)})} F(p)$$

$$\int d^4y_v\, e^{i\sum_{\ell\in v}\varepsilon_\ell p_\ell y_v} = (2\pi)^4\delta^{(4)}\Big(\sum_{\ell\in v}\varepsilon_\ell p_\ell\Big)$$

Sum over lines connected to v

$$\varepsilon_\ell = \begin{cases} +1 & \text{if } \ell \text{ goes out of } v \\ -1 & \text{if } \ell \text{ comes in to } v \end{cases}$$

e.g.



$$\sim (2\pi)^4\delta^{(4)}(p_1-p_2+p_3+p_4)$$

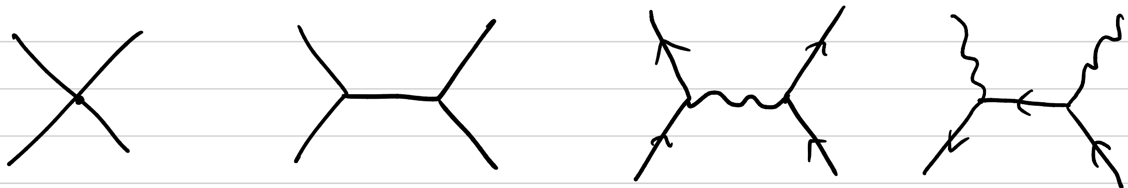
$$= \int \prod_{e\in E} d^4p_e\, e^{-ip_e x_e} \int \prod_{i\in I} d^4p_i\, \underbrace{\prod_{v\in V}(2\pi)^4\delta^{(4)}\Big(\sum_{\ell\in v}\varepsilon_\ell p_\ell\Big)}_{(2\pi)^4\delta^{(4)}\left(\sum_{e\in E} p_e\right)\prod_{v=1}^{V-1}(2\pi)^4\delta^{(4)}\left(\sum_{\ell\in v}\varepsilon_\ell p_\ell\right)} F(p)$$

Overall momentum conservation

$\therefore$ net # of momentum integrals

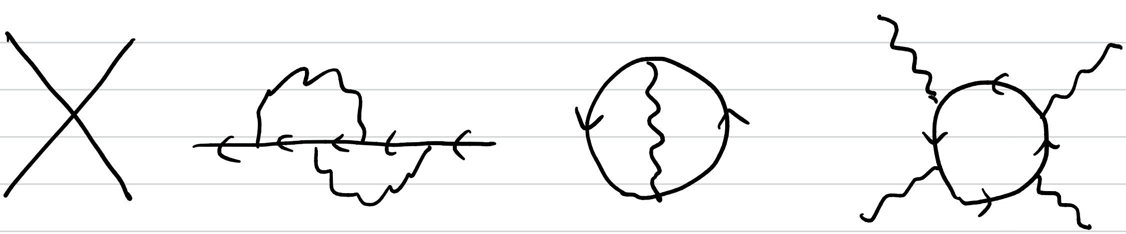
$$= I-(V-1) = L.$$

- A diagram without loop ($L=0$) is called
  a <u>tree diagram</u> :

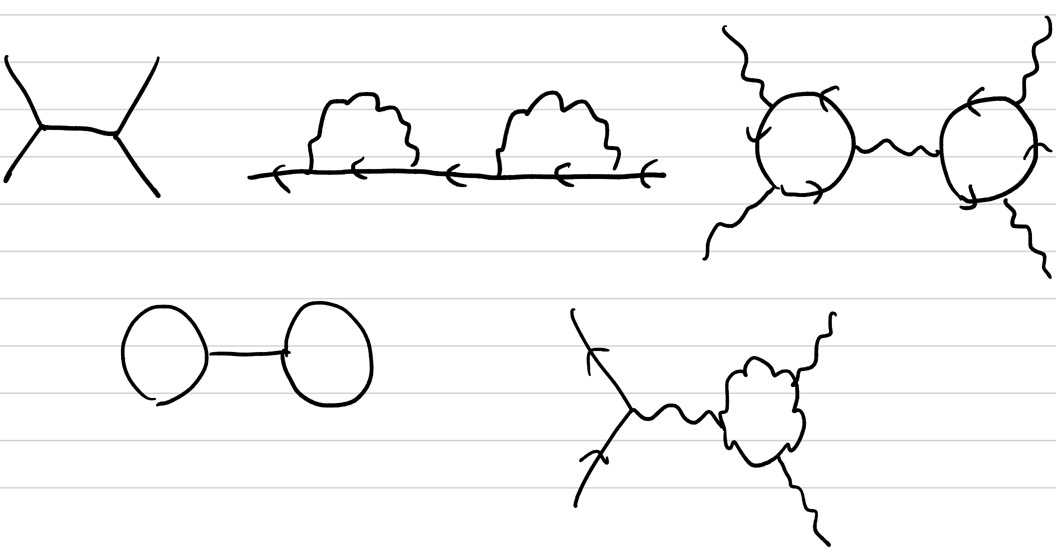


  ···· no momentum integral

- A connected diagram ($\neq$ a propagator) is
  <u>one particle irreducible</u> (<u>1PI</u>) if it is still connected
  when any internal line is cut.

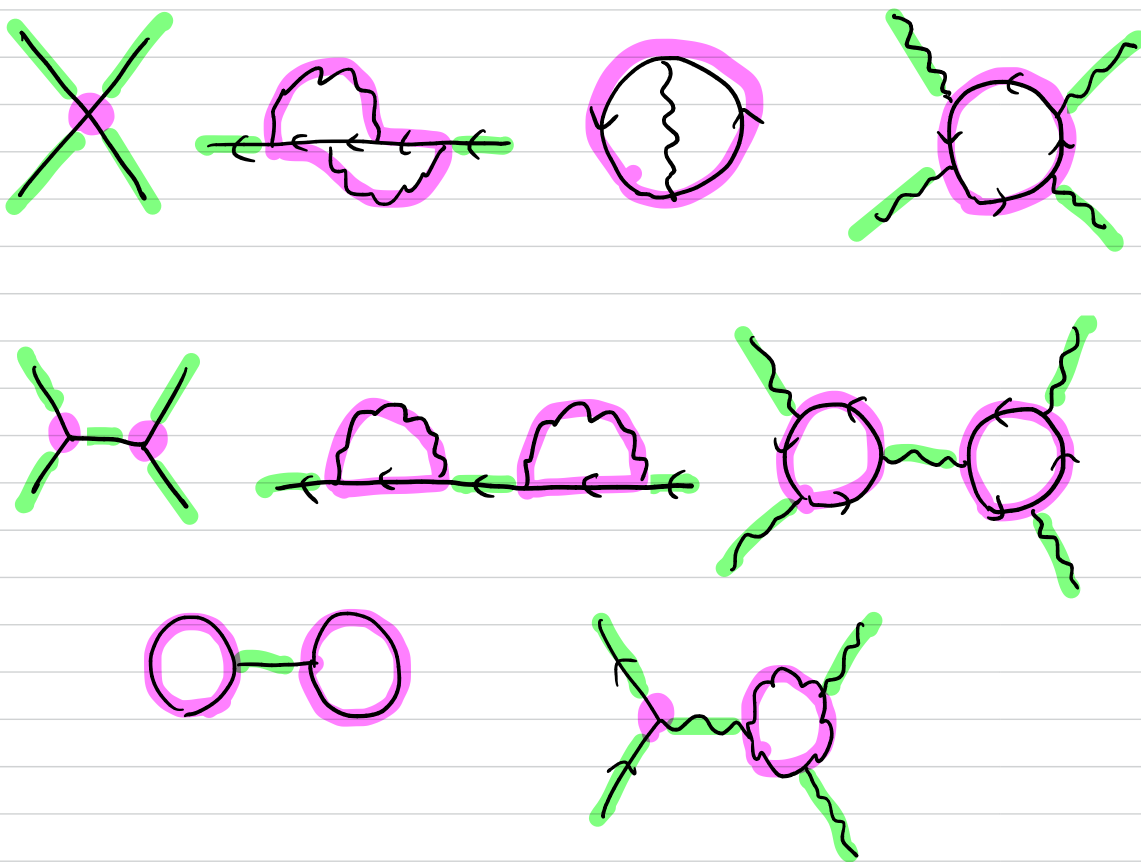


1PI



not 1PI

- Any diagram is uniquely decomposed into 1PI blocks and separating lines



Z_{pert} & $\langle f \rangle_{pert}$ is the sum of tree diagrams with 1PI blocks as the vertices.

1PI effective action

Consider a theory of variables $\phi = (\phi_1, \cdots, \phi_N)$

measure $d\phi$ and action $S_E(\phi)$ (omit "E" below).

$$e^{-W(J)} = \int d\phi \, e^{-S(\phi) + J\cdot\phi}$$

Decompose $S(\phi) - J\cdot\phi = \underbrace{\frac{1}{2}\sum_{i,j} \phi_i A^{ij} \phi_j}_{\text{free part}} + \underbrace{\text{else}}_{\text{interaction part}}$

and evaluate $W(J)$ perturbatively.

※ Everything below is perturbative but we omit "pert".

e.g. $W(J) = W_{\text{pert}}(J)$ is the sum of connected diagram.

$$-\frac{\partial}{\partial J_i} W(J) = \frac{\int d\phi \, e^{-S(\phi)+J\cdot\phi} \phi_i}{\int d\phi \, e^{-S(\phi)+J\cdot\phi}} =: \langle \phi_i \rangle_J$$

Solve $\langle \phi_i \rangle_J \overset{!}{=} \phi_i \quad i=1,\cdots,N$ for J, write the solution

Unique in perturbation theory

as $J = J(\phi)$ and put

$$\Gamma(\phi) := W(J(\phi)) + J(\phi)\cdot\phi$$

$\cdots\cdots$ Legendre transform of $W(J)$.

$$\frac{\partial \Gamma(\phi)}{\partial \phi_i} = \cancel{\frac{\partial J^j(\phi)}{\partial \phi_i} \cdot \underbrace{\frac{\partial W}{\partial J^j}(J(\phi))}_{\phi_j}} + \cancel{\frac{\partial J^j(\phi)}{\partial \phi_i} \cdot \phi_j} + J^i(\phi) = J^i(\phi),$$

$$i = 1, \cdots, N.$$

Thus,

$$\phi_i^* := \langle \phi_i \rangle_{J=0} \quad \Rightarrow \quad J(\phi^*) = 0 \quad \therefore \quad \frac{\partial \Gamma}{\partial \phi_i}(\phi^*) = 0.$$

VEV of ϕ at $J = 0$ is a critical point of $\Gamma(\phi)$.

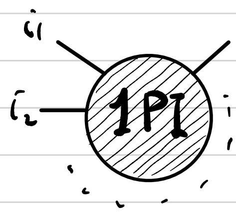
Properties of $\Gamma(\phi)$

① It is a generating series of 1PI vertices

$$\Gamma(\phi) = \frac{1}{2} \log \det(A/2\pi) + \frac{1}{2} \sum_{i,j} \phi_i A^{ij} \phi_j$$

$$- \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{i_1 \cdots i_n} \lambda_{1PI}^{i_1 \cdots i_n} \phi_{i_1} \cdots \phi_{i_n}$$

where $\lambda_{1PI}^{i_1 \cdots i_n}$ is the 1PI vertex defined by

$$\langle \phi_{i_1} \cdots \phi_{i_n} \rangle_{1PI} =$$ 

$$= \sum_{j_1 \cdots j_n} \overbrace{\phi_{i_1} \phi_{j_1}} \cdots \overbrace{\phi_{i_n} \phi_{j_n}} \, \lambda_{1PI}^{j_1 \cdots j_n}$$

For this reason, $\Gamma(\phi)$ is called 1PI effective action.

② $\Gamma(\varphi) = \frac{1}{2}\log\det(A/2\pi)$ — The sum of 1PI vacuum diagrams of $\mathcal{T}(\varphi)$, the theory with background ϕ:

$$\begin{cases} \text{variables} & \xi = (\xi_1, \dots, \xi_N) \\ \text{measure} & d_\phi \xi = d(\phi+\xi) \\ \text{action} & S_\phi(\xi) = S(\phi+\xi) \end{cases}$$

$$\int d_\phi \xi \; e^{-S_\phi(\xi)} = \sqrt{\frac{(2\pi)^N}{\det A}}\; e^{\text{connected vacuum diagrams}}$$

$$= e^{-\Gamma(\phi) + \text{non-1PI conn. vac. diagrams}}$$

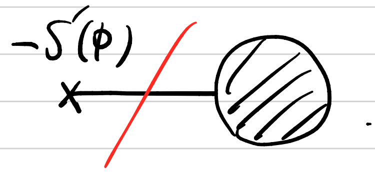
Here we take

$$S_\phi(\xi) = \underbrace{\frac{1}{2}\xi_i A^{ij}\xi_j}_{\text{free part}} + \underbrace{\text{else}}_{\text{interaction part}}$$

③ This holds for any decomposition of $S_\phi(\xi)$ into free + interaction. In particular, for the expansion in powers of ξ, we can take the ξ-quadratic part $\frac{1}{2}\sum_{ij}\xi_i\xi_j\partial_i\partial_j S(\varphi)$ as the free part.

$$S_\phi(\xi) = \underbrace{S(\phi)} + \underbrace{S'(\phi)\xi} + \underbrace{\frac{1}{2}S''(\phi)\xi^2}_{\text{free part}} + \underbrace{\frac{1}{3!}S'''(\phi)\xi^3 + \cdots}_{\text{interaction}}$$

- $S(\phi)$ is outside the ξ integral.
- Any diagram involving the vertex $-S'(\phi)\cdot\xi$ is not 1PI:



Thus, we can take only the cubic or higher powers in ξ as the interaction part to produce vertices.

With this understanding,

$$e^{-\Gamma(\phi)} = e^{-S(\phi)} \cdot \sqrt{\frac{(2\pi)^n}{\det S''(\phi)}} \cdot \exp\left(1PI \text{ vacuum diagrams}\right).$$

That is,

$$\Gamma(\phi) = S(\phi) + \underbrace{\frac{1}{2}\log\det\left(\frac{S''(\phi)}{2\pi}\right)}_{\frac{1}{2}\operatorname{tr}\log\left(\frac{S''(\phi)}{2\pi}\right)} - 1PI \text{ vacuum diagrams}.$$

Consequence of 2 :

recover $\hbar$ $\quad \int d_\phi \xi \, e^{-\frac{1}{\hbar} S_\phi(\xi)} = e^{-\frac{1}{\hbar}\Gamma(\phi,\hbar) + \text{others}}$

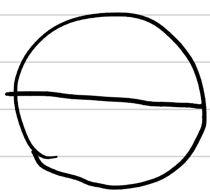
$\rightsquigarrow$ propagator $\propto \hbar$, vertex $\propto \hbar^{-1}$

A 1PI vacuum diagram with # propagator $= P$

vertices $= V$

$\propto \hbar^{P-V} = \hbar^{L-1}$

where $L = P - V + 1$ is # loops

e.g.  $P = 3$, $V = 2$ $\quad L = 3 - 2 + 1 = 2$

Thus, $\Gamma(\phi,\hbar) = \sum_{L=0}^{\infty} \hbar^L \Gamma_L(\phi)$

$\Rightarrow$ $-\Gamma_L(\phi) =$ the sum of 1PI vacuum diagrams with # loops $= L$

($\log\det(A/2\pi\hbar)$ is included in $L = 1$)

$\therefore$ $\hbar$-expansion = loop expansion.

Parameter dependence

The action may depend on parameters $g = (g_I)$ such as coupling constants & external fields, and the dependence can be made explicit as $S(\phi, g)$, and similarly for $W(J, g)$ & $\Gamma(\phi, g)$. I.e.

$$e^{-W(J,g)} = \int d\phi \; e^{-S(\phi, g) + J\cdot\phi}$$

$$\langle \phi_i \rangle_{J,g} = -\frac{\partial W}{\partial J^i}(J, \phi) \stackrel{!}{=} \phi_i \quad \rightsquigarrow \quad J = J_g(\phi)$$

$$\Gamma(\phi, g) := W(J_g(\phi), g) + J_g(\phi)\cdot\phi$$

Then $\frac{\partial \Gamma}{\partial \phi_i}(\phi, g) = J_g^i(\phi)$, ①, ②, ③, ... remain to hold.

Also

$$\frac{\partial \Gamma}{\partial g_I}(\phi, g) = \cancel{\frac{\partial J_g^j(\phi)}{\partial g_I}\frac{\partial W}{\partial J^j}}(J_\phi(\phi), g) + \frac{\partial W}{\partial g_I}(J_g(\phi), g) + \cancel{\frac{\partial J_g^j(\phi)}{\partial g_I}\phi_j}$$

$$= \left\langle \frac{\partial S}{\partial g_I}(\phi, g) \right\rangle_{J_g(\phi),\, g}$$

Ward identity for 1PI effective action

Suppose $\phi \to \phi + \delta\phi$ is a symmetry, $\delta(d\phi\, e^{-S(\phi)}) = 0$.

Then, we have Ward identity

$$0 = \int \delta(d\phi\, e^{-S(\phi)+J\cdot\phi})$$

$$= \int d\phi\, e^{-S(\phi)+J\cdot\phi}\, J\cdot\delta\phi = e^{-W(J)}\, J\cdot\langle\delta\phi\rangle_J$$

Set $J = J(\phi)$ and use $\frac{\partial\Gamma}{\partial\phi_i}(\phi) = J^i(\phi)$.

We obtain

$$\boxed{\sum_i \langle\delta\phi_i\rangle_{J(\phi)} \frac{\partial\Gamma}{\partial\phi_i}(\phi) = 0.}$$

Slavnov-Taylor identity

i.e. $\Gamma(\phi)$ is invariant under $\phi \to \phi + \langle\delta\phi\rangle_{J(\phi)}$.

For an at most linear symmetry: $\delta\phi_i = M_{ij}\phi_j + C_i$,

$$\langle\delta\phi_i\rangle_{J(\phi)} = M_{ij}\langle\phi_j\rangle_{J(\phi)} + C_i = M_{ij}\phi_j + C_i = \delta\phi_i.$$

So $\Gamma(\phi)$ is invariant under the original symmetry.

A variant: Suppose $S(\phi, g)$ is invariant under

$$\phi \to \phi + \delta\phi \quad \text{and} \quad g \to g + \delta g.$$

$$0 = \int \underset{\text{for } \phi \text{ only}}{\delta} \left(d\phi \, e^{-S(\phi,g) + J\cdot\phi} \right)$$

$$= \int d\phi \, e^{-S(\phi,g)+J\cdot\phi} \Big(\underbrace{-\delta\phi_i \frac{\partial S}{\partial \phi_i}(\phi, g)}_{= \delta g_I \frac{\partial S}{\partial g_I}(\phi, g)} + J \cdot \delta\phi \Big)$$

$$= e^{-W(J,g)} \left(\delta g \cdot \left\langle \frac{\partial S}{\partial g}(\phi, g) \right\rangle_{J,g} + J \cdot \langle \delta\phi \rangle_{J,g} \right)$$

Set $J = J_g(\phi)$ & use $\frac{\partial \Gamma}{\partial \phi_i}(\phi, g) = J_g^i(\phi)$,

$$\frac{\partial \Gamma}{\partial g_I}(\phi, g) = \left\langle \frac{\partial S}{\partial g_I}(\phi, g) \right\rangle_{J_g(\phi), g}$$

$$\boxed{\sum_i \langle \delta\phi_i \rangle_{J_g(\phi), g} \frac{\partial \Gamma}{\partial \phi_i}(\phi, g) + \sum_I \delta g_I \frac{\partial \Gamma}{\partial g_I}(\phi, g) = 0}$$

i.e. $\Gamma(\phi, g)$ is invariant under

$$\phi \to \phi + \langle \delta\phi \rangle_{J_g(\phi), g}, \quad g \to g + \delta g.$$

Ward identities in gauge theory

Consider a gauge theory with variable ϕ,

action $S_E(\phi)$, gauge symmetry $\phi \to \phi + \delta_\epsilon \phi$.

Choose a gauge fixing function $\chi(\phi)$

$\rightsquigarrow$ gauge fixed system: variable $X = (\overset{b/f}{\phi}, \overset{f}{c}, \overset{f}{\bar{c}}, \overset{b}{B})$

action $\tilde{S}_E(X) = S_E(\phi) + \frac{\xi}{2} B^2 - iB \cdot \chi(\phi) + \bar{c}\, \delta_c \chi(\phi)$

BRST symmetry (fermionic & nilpotent $\delta_B^2 = 0$):

$$\delta_B \phi = \delta_c \phi, \quad \delta_B c = -\tfrac{1}{2}[c, c], \quad \delta_B \bar{c} = iB, \quad \delta_B B = 0$$

Introduce an external field $K = (\overset{f/b}{K^\phi}, \overset{b}{K^c}, \overset{b}{K^{\bar{c}}}, \cancel{K^B})$

coupled to $\delta_B X = (\overset{f/b}{\delta_B \phi}, \overset{b}{\delta_B c}, \overset{b}{\delta_B \bar{c}}, 0)$, and consider

$$S(X, K) = \tilde{S}_E(X) - K \cdot \delta_B X$$

Define $W(J, K)$ & $\Gamma(\phi, K)$ as before.

I.e. $$e^{-W(J,K)} := \int dX\, e^{-\tilde{S}(X) + J \cdot X + K \cdot \delta_B X}$$

$$J = (\overset{b/f}{J^\phi}, \overset{f}{J^c}, \overset{f}{J^{\bar{c}}}, \overset{b}{J^B})$$

$$\Gamma(X,K) := W(J_K(X),K) + J_K(X)\cdot X$$

where $J = J_K(X)$ soln of $\dfrac{\partial W}{\partial J}(J,K) \overset{!}{=} -X$

$$\frac{\partial \Gamma}{\partial X_i}(X,K) = \cancel{\frac{\partial J_K(X)}{\partial x_i}\cdot \underbrace{\frac{\partial W}{\partial J}(J_K(X),K)}_{-X}} + \cancel{\frac{\partial J_K(X)}{\partial X_i}\cdot X} + \epsilon_i J_K^i(X)$$

$$= \epsilon_i J_K^i(X) \quad ; \quad \epsilon_i = \begin{cases} 1 & X_i \text{ bosonic} \\ -1 & X_i \text{ fermionic} \end{cases}$$

$$\frac{\partial \Gamma}{\partial K^i}(X,K) = \left\langle \frac{\partial S}{\partial K^i}(X,K) \right\rangle_{J_K(X),K} = -\left\langle \delta_B X_i \right\rangle_{J_K(X),K}$$

Ward identity for BRST symmetry:

$$0 = \int \delta_B \left(dX \, e^{-\tilde{S}(X) + J\cdot X + K\cdot \delta_B X} \right)$$

$$= \int dX \, e^{-\tilde{S}(X) + J\cdot X + K\cdot \delta_B X} \sum_i \epsilon_i J^i \delta_B X_i$$

$$= e^{-W(J,K)} \sum_i \epsilon_i J^i \langle \delta_B X_i \rangle_{J,K}$$

Set $J = J_K(X)$ & use above :

$$\boxed{\sum_i \frac{\partial \Gamma}{\partial X_i}(X,K) \cdot \frac{\partial \Gamma}{\partial K^i}(X,K) = 0}$$

<u>Zinn-Justin equation</u>

At $K=0$, it reduces to BRST-Ward identity :

$$\sum_i \langle \delta_B X_i \rangle_{J(X)} \cdot \frac{\partial \Gamma}{\partial X_i}(X) = 0 .$$

Ghost number symmetry

The extended system $S(X,K) = \tilde{S}_E(X) - K \cdot \delta_B X$

has ghost number symmetry :

	ϕ	c	$\bar{c}$	B	K^ϕ	K^c	$K^{\bar{c}}$
ghost #	0	1	-1	0	-1	-2	0

I.e. it is invariant under

$$X_i \to e^{Q_{X_i}\alpha} X_i , \quad K^i \longrightarrow e^{Q_{K^i}\alpha} K^i$$

$\rightsquigarrow$ $\Gamma(X,K)$ also has the same symmetry.

B & $K^{\bar{c}}$ dependence

$K^{\bar{c}} \cdot iB$

$$0 = \int dX \frac{\partial}{\partial B} \left(e^{-\tilde{S}_E(X) + J\cdot X + K\cdot \delta_B X} \right)$$

$$= \int dX \, e^{-\tilde{S}_E(X) + J\cdot X + K\cdot \delta_B X} \times$$

$$\left(-\xi B + i\chi(\phi) + J^B + iK^{\bar{c}} \right)$$

Set $J = J_K(X)$. Also, assume $\chi(\phi)$ is at most linear in ϕ.

Then

$$-\xi B + i\chi(\phi) + \frac{\partial \Gamma}{\partial B} + iK^{\bar{c}} = 0$$

$$\therefore \frac{\partial \Gamma}{\partial B} = \xi B - i\chi(\phi) - iK^{\bar{c}}.$$

Also, $\dfrac{\partial \Gamma}{\partial K^{\bar{c}}} = -\langle iB \rangle_{J_K(X), K} = -iB.$

$$\therefore \Gamma(X, K) = \frac{1}{2}\xi B^2 - iB\cdot\chi(\phi) - iK^{\bar{c}}\cdot B$$

$$+ \; B \;\&\; K^{\bar{c}} \text{ independent terms.}$$

I.e. $\nexists$ quantum correction to B & $K^{\bar{c}}$ dependence.

These Ward identities will be used in the proof of renormalizability of gauge theory.