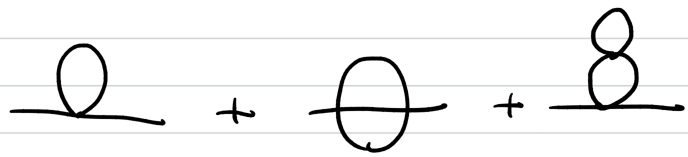
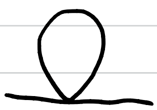
Regularization and Renormalization

Divergences

$$\mathcal{L}_E = \frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 \qquad \text{in } d=4.$$

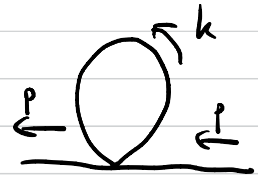
$$\langle \phi(x_1)\phi(x_2)\rangle_{1PI} = $$

$$+ \cdots$$

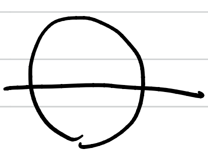


$$= -\frac{\lambda}{2}\int d^4y\, \phi(x_1)\phi(y)\,\phi(y)\phi(y)\,\phi(y)\phi(x_2)$$

$$(2\pi)^4\delta^4(p_1-p_2) \qquad \int\frac{d^4p_1}{(2\pi)^4}\frac{e^{-ip_1(x_1-y)}}{p_1^2+m^2} \int\frac{d^4k}{(2\pi)^4}\frac{e^{-ik(y-y)}}{k^2+m^2} \int\frac{d^4p_2}{(2\pi)^4}\frac{e^{-ip_2(y-x_2)}}{p_2^2+m^2}$$

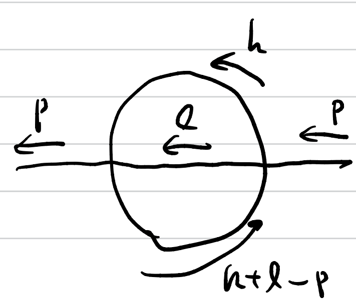
$$= \int\frac{d^4p}{(2\pi)^4}\frac{e^{-ipx_1}}{p^2+m^2}\left(-\frac{\lambda}{2}\underbrace{\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2+m^2}}_{\text{quadratically divergent}}\right)\frac{e^{ipx_2}}{p^2+m^2}$$



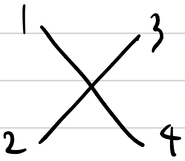


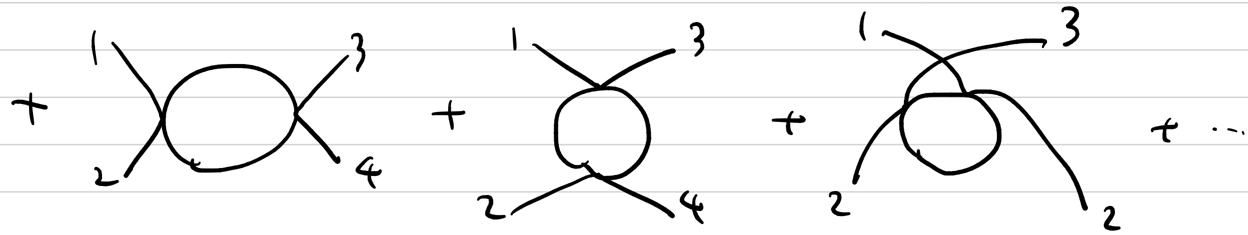
$$= \frac{\lambda^2}{6}\int d^4y_1\, d^4y_2\, \phi(x_1)\phi(y_1)\,\left[\phi(y_1)\phi(y_2)\right]^3\,\phi(y_2)\phi(x_2)$$

$$= \int\frac{d^4p}{(2\pi)^4}\frac{e^{-ipx_1}}{p^2+m^2}\left(\frac{\lambda^2}{6}\underbrace{\int\frac{d^4k}{(2\pi)^4}\frac{d^4l}{(2\pi)^4}\frac{1}{k^2+m^2}\,\frac{1}{l^2+m^2}\,\frac{1}{(k+l-p)^2+m^2}}_{\text{quadratically divergent}}\right)\frac{e^{ipx_2}}{p^2+m^2}$$



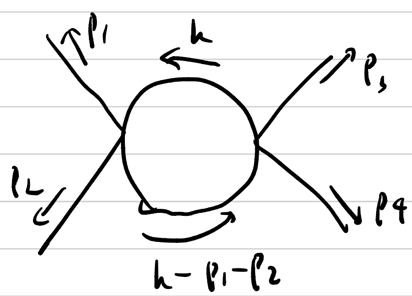
$$\langle \phi(x_1)\,\phi(x_2)\,\phi(x_3)\,\phi(x_4)\rangle_{1PI} =$$





$$= \int \prod_{a=1}^{4} \frac{d^4 p_a}{(2\pi)^4} \frac{e^{-i p_a x_a}}{p_a^2+m^2} (2\pi)^4 \delta^4(p_1+p_2+p_3+p_4) \times \Big\{ -\lambda$$

$$+ \frac{\lambda^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2+m^2} \frac{1}{(k-p_1-p_2)^2+m^2} + (2\leftrightarrow 3) + (2 \leftrightarrow 4) + \cdots \Big\}$$



logarithmically divergent

The integral over the loop momenta k's can be divergent at $|k| \to \infty$

..... ultra-violet (= short distance) divergence

superficial degree of divergence D

$:=$ power of momenta k of the integral

$=$ (power in numerator) $-$ (power in denominator)

e.g. from d^4k, vertex ...   from propagator

$E = \#$ external lines, $I = \#$ internal lines, $V = \#$ vertices,

$L = \#$ loops $= I - V + 1 \underset{\text{Lecture 4}}{=}$ net $\#$ of momentum integrals

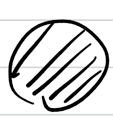
Theory of scalar ϕ in d-dimensions : $D = \underbrace{dL}_{d^dk} - \underbrace{2I}_{\frac{1}{k^2+m^2}}$

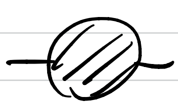
If $\mathcal{L}_{int} \propto \phi^4$: $2I + E = 4V$

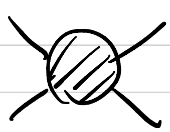
$$D = d(I - V + 1) - 2I$$

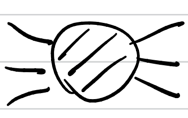
$$\overset{\phi^4}{=} d(-V+1) + (d-2)\frac{4V-E}{2} \overset{d=4}{=} 4 - E$$

4d ϕ^4 theory

$E = 0$  : $D = 4$ quartic divergence

$E = 2$  : $D = 2$ quadratic divergence

$E = 4$  : $D = 0$ logarithmic divergence

$E \geq 6$  : $D < 0$ (superficially) convergent

For $E=0,2,4$, the divergence occurs for any number V of vertices,

i.e. at all orders in perturbative expansion

ϕ^4 theory in other d : $D = d + (d-4)V - \frac{d-2}{2}E$

$\underline{d<4}$ $D<0$ for large enough V.

Only a finite number of Feynman diagrams are (superficially) divergent.

$\underline{d>4}$ For each E, $D>0$ for large enough V.

Any correlator is (superficially) divergent at sufficiently high orders in perturbative expansion.

How do we deal with such divergences?

— At least, we need a

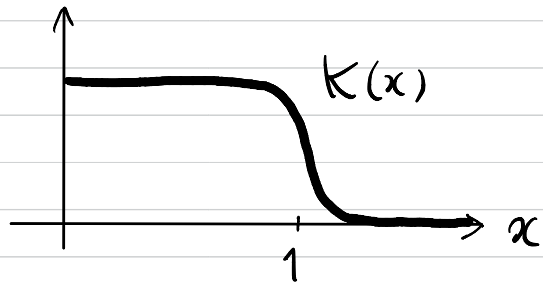
regularization:

a systematic change of the theory so that the loop integrals are all finite.

Regularizations

① Change of propagator $\dfrac{1}{p^2+m^2} \rightsquigarrow \dfrac{K(p^2/\Lambda^2)}{p^2+m^2}$

$$K(x) = \begin{cases} 1 & x \ll 1 \\ 0 & x \gg 1 \end{cases}$$



The propagator remains the same as the original at low $|p|$ compared to Λ, but is significantly modified at $|p| \gtrsim \Lambda$.

Λ: ultra-violet cut-off (UV cut-off)

e.g. $$\frac{1}{p^2+m^2} \longrightarrow \begin{cases} \dfrac{1}{p^2+m^2} & p^2 < \Lambda^2 \\ 0 & p^2 > 0 \end{cases}$$ sharp cut-off

e.g. $$\frac{1}{p^2+m^2} = \int_0^\infty d\alpha\, e^{-\alpha(p^2+m^2)}$$

$$\longrightarrow \int_{1/\Lambda^2}^\infty d\alpha\, e^{-\alpha(p^2+m^2)} = \frac{e^{-\frac{p^2+m^2}{\Lambda^2}}}{p^2+m^2}$$

$\longleftrightarrow$ change of Lagrangian :

$$\mathcal{L}_{E,\Lambda} = \frac{1}{2}\phi(-\partial^2+m^2)\underbrace{e^{\frac{-\partial^2+m^2}{\Lambda^2}}}_{}\phi + \frac{\lambda}{4!}\phi^4$$

$K(-\partial^2/\Lambda^2)^{-1}$ more generally

①′ Pauli-Villars regularization (⊂ ①)

$$\frac{1}{p^2+m^2} \to \frac{1}{p^2+m^2} - \frac{1}{p^2+\Lambda^2} = \frac{\Lambda^2-m^2}{(p^2+m^2)(p^2+\Lambda^2)}, \quad \text{or}$$

$$\frac{1}{p^2+m^2} \to \frac{1}{p^2+m^2} - \frac{\alpha_1}{p^2+\Lambda_1^2} - \frac{\alpha_2}{p^2+\Lambda_2^2} - \cdots = \frac{\text{Const}}{(p^2)^N + \text{lower}}$$

One can choose $\Lambda_1, \alpha_1, \Lambda_2, \alpha_2, \cdots$, to make the power $2N$ of denominator as large as possible.

$\leftrightarrow$ introduce new field variables $\phi_1, \phi_2, \cdots$ (regulators) and consider the system with Lagrangian

$$\mathcal{L}_{E,\text{reg}} = \frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + \sum_{i=1,2,\cdots} \frac{1}{2}(\partial\phi_i)^2 + \frac{\Lambda_i^2}{2}\phi_i^2 \quad \text{] free part}$$

$$+ \frac{\lambda}{4!}\left(\phi + \sum_{i=1,2,\cdots}\sqrt{-\alpha_i}\,\phi_i\right)^4 \quad \text{] interaction}$$

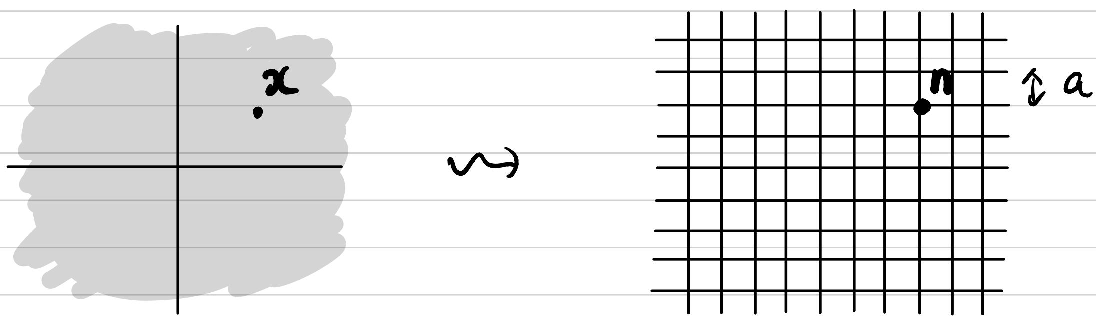
The internal propagators are only for $\Phi = \phi + \sum_i \sqrt{-\alpha_i}\,\phi_i$:

$$\overbrace{\Phi(x)\,\Phi(y)} = \overbrace{\phi(x)\,\phi(y)} + \sum_i(-\alpha_i)\,\overbrace{\phi_i(x)\,\phi_i(y)}$$

$$= \int\frac{d^dk}{(2\pi)^d}\, e^{-ik(x-y)}\left(\frac{1}{k^2+m^2} - \sum_i \frac{\alpha_i}{k^2+\Lambda_i^2}\right)$$

② Lattice

$$x \in \mathbb{R}^d \mapsto \phi(x) \quad \rightsquigarrow \quad n \in \mathbb{Z}^d \mapsto \phi_n$$



$$S_{E,reg} = \sum_n a^d \left(\frac{1}{2} \sum_\mu \left(\frac{\phi_{n+e_\mu} - \phi_n}{a} \right)^2 + \frac{m^2}{2} \phi_n^2 + \frac{\lambda}{4!} \phi_n^4 \right)$$

Advantage : momentum integral is over compact space

$$\phi_n = \int_0^{2\pi/a} \cdots \int_0^{2\pi/a} \frac{d^d p}{(2\pi)^d} e^{-ipna} \phi(p)$$

$$\phi_n \phi_{n'} = \int_0^{2\pi/a} \cdots \int_0^{2\pi/a} \frac{d^d p}{(2\pi)^d} \frac{e^{-ip(na-n'a)}}{\sum_\mu \left(\frac{e^{-ip_\mu a} - 1}{a} \right) \left(\frac{e^{ip_\mu a} - 1}{a} \right) + m^2}$$

③ Dimensional regularization

dimension $d \in \mathbb{Z}$ say 4 $\rightsquigarrow$ $d \in \mathbb{C}$

$$\int_{\mathbb{R}^4} \frac{d^4k}{(2\pi)^4} f(k^2) \rightsquigarrow \mu_{DR}^{4-d} \int_{\mathbb{R}^d} \frac{d^dk}{(2\pi)^d} f(k^2)$$

μ_{DR}: a parameter of mass dimension 1

$$= \mu_{DR}^{4-d} \frac{Vol(S^{d-1})}{(2\pi)^d} \int_0^\infty k^{d-1} k\, f(k^2) = \frac{1}{2}\int_0^\infty (k^2)^{\frac{d}{2}-1} dk^2 f(k^2)$$

$$\underbrace{\left(\int_{\mathbb{R}} \frac{dx}{2\pi} e^{-x^2}\right)^d}_{\left(\frac{1}{2\pi}\sqrt{\pi}\right)^d = \frac{1}{(4\pi)^{d/2}}} = \int_{\mathbb{R}^d} \frac{d^dx}{(2\pi)^d} e^{-\|x\|^2} = \frac{Vol(S^{d-1})}{(2\pi)^d} \underbrace{\int_0^\infty r^{d-1} dr\, e^{-r^2}}_{\underbrace{\frac{1}{2}\int_0^\infty (r^2)^{\frac{d}{2}-1} dr^2\, e^{-r^2}}_{\frac{1}{2}\Gamma\left(\frac{d}{2}\right)}}$$

$$\therefore \frac{Vol(S^{d-1})}{2(2\pi)^d} = \frac{1}{(4\pi)^{d/2}\Gamma(d/2)}$$

$$= \frac{\mu_{DR}^{4-d}}{(4\pi)^{d/2}\Gamma(d/2)} \int_0^\infty (k^2)^{\frac{d}{2}-1} dk^2 f(k^2)$$

This makes sense also for $d \in \mathbb{C}$

e.g. $I = \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2+m^2}$ & $V = \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2+m^2} \frac{1}{(k-p)^2+m^2}$

Via ① $\frac{1}{p^2+m^2} \sim \int_{1/\Lambda^2}^{\infty} d\alpha \, e^{-\alpha(p^2+m^2)}$ & ③ dim reg : $4 \to d = 4-\epsilon$

$$I_{①} = \frac{1}{(4\pi)^2}\left[\Lambda^2 - m^2\left(\log\left(\frac{\Lambda^2}{m^2}\right)+1-\gamma\right) + m^2 \, O\left(\frac{m^2}{\Lambda^2}\right)\right]$$

$\gamma := \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log n\right) = 0.57721\ldots$ Euler's constant

$$I_{③} = \frac{\mu_{DR}^{4-d}\, m^{d-2}}{(4\pi)^{d/2}}\Gamma\left(1-\frac{d}{2}\right)$$

… divergent for $d=4$, but for $d=4-\epsilon$:

$$= -\frac{m^2}{(4\pi)^2}\left[\frac{2}{\epsilon} + \log\left(\frac{4\pi\mu_{DR}^2}{m^2}\right) + 1 - \gamma + O(\epsilon)\right]$$

$$V_{①} = \frac{1}{(4\pi)^2}\left[\log\left(\frac{\Lambda^2}{2m^2}\right) - \gamma - 1 - \int_0^1 dx \log\left(1 + x(1-x)\frac{p^2}{m^2}\right) + O\left(\frac{m^2}{\Lambda^2}, \frac{p^2}{\Lambda^2}\right)\right]$$

$$V_{③} = \frac{\mu_{DR}^{4-d}\,\Gamma\left(2-\frac{d}{2}\right)}{(4\pi)^{d/2}} \int_0^1 dx \left(x(1-x)p^2 + m^2\right)^{\frac{d}{2}-2}$$

… divergent for $d=4$, but for $d=4-\epsilon$:

$$= \frac{1}{(4\pi)^2}\left[\frac{2}{\epsilon} + \log\left(\frac{4\pi\mu_{DR}^2}{m^2}\right) - \gamma - \int_0^1 dx \log\left(1 + x(1-x)\frac{p^2}{m^2}\right) + O(\epsilon)\right]$$

∵ Exercise.

Renormalization

After regularization, we let the couplings to depend on the cut-off (Λ in ①, a in ②, (ϵ, μ_{DR}) in ③)

so that the correlation function of properly normalized fields are finite, as we remove the cut-off

$(\Lambda \to \infty\ ;\ a \to 0\ ;\ \epsilon \to 0)$.

regularization

$$S_\Lambda = \left[\int d^4x \left(\frac{1}{2} (\partial \phi_0)^2 + \frac{m_0(\Lambda)^2}{2} \phi_0^2 + \frac{\lambda_0(\Lambda)}{4!} \phi_0^4 \right) \right]_\Lambda$$

Λ ← cutoff

$$\phi_0 = \sqrt{Z_0(\Lambda)}\, \phi$$

$$= \left[\int d^4x \left(\frac{1}{2} Z_0(\Lambda) (\partial \phi)^2 + \frac{m_0(\Lambda)^2}{2} Z_0(\Lambda) \phi^2 + \frac{\lambda_0(\Lambda)}{4!} Z_0(\Lambda)^2 \phi^4 \right) \right]_\Lambda$$

Choose $Z_0(\Lambda)$, $m_0(\Lambda)$, $\lambda_0(\Lambda)$ so that

$\langle \phi(x_1) \cdots \phi(x_s) \rangle$ are all finite as Λ is removed

We do this order by order in perturbation theory.

$$Z_0(\Lambda) = 1 + \lambda a_1(\Lambda) + \lambda^2 a_2(\Lambda) + \cdots$$

$$Z_0(\Lambda) m_0(\Lambda)^2 = m^2 + \lambda b_1(\Lambda) + \lambda^2 b_2(\Lambda) + \cdots$$

$$Z_0(\Lambda)^2 \lambda_0(\Lambda) = \lambda + \lambda^2 c_1(\Lambda) + \lambda^3 c_2(\Lambda) + \cdots$$

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2 + \cdots$$

$$\mathcal{L}_0 = \frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4$$

$$\left.\begin{aligned} \mathcal{L}_1 &= \frac{1}{2}\lambda a_1(\Lambda)(\partial\phi)^2 + \frac{1}{2}\lambda b_1(\Lambda)\phi^2 + \frac{\lambda^2}{4!}c_1(\Lambda)\phi^4 \\ \mathcal{L}_2 &= \frac{1}{2}\lambda^2 a_2(\Lambda)(\partial\phi)^2 + \frac{1}{2}\lambda^2 b_2(\Lambda)\phi^2 + \frac{\lambda^3}{4!}c_2(\Lambda)\phi^4 \\ &\vdots \end{aligned}\right\}$$

called the counter terms

Do perturbation theory with

$$\mathcal{L}_{free} = \frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 \; ; \; \mathcal{L}_{int} = \frac{\lambda}{4!}\phi^4 + \mathcal{L}_1 + \mathcal{L}_2 + \cdots$$

$\mathcal{L}_0 \leftrightarrow$ tree

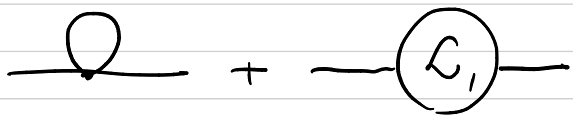
$\mathcal{L}_1 \leftrightarrow$ 1-loop

$\mathcal{L}_2 \leftrightarrow$ 2-loop

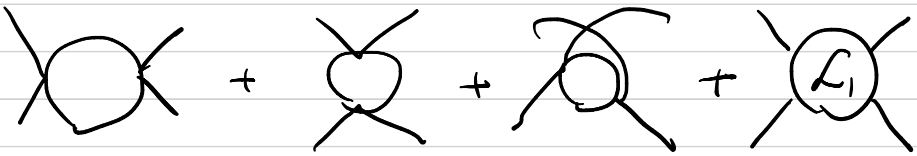
$\vdots$

Find $a_n(\Lambda)$, $b_n(\Lambda)$, $c_n(\Lambda)$ recursively so that the correlation functions of ϕ's are finite at each order.

@ 1-loop



$$= -\frac{\lambda}{2}\frac{1}{(4\pi)^2}\left[\Lambda^2 - m^2 \log\left(\frac{\Lambda^2}{m^2}\right) + \text{finite}\right] - \lambda a_1(\Lambda) p^2 - \lambda b_1(\Lambda)$$



$$\frac{\lambda}{2}\frac{1}{(4\pi)^2}\log\left(\frac{\Lambda^2}{2m^2}\right) \times 3 + \text{finite} - \lambda^2 c_1(\Lambda)$$

Can these be made finite?

Yes,

$$a_1(\Lambda) = \text{finite}$$

$$b_1(\Lambda) = -\frac{1}{2(4\pi)^2}\left(\Lambda^2 - m^2\log\left(\frac{\Lambda^2}{m^2}\right)\right) + \text{finite}$$

$$c_1(\Lambda) = \frac{3}{(4\pi)^2}\log\left(\frac{\Lambda^2}{2m^2}\right) + \text{finite}$$

will do the job!

<u>Claim</u> For each $n \geq 1$, it is possible to find $a_n(\Lambda)$, $b_n(\Lambda)$, $c_n(\Lambda)$ so that $L \leq n$ loop contributions to all the correlation functions of ϕ are finite.

Such a theory is said to be <u>renormalizable</u>.

$\phi_0 / m_0(\Lambda) / \lambda_0(\Lambda)$: <u>bare</u> field / mass / coupling

$\phi / m / \lambda$: <u>renormalized</u> field / mass / coupling

<u>Claim</u> A theory is renormalizable when the superficial degree of divergence D is ≥ 0 only for a finite number of correlation functions.

Eg. ϕ^4 theory

$d \leq 4$: Yes $\Rightarrow$ renormalizable

($d < 4$: No divergence at high enough loops $\Rightarrow$ <u>Superrenormalizable</u>)

$d > 4$: No $\Rightarrow$ not renormalizable.

Criterion: mass dimension of couplings

$$S = \int d^d x \, \mathcal{L} = \int d^d x \left(\frac{1}{2} (\partial \phi)^2 + \frac{m^2}{2} \phi^2 + \frac{\lambda}{4!} \phi^4 \right)$$

mass-dimension of $S = 0$ so that e^{-S} makes sense.

$$[S] = 0. \quad [d^d x] = -d \quad \therefore \quad [\mathcal{L}] = d.$$

$$[\partial_\mu] = 1 \quad \Rightarrow \quad [\phi] = \frac{d-2}{2}$$

$$[m^2] = 2$$

$$[\lambda] = d - 4\left(\frac{d-2}{2}\right) = 4-d.$$

The theory is

renormalizable $\Leftrightarrow$ $[\text{coupling}] \geq 0$

superrenormalizable $\Leftrightarrow$ $[\text{coupling}] > 0$

not renormalizable $\Leftrightarrow$ $[\text{coupling}] < 0.$

Recall: any diagram is a tree diagram with 1PI vertices.

So, to carry out renormalization, it is enough to make the 1PI effective action finite as a function of renormalized fields/masses/couplings as the cut-off is removed.

e.g. $\Gamma_0(\phi_0, m_0(\Lambda), \lambda_0(\Lambda); \Lambda) = \Gamma(\phi, m, \lambda; \Lambda)$

is finite as a function of ϕ, m, λ as $\Lambda \to \infty$.

<u>Now, an important point:</u>

Even when this is possible, there is an <u>ambiguity</u> in the choice of renormalized fields/masses/couplings.

e.g. $a_1(\Lambda) = \text{finite}$

$b_1(\Lambda) = \cdots + \text{finite}$

$c_1(\Lambda) = \cdots + \text{finite}$

To fix the ambiguity, impose <u>renormalization condition</u> :

For example

$$\Gamma(\phi) = \Gamma(\phi, m, \lambda ; \Lambda)$$

$$= \sum_{\substack{n=0 \\ \text{even}}}^{\infty} \frac{1}{n!} \int \frac{d^4 p_1}{(2\pi)^4} \cdots \frac{d^4 p_n}{(2\pi)^4} (2\pi)^4 \delta^{(4)}(p_1 + \cdots + p_n)$$
$$\Gamma(p_1, \cdots, p_n)\, \tilde{\phi}(p_1) \cdots \tilde{\phi}(p_n)$$

$$\begin{cases} \Gamma(-p, p)\big|_{p^2=-m^2} = 0 \\ \frac{d}{dp^2}\Gamma(-p,p)\big|_{p^2=-m^2} = 1 \\ \Gamma(p_1, \cdots, p_4)\big|_{p_i \cdot p_j = \begin{cases} -m^2 & i=j \\ m^2/3 & i \neq j \end{cases}} = \lambda \end{cases}$$

"On shell renormalization"

or

$$\begin{cases} \Gamma(-p, p)\big|_{p^2=0} = m^2 \\ \frac{d}{dp^2}\Gamma(-p,p)\big|_{p^2=0} = 1 \\ \Gamma(p_1, \cdots, p_4)\big|_{p_i \cdot p_j = 0} = \lambda \end{cases}$$

"intermediate renormalization"

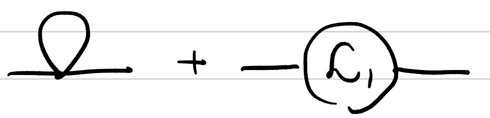
Or (μ = some mass scale)

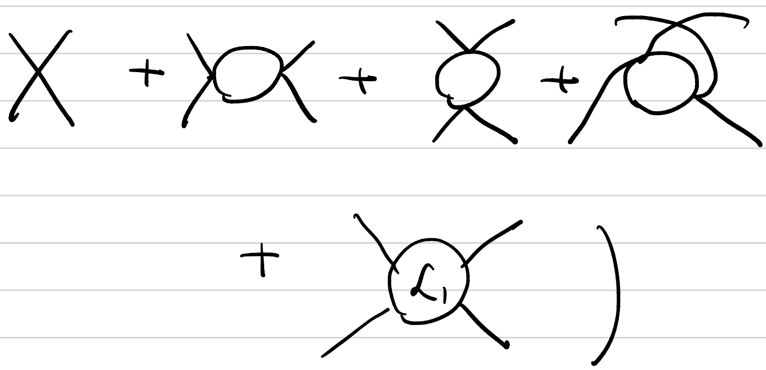
$$
\begin{cases}
\Gamma(-p,p)\Big|_{p^2=\mu^2} = \mu^2+m^2 \\
\dfrac{d}{dp^2}\Gamma(-p,p)\Big|_{p^2=\mu^2} = 1 \\
\Gamma(p_1,\cdots,p_4)\Big|_{p_i\cdot p_j = \begin{cases} \mu^2 & i=j \\ -\mu^2/3 & i\neq j \end{cases}} = \lambda
\end{cases}
$$

"another R.C."

When the renormalization condition is imposed, the ambiguity is completely fixed.

Let us confirm this at 1-loop

$\Gamma_1(-p,p) = p^2+m^2 -\Big($  $\Big)$

$\Gamma_1(p_1,\cdots,p_4) = -\Big($  $\Big)$

For ① momentum cut-off $\frac{1}{p^2+m^2} \to \frac{e^{-\frac{p^2+m^2}{\Lambda^2}}}{p^2+m^2}$

$$\Gamma_1(-p,p) = p^2+m^2 + \frac{\lambda m^2}{2(4\pi)^2}\left(\frac{\Lambda^2}{m^2} - \left(\log\left(\frac{\Lambda^2}{m^2}\right)+1-\gamma\right) + O\left(\frac{m^2}{\Lambda^2}\right)\right)$$

$$+ \lambda a_1(\Lambda)p^2 + \lambda b_1(\Lambda)$$

$$\Gamma_1(p_1,\cdots,p_4) = \lambda - \frac{\lambda^2}{2(4\pi)^2}\left[\log\left(\frac{\Lambda^2}{2m^2}\right) - \gamma - 1 - \int_0^1 dx \log\left(1+x(1-x)\frac{p_{12}^2}{m^2}\right) + O\left(\frac{p_{12}^2}{\Lambda^2}, \frac{m^2}{\Lambda^2}\right)\right]$$

$p_{12} = p_1+p_2$ etc

$$-(2\leftrightarrow 3) - (2\leftrightarrow 4)$$

$$+ \lambda^2 c_1(\Lambda)$$

Solution to the renormalization condition:

$$a_1(\Lambda) = 0, \quad b_1(\Lambda) = \frac{m^2}{2(4\pi)^2}\left[-\frac{\Lambda^2}{m^2} + \log\left(\frac{\Lambda^2}{m^2}\right) + 1 - \gamma + O\left(\frac{m^2}{\Lambda^2}\right)\right]$$

$$c_1(\Lambda) = \frac{3}{2(4\pi)^2}\left[\log\left(\frac{\Lambda^2}{2m^2}\right) - \gamma - 1 - K + O\left(\frac{m^2}{\Lambda^2}\right)\right]$$

$$K = \begin{cases} \int_0^1 dx \log\left(1-\frac{4}{3}x(1-x)\right) = 2\sqrt{2}\,\mathrm{Arg}(\sqrt{2}+i) - 2 & \underline{\text{On shell R.C.}} \\ 0 & \underline{\text{intermediate R.C.}} \\ \int_0^1 dx \log\left(1+x(1-x)\frac{4M^2}{3m^2}\right) & \underline{\text{"another" R.C.}} \end{cases}$$

For ③ dimensional regularization

$$\Gamma_i(-p,p) = p^2+m^2 - \frac{\lambda m^2}{2(4\pi)^2}\left(\frac{2}{\epsilon} + \log\left(\frac{4\pi \mu_{DR}^2}{m^2}\right) + 1 - \gamma + O(\epsilon)\right)$$

$$+ \lambda a_1(\epsilon) p^2 + \lambda b_1(\epsilon)$$

$$\Gamma_i(p_1,\cdots,p_4) = \lambda - \frac{\lambda^2}{2(4\pi)^2}\left[\frac{2}{\epsilon} + \log\left(\frac{4\pi \mu_{DR}^2}{m^2}\right) - \gamma - \int_0^1 dx \log\left(1 + x(1-x)\frac{p_{12}^2}{m^2}\right)\right.$$

$$\left. + O(\epsilon)\right]$$

$$- (2\leftrightarrow 3) - (2\leftrightarrow 4)$$

$$+ \lambda^2 c_1(\epsilon)$$

Solution to the renormalization condition:

$$a_1(\epsilon) = 0, \quad b_1(\epsilon) = \frac{m^2}{2(4\pi)^2}\left[\frac{2}{\epsilon} + \log\left(\frac{4\pi \mu_{DR}^2}{m^2}\right) + 1 - \gamma + O(\epsilon)\right]$$

$$c_1(\epsilon) = \frac{3}{2(4\pi)^2}\left[\frac{2}{\epsilon} + \log\left(\frac{4\pi \mu_{DR}^2}{m^2}\right) - \gamma - K + O(\epsilon)\right]$$

K = same as in ①