Renormalization and linear symmetry

Consider a classical system with a linear symmetry

$$\delta S[\phi] = 0 \quad ; \quad \delta\phi_i = \sum_j M_{ij}\,\phi_j$$

Suppose we have a regularization that respects this:

$$\delta\left(\mathcal{D}_\Lambda\phi\; e^{-S[\phi;\Lambda]}\right) = 0.$$

Then, we have Slaunov-Taylor identity

$$\delta\Gamma[\phi;\Lambda] = 0,$$

and this holds at each order in loop expansion.

In particular, divergent terms are also symmetric.

So, we can choose counter terms that are also symmetric.

That is, we can renormalize the system while respecting the symmetry. I.e.

<u>a linear symmetry is protected from renormalization.</u>

What about <u>gauge symmetry</u>?

Renormalization of gauge theories

Consider a 4d gauge theory with gauge group G, a Dirac fermion ψ in a representation V, and Lagrangian

$$\mathcal{L}_E = \frac{1}{4e^2} F^{\mu\nu} \cdot F_{\mu\nu} - i \bar{\psi} \not{D}_A \psi + \bar{\psi} m \psi$$

Dimensions: $[e]=0$, $[m]=1$. So the theory is renormalizable from the viewpoint of power counting.

The question is whether it is possible to renormalize the system while preserving gauge symmetry.

I.e. whether we do not need to introduce gauge symmetry violating counter terms.

Answer: Yes, as long as $\exists$ regularization that respects the gauge symmetry.

Remark The same can be done for theories with

- Charged scalar fields
- chiral fermions

Gauge fixed Lagrangian :

$$\tilde{\mathcal{L}}_E = \mathcal{L}_E + \frac{\xi}{2} B^2 - iB \cdot \partial^\mu A_\mu + \bar{c}\, \partial^\mu D_\mu c.$$

It has BRST symmetry δ_B

$$\begin{cases} \delta_B A_\mu = D_\mu c, \quad \delta_B \psi = -c\psi \\ \delta_B c = -\frac{1}{2}[c,c], \quad \delta_B \bar{c} = iB, \quad \delta_B B = 0 \end{cases}$$

<u>Assumption</u> $\exists$ a regularization that respects the BRST symmetry.

Λ : the cut-off.

<u>Claim</u> The 1PI effective action is finite as $\Lambda \nearrow \infty$, as a functional of properly defined fields/couplings.

$$\Gamma(A_0, \psi_0, c_0, \bar{c}_0, B_0; e_0, m_0, \xi_0; \Lambda)$$

$$\begin{cases} A_0 = \sqrt{Z_A}\, A, \ \psi_0 = \sqrt{Z_\psi}\, \psi, \ c_0 = \sqrt{Z_c}\, c, \ \bar{c}_0 = \sqrt{Z_{\bar{c}}}\, \bar{c}, \ B_0 = \sqrt{Z_B}\, B \\ e_0 = e_0(e, m; \Lambda), \ m_0 = m_0(e, m; \Lambda), \ \xi_0 = \xi_0(e, m, \xi; \Lambda) \end{cases}$$

$$= \Gamma(A, \psi, c, \bar{c}, B; e, m, \xi; \Lambda) \ : \text{finite as } \Lambda \nearrow \infty$$

if we choose $Z_i = Z_i(e, m; \Lambda)$

$e_0(e, m; \Lambda)$, $m_0(e, m; \Lambda)$, $\xi_0(e, m, \xi; \Lambda)$

appropriately.

<u>Remark</u> $Z_A, Z_\psi, Z_c, Z_B, e_0, m_0, \xi_0$ may be different for different components:

$G = G_1 \times G_2 \times \cdots \times G_k$ decomposition to simple or abelian factors

$V = V_1 \oplus V_2 \oplus \cdots \oplus V_N$ irreducible decomposition

Then $A_0 = (\sqrt{Z_{A_1}} A_1, \sqrt{Z_{A_2}} A_2, \ldots, \sqrt{Z_{A_k}} A_k)$

similarly for $c_0, \bar{c}_0, B_0$

$\psi_0 = (\sqrt{Z_{\psi_1}} \psi_1, \sqrt{Z_{\psi_2}} \psi_2, \cdots, \sqrt{Z_{\psi_N}} \psi_N)$

$e_0 = \cdots$

$m_0 = \cdots$

$\xi_0 = \cdots$

For simplicity, G is assumed to be <u>simple</u>, so that you just have to consider a single $Z_A, Z_c, Z_B, e_0, \xi_0$

V may not be irreducible:

$Z_\psi = (Z_{\psi_i})_{i=1}^N$, $m_0 = (m_{0i})_{i=1}^N$ is understood.

Extended 1PI effective action

$$X = (A_\mu, \psi, c, \bar{c}, B)$$

$$K = (K^\mu, K^\psi, K^c, K^{\bar{c}}, -)$$

$$S[X,K] = \tilde{S}_E[X] + K \cdot \delta_B X$$

the sign is opposite to Lecture 4

$$e^{-W[J,K]} := \int \mathcal{D}X \, e^{-S[X,K] + J \cdot X}$$

$$\Gamma[X,K] := W[J_K[X],K] + J_K[X] \cdot X$$

where $J = J_K[X]$ is the soln of $\frac{\delta W}{\delta J}[J,K] \overset{!}{=} -X$.

Then

$$\frac{\delta \Gamma}{\delta X_i}[X,K] = \epsilon_i \, J_K^i[X] \qquad \epsilon_i = \begin{cases} 1 & X_i \text{ bosonic} \\ -1 & X_i \text{ fermionic} \end{cases}$$

$$\frac{\delta \Gamma}{\delta K^i}[X,K] = +\langle \delta_B X_i \rangle_{J_K[X],K}$$

$$0 = \int \delta_B \left(\mathcal{D}X \, e^{-S[X,K] + J \cdot X} \right) \rightsquigarrow$$

$$\boxed{\sum_i \frac{\delta \Gamma}{\delta X_i} \cdot \frac{\delta \Gamma}{\delta K^i} = 0}$$

Zinn-Justin equation

$$S[X,K] \supset K^{\bar{c}} \cdot iB$$

- $$0 = \int \mathcal{D}X \frac{\delta}{\delta B}\left(e^{-S[X,K]+J\cdot X}\right)$$

$$= \int \mathcal{D}X \, e^{-S[X,K]+J\cdot X}\left(-\xi B + i\partial^\mu A_\mu - iK^{\bar{c}} + J^B\right)$$

Set $J = J_K[X]$. Then

$$-\xi B + i\partial^\mu A_\mu - iK^{\bar{c}} + \frac{\delta\Gamma}{\delta B} = 0$$

$$\therefore \frac{\delta\Gamma}{\delta B} = \xi B - i\partial^\mu A_\mu + iK^{\bar{c}}$$

Also, $\dfrac{\delta\Gamma}{\delta K^{\bar{c}}} = \langle \delta_B \bar{c} \rangle_{J_K[X],K} = \langle iB \rangle_{J_K[X],K} = iB$

$$\boxed{\Gamma[X,K] = \int d^4x\left(\frac{\xi}{2}B^2 - iB\cdot\partial^\mu A_\mu + K^{\bar{c}}\cdot iB\right) + B \,\&\, K^{\bar{c}} \text{ independent terms.}} \quad (\star)$$

I.e. "$\nexists$ quantum correction to B & $K^{\bar{c}}$ dependence."

$$\bullet \quad 0 = \int \mathcal{D}X \frac{\delta}{\delta \bar{c}} \left(e^{-S[X,K] + J\cdot X} \right)$$

$$= \int \mathcal{D}X \, e^{-S[X,K]} \left(-\partial^\mu \underbrace{D_\mu c}_{\delta_B A_\mu} - J^{\bar{c}} \right) \qquad \left(J^{\bar{c}} \text{ is fermionic} \right)$$

$$\therefore \quad \partial^\mu \langle \delta_B A_\mu \rangle + J^{\bar{c}} = 0$$

$$J = J_K[X] \qquad \boxed{\partial^\mu \frac{\delta \Gamma}{\delta K^\mu} - \frac{\delta \Gamma}{\delta \bar{c}} = 0} \qquad (\star\star)$$

($\star$) and ($\star\star$) may be interpreted as

"gauge fixing condition is not renormalized."

(These hold as long as the gauge fixing function is linear in fields so that $\langle \chi(\phi) \rangle_{J_K[X], K} = \chi(\phi)$.)

To be precise, the above identities (ZJ, $\star$, $\star\star$) hold for bare fields X_0, K_0. However, with an appropriate relation among Z_{X_i} & Z_{K^i}'s where $K_0^i = \sqrt{Z_{K^i}} K_i$, they also hold for renormalized X, K :

- If Z_{K^i} are chosen so that

$$Z_{X_i} \cdot Z_{K^i} \quad \text{is} \quad i\text{-independent} =: \#$$

$$\sum_i \frac{\delta\Gamma}{\delta K_0^i} \cdot \frac{\delta\Gamma}{\delta X_{0i}} = 0 \quad \Rightarrow \quad \boxed{\sum_i \frac{\delta\Gamma}{\delta K^i} \cdot \frac{\delta\Gamma}{\delta X_i} = 0}$$

- $K^{\bar{c}}, B$ dependent terms remains unchanged

$$\frac{\xi_0}{2} B_0^2 - i B_0 \cdot \partial^\mu A_{0\mu} + K_0^{\bar{c}} \cdot i B_0 \overset{!}{=} \frac{\xi}{2} B^2 - i B \cdot \partial^\mu A_\mu + K^{\bar{c}} \cdot i B$$

$$\rightsquigarrow \quad \xi_0 Z_B = \xi, \quad Z_B \cdot Z_A = 1, \quad Z_{K^{\bar{c}}} \cdot Z_B = 1$$

$$\rightsquigarrow \quad Z_A = Z_B^{-1} = Z_{K^{\bar{c}}} = Z_c^{-1} \cdot \# \qquad \therefore \; \# = Z_A \cdot Z_c$$

$$\boxed{\begin{aligned} Z_{K^i} &= Z_{X_i}^{-1} \cdot Z_A Z_c \\ \xi_0 &= Z_A \xi \end{aligned}}$$

Then $Z_{K^\mu} = Z_c$ so that

$$\partial^\mu \frac{\delta\Gamma}{\delta K_0^\mu} = \frac{\delta\Gamma}{\delta \bar{c}_0} \quad \Rightarrow \quad \boxed{\partial^\mu \frac{\delta\Gamma}{\partial K^\mu} = \frac{\delta\Gamma}{\delta \bar{c}}}$$

$$\mathcal{L}_E = \frac{1}{4e^2} F^{\mu\nu} \cdot F_{\mu\nu} - i\bar{\psi} \not{D}_A \psi + \bar{\psi} m \psi$$

$$\tilde{\mathcal{L}}_E = \mathcal{L}_E + \frac{\xi}{2} B^2 - i B \cdot \partial^\mu A_\mu + \bar{c} \cdot \partial^\mu D_\mu c$$

Suppose

$$\left.\begin{aligned} e_0 &= e + \hbar e^{(1)} + \hbar^2 e^{(2)} + \cdots + \hbar^{N-1} e^{(N-1)} \\ m_0 &= m + \hbar m^{(1)} + \hbar^2 m^{(2)} + \cdots + \hbar^{N-1} m^{(N-1)} \\ Z_i &= 1 + \hbar Z_i^{(1)} + \hbar^2 Z_i^{(2)} + \cdots + \hbar^{N-1} Z_i^{(N-1)} \end{aligned}\right\} @_{N-1}$$

$i = A_\mu, \psi, c$

have been chosen so that the loop $\leq N-1$ terms of its 1PI effective action is made finite :

$$\Gamma_{@_{N-1}} = \underbrace{\Gamma_0 + \hbar \Gamma_1 + \cdots + \hbar^{N-1} \Gamma_{N-1}}_{\text{finite.}} + \underbrace{\hbar^N \Gamma_N' + \hbar^{N+1} \Gamma_{N+1}' + \cdots}_{\text{possibly divergent}}$$

The divergence comes from $\left\{ \begin{array}{l} \cdot \text{ the loop integrals} \\ \cdot \text{ Counter terms} \end{array} \right.$

Let $\Gamma_{N,\infty}$ be a divergent part of Γ_N'

$$\Gamma_N' = \Gamma_{N,\infty} + \text{finite.}$$

We shall determine a possible form of $\Gamma_{N,\infty}$ and show that $e_0^{(N)}$, $m_0^{(N)}$, $Z_i^{(N)}$ $(i = A, \psi, c)$ can be chosen so that

$$\Gamma_N = \Gamma_N' + \text{new counter term} \quad \text{is finite.}$$

- $\Gamma_{N,\infty}$ is independent of B & $K^{\bar{c}}$

- $\Gamma_{N,\infty}$ can be chosen to the integral of a "local expression"

$$\Gamma_{N,\infty} = \int d^4x\, \mathcal{L}_{N,\infty}$$

polynomial of fields & their derivatives with coefficients of mass dim ≥ 0

- $\Gamma_{N,\infty}$ is invariant under linear symmetries :
  - rigid gauge symmetry : $g \in G$ constant

$$\mathcal{O} \to g\mathcal{O}g^{-1} \text{ for } \mathcal{O} = A, c, \bar{c}, K^\mu, K^c$$
$$\psi \to g\psi, \quad \bar{\psi} \to \bar{\psi}g^{-1}, \quad K^\psi \to K^\psi g^{-1}, \quad \overline{K^\psi} \to g\overline{K^\psi}$$

  - flavor symmetry : $h : V \to V$ commuting with G, m

$$\psi \to h\psi, \quad \bar{\psi} \to \bar{\psi}h^{-1}, \quad K^\psi \to K^\psi h^{-1}, \quad \overline{K^\psi} \to h\overline{K^\psi}$$

$[\delta_B X] = [X] + 1$

- Canonical dimension of fields

A_μ	ψ	c	$\bar{c}$	K^μ	K^ψ	K^c
1	3/2	1	1	2	3/2	2

$\rightsquigarrow$ $\Gamma_{N,\infty}$ is at most quadratic in K^i's

$$\mathcal{L}_{N,\infty} = \underbrace{\alpha_{ij} K^i K^j}_{} + \beta_i K^i + K\text{-indep}$$

$\hookrightarrow$ may include $K^\psi \not{A} K^{\bar\psi}$, $K^\psi \not{\partial} K^{\bar\psi}$

- Ghost number symmetry

A_μ	ψ	c	$\bar{c}$	K^μ	K^ψ	K^c
0	0	1	-1	-1	-1	-2

This forbids $\alpha_{ij} K^i K^j$.

$\therefore$ $\mathcal{L}_{N,\infty}$ is at most linear in K^i's $(i = A_\mu, \psi, \bar\psi, c)$:

$$\Gamma_{N,\infty} = \Gamma^{(0)}_{N,\infty}[A, \psi, c, \bar{c}]$$

$$+ K^\mu \cdot \delta_{N,\infty} A_\mu + \left(K^\psi \cdot \delta_{N,\infty} \psi + \text{c.c.}\right) + K^c \cdot \delta_{N,\infty} c$$

These define local expressions (modulo finite ones)

$$\delta_{N,\infty} A_\mu,\quad \delta_{N,\infty}\psi,\quad \delta_{N,\infty}\bar{\psi},\quad \delta_{N,\infty} c.$$

We also put $\delta_{N,\infty}\bar{c} := 0$, $\delta_{N,\infty} B := 0$.

Then, we have

$$\Gamma_{N,\infty} = \Gamma^{(0)}_{N,\infty}[A,\psi,c,\bar{c}] + K^i \cdot \delta_{N,\infty} X_i .$$

Also, we define "<u>variation $\delta_{N,\infty}$</u>" by

$$\delta_{N,\infty}\mathcal{F} := \sum_i \delta_{N,\infty} X_i \cdot \frac{\delta \mathcal{F}}{\delta X_i}$$

Zinn-Justin eqn for $\Gamma_{@N-1} = \Gamma_0 + \hbar\Gamma_1 + \cdots + \hbar^{N-1}\Gamma_{N-1} + \hbar^N \Gamma_N' + \cdots$

at $O(\hbar^N)$:

$$\frac{\delta\Gamma_0}{\delta K^i}\cdot\frac{\delta\Gamma_N'}{\delta X_i} + \underbrace{\frac{\delta\Gamma_1}{\delta K^i}\cdot\frac{\delta\Gamma_{N-1}}{\delta X_i} + \cdots + \frac{\delta\Gamma_{N-1}}{\delta K^i}\cdot\frac{\delta\Gamma_1}{\delta X_i}}_{\text{finite}} + \frac{\delta\Gamma_N'}{\delta K^i}\cdot\frac{\delta\Gamma_0}{\delta X_i} = 0$$

Write $\cdots = \text{finite}$ as $\cdots \doteq 0$. Then $\Gamma_N' \doteq \Gamma_{N,\infty}$ and

$$\frac{\delta\Gamma_0}{\delta K^i}\cdot\frac{\delta\Gamma_{N,\infty}}{\delta X_i} + \frac{\delta\Gamma_{N,\infty}}{\delta K^i}\cdot\frac{\delta\Gamma_0}{\delta X_i} \doteq 0 .$$

Recall $\Gamma_0 = \tilde{S}_E + K^i \cdot \delta_B X_i$, $\Gamma_{N\infty} = \Gamma^{(0)}_{N\infty}[A, \varphi, c, \bar{c}] + K^i \cdot \delta_{N\infty} X_i$

$$\rightsquigarrow \quad \frac{\delta \Gamma_0}{\delta K^i} = \delta_B X_i , \quad \frac{\delta \Gamma_{N\infty}}{\delta K^i} = \delta_{N\infty} X_i$$

$$\therefore \quad \delta_B X_i \cdot \frac{\delta \Gamma_{N,\infty}}{\delta X_i} + \delta_{N,\infty} X_i \frac{\delta \Gamma_0}{\delta X_i} \doteq 0 , \quad \text{that is,}$$

$$\underline{\delta_B \Gamma_{N,\infty} + \delta_{N,\infty} \Gamma_0 \doteq 0}$$

(1) $\delta_B \Gamma^{(0)}_{N,\infty} + \delta_{N,\infty} \tilde{S}_E \doteq 0$

(2) $\delta_B \delta_{N,\infty} X_i + \delta_{N,\infty} \delta_B X_i = 0$ $\leftarrow$ non-trivial only for $X_i = A, \varphi, c$

By $\partial^\mu \frac{\delta \Gamma}{\delta K^\mu} = \frac{\delta \Gamma}{\delta \bar{c}}$ at $O(\hbar^n)$

$$\partial^\mu \underbrace{\frac{\delta \Gamma_{N,\infty}}{\delta K^\mu}}_{\delta_{N,\infty} A_\mu} \doteq \frac{\delta \Gamma_{N,\infty}}{\delta \bar{c}} = \frac{\delta \Gamma^{(0)}_{N,\infty}}{\delta \bar{c}} + \frac{\delta}{\delta \bar{c}} \left(K^i \cdot \delta_{N,\infty} X_i \right)$$

(3) $\frac{\delta \Gamma^{(0)}_{N,\infty}}{\delta \bar{c}} \doteq \partial^\mu \delta_{N,\infty} A_\mu$

(4) $\frac{\delta}{\delta \bar{c}} \delta_{N,\infty} X_i \doteq 0$ $\quad X_i = A, \varphi, c$

$$\therefore \quad \Gamma^{(0)}_{N,\infty}[A,\psi,c,\bar{c}] \overset{(3)}{\doteq} \bar{c}\cdot\partial^\mu \delta_{N,\infty} A_\mu + \underbrace{\bar{c}\text{-independent}}_{}$$

also c-independent by ghost # symmetry

$$\doteq \tilde{\Gamma}^{(0)}_{N,\infty}[A,\psi] + \bar{c}\cdot\partial^\mu \delta_{N\infty} A_\mu .$$

By Zinn-Justin (2): $\{\delta_B, \delta_{N,\infty}\} X_i \doteq 0$,

(4) $\frac{\delta}{\delta \bar{c}} \delta_{N,\infty} X_i \doteq 0$, and symmetries & dimensional analysis,

one can show that

$$\boxed{\begin{aligned} \delta_{N,\infty} A_\mu &\doteq \xi_N \partial_\mu c + \eta_N [A_\mu, c] \\ &= \eta_N \delta_B A_\mu + (\xi_N - \eta_N)\partial_\mu c \\ \delta_{N,\infty}\psi &\doteq -\eta_N c\psi = \eta_N \delta_B \psi \\ \delta_{N,\infty} c &\doteq -\tfrac{1}{2}\eta_N [c,c] = \eta_N \delta_B c \end{aligned}}$$

for some constants η_N and ξ_N.

See the additional note.

$$\Gamma^{(0)}_{N,\infty} = \tilde{\Gamma}_{N,\infty}[A,\psi] + \bar{c}\cdot\partial^\mu \delta_{N,\infty} A_\mu$$

$$\tilde{S}_E = S_E[A,\psi] + \frac{\xi}{2}B^2 - iB\cdot\partial^\mu A_\mu + \bar{c}\cdot\partial^\mu D_\mu c$$

Zinn-Justin (1):

$$0 \doteq \delta_B \Gamma^{(0)}_{N,\infty} + \delta_{N,\infty}\tilde{S}_E$$

$$= \delta_B \tilde{\Gamma}^{(0)}_{N,\infty}[A,\psi] + \cancel{iB\cdot\partial^\mu \delta_{N,\infty} A_\mu} - \bar{c}\cdot\partial^\mu \boxed{\delta_B \delta_{N,\infty} A_\mu}$$

$$+ \delta_{N,\infty} S_E[A,\psi] - \cancel{iB\cdot\partial^\mu \delta_{N,\infty} A_\mu} - \bar{c}\cdot\partial^\mu \boxed{\delta_{N,\infty}\delta_B A_\mu} = 0$$

$$= \delta_B \tilde{\Gamma}^{(0)}_{N,\infty}[A,\psi] + (\xi_N - \eta_N)\,\partial_\mu c\cdot\frac{\delta S_E}{\delta A_\mu}$$

On the other hand

$$\partial_\mu c\cdot\frac{\delta S_E}{\delta A_\mu} = \int d^4x \left\{ \frac{1}{e^2} F^{\mu\nu}\cdot D_\mu \partial_\nu c + i\bar{\psi}\,\partial\!\!\!/ c\,\psi \right\}$$

$$= \delta_B \int d^4x \left\{ \frac{1}{2e^2} F^{\mu\nu}\cdot[A_\mu, A_\nu] - i\bar{\psi} A\!\!\!/ \psi \right\}$$

$\uparrow$ Exercise

$$\therefore\ \tilde{\Gamma}^{(0)}_{N,\infty}[A,\psi] \doteq -(\xi_N - \eta_N)\int d^4x \left\{ \frac{1}{2e^2} F^{\mu\nu}\cdot[A_\mu, A_\nu] - i\bar{\psi} A\!\!\!/ \psi \right\}$$

$$+ \int d^4x\, \mathcal{L}_{G.I.}$$

$\uparrow$ gauge invariant, dim = 4.

$$\mathcal{L}_{G.I} = \frac{1}{4} \underset{\substack{\uparrow \\ \text{number}}}{X_N} F^{\mu\nu} \cdot F_{\mu\nu} - \underset{\substack{\uparrow \\ \text{number}}}{Y_N} i \bar{\Psi} \not{D} \Psi + \bar{\Psi} \underset{\substack{\uparrow \\ \text{mass matrix commuting} \\ \text{with gauge/flavor sym}}}{M_N} \Psi$$

$$\begin{aligned}
\mathcal{L}_{N,\infty} \doteq \mathcal{L}_{G.I.} & \\
& - (\xi_N - \eta_N) \frac{1}{2e^2} F^{\mu\nu} \cdot [A_\mu, A_\nu] + (\xi_N - \eta_N) i \bar{\Psi} \not{A} \Psi \\
& + \bar{c} \cdot \partial^\mu (\xi_N \partial_\mu c + \eta_N [A_\mu, c]) \\
& + K^\mu \cdot (\xi_N \partial_\mu c + \eta_N [A_\mu, c]) \\
& + K^\Psi (-\eta_N c \Psi) + c.c. \\
& + K^c \cdot \left(-\tfrac{1}{2} \eta_N [c, c]\right)
\end{aligned}$$

<u>Claim</u> This $\hbar^N \Gamma_{N,\infty}$ can be cancelled by the new $O(\hbar^N)$ terms that appears in $S[X_0, K_0; e_0, m_0, \xi_0]$ when we add an appropriate $O(\hbar^N)$ terms to $Z_A, Z_\Psi, Z_c, e_0, m_0$.

Indeed, if we write $\sqrt{Z_A}_{\text{old}} = \sqrt{Z_A \text{ in } \mathcal{O}_{N-1}}$ and $\hbar^N \sqrt{Z_A}^{(N)}_{\text{new}}$ for the $O(\hbar^N)$ effect of $\hbar^N Z_{A\,\text{new}}^{(N)}$ in $\sqrt{Z_A}$ (similarly for others):

$$\sqrt{Z_A} = \sqrt{Z_A}_{\text{old}} + \hbar^N \sqrt{Z_A}^{(N)}_{\text{new}} + O(\hbar^{N+1}),$$

$$Z_\psi = Z_{\psi\,\text{old}} + \hbar^N Z^{(N)}_{\psi\,\text{new}},$$

$$Z_c = Z_{c\,\text{old}} + \hbar^N Z^{(N)}_{c\,\text{new}},$$

$$\frac{Z_A}{e_0^2} = \left(\frac{Z_A}{e_0^2}\right)_{\text{old}} + \hbar^N \left(\frac{Z_A}{e_0^2}\right)^{(N)}_{\text{new}} + O(\hbar^{N+1}),$$

$$Z_\psi m_0 = (Z_\psi m_0)_{\text{old}} + \hbar^N (Z_\psi m_0)^{(N)}_{\text{new}} + O(\hbar^{N+1}),$$

Then

$$S[X_0, K_0; e_0, m_0, \xi_0] = O(1, \hbar, \cdots, \hbar^{N-1}, \hbar^N) \text{ from old}$$

$$+ \hbar^N \int d^4x \Big\{ \frac{1}{4}\left(\frac{Z_A}{e_0^2}\right)^{(N)}_{\text{new}} F^{\mu\nu} \cdot F_{\mu\nu} + \frac{1}{2} \frac{\sqrt{Z_A}^{(N)}_{\text{new}}}{e^2} F^{\mu\nu} \cdot [A_\mu, A_\nu]$$

$$- i Z^{(N)}_{\psi\,\text{new}} \bar{\psi} \not{\partial} \psi - i \sqrt{Z_A}^{(N)}_{\text{new}} \bar{\psi} \not{A} \psi + \bar{\psi} (Z_\psi m_0)^{(N)}_{\text{new}} \psi$$

$$+ \bar{c}\, \partial^\mu \left(Z^{(N)}_{c\,\text{new}} \partial_\mu c + \left(Z^{(N)}_{c\,\text{new}} + \sqrt{Z_A}^{(N)}_{\text{new}}\right)[A_\mu, c] \right)$$

$$+ K^\mu \left(Z^{(N)}_{c\,\text{new}} \partial_\mu c + \left(Z^{(N)}_{c\,\text{new}} + \sqrt{Z_A}^{(N)}_{\text{new}}\right)[A_\mu, c] \right)$$

$$+ K^\psi \left(Z^{(N)}_{c\,\text{new}} + \sqrt{Z_A}^{(N)}_{\text{new}} \right)(-c\psi) + \text{c.c.}$$

$$+ K^c \left(Z^{(N)}_{c\,\text{new}} + \sqrt{Z_A}^{(N)}_{\text{new}} \right)\left(-\frac{1}{2}[c, c]\right) \Big\} + O(\hbar^{N+1}).$$

See the additional note.

If $\left(\frac{Z_A}{e_0^2}\right)^{(N)}_{new} = -X_N$, $\quad Z_{\psi\,new}^{(N)} = -Y_N$, $\quad (Z_\psi m_0)^{(N)}_{new} = -M_N$

$$\sqrt{Z_A}^{(N)}_{new} = \xi_N - \eta_N, \quad Z_{c\,new}^{(N)} = -\xi_N,$$

$$\hookrightarrow \; Z_{c\,new}^{(N)} + \sqrt{Z_A}^{(N)}_{new} = -\eta_N,$$

then

$$\hbar^N \Gamma_{N,\infty} + \text{the new } O(\hbar^N) \text{ counter terms} \doteq 0 .$$

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