

Renormalization group

Choices of renormalization conditions:

"On shell", "intermediate", "Another (μ)", ...

All these originate from the same classical Lagrangian

→ same physics.

But we need a dictionary:

renormalization condition I → Φ_I, λ_I, \dots
renormalization condition II → $\Phi_{II}, \lambda_{II}, \dots$ } relation?

e.g. in 4d ϕ^4 theory

"another R.C." parametrized by a mass scale μ
renormalization point

$$\left\{ \begin{array}{l} \Gamma(-p, p) \Big|_{p^2 = \mu^2} = \mu^2 + m^2 \\ \frac{d}{dp^2} \Gamma(-p, p) \Big|_{p^2 = \mu^2} = 1 \\ \Gamma(p_1, p_2, p_3, p_4) \Big|_{p_i \cdot p_j = \begin{cases} \mu^2 & i=j \\ -\mu^2/3 & i \neq j \end{cases}} = \lambda \end{array} \right.$$

at μ : ϕ, m, λ

at μ' : ϕ', m', λ'

To describe the same physics,
how are they related?

Answer: so that the bare fields/couplings are the same.

$$Z_0^{\frac{1}{2}}(m, \lambda; \mu, \Lambda) \phi = \phi_0 = Z_0^{\frac{1}{2}}(m', \lambda'; \mu', \Lambda) \phi'$$

$$m_0(m, \lambda; \mu, \Lambda) = m_0 = m_0(m', \lambda'; \mu', \Lambda)$$

$$\lambda_0(m, \lambda; \mu, \Lambda) = \lambda_0 = \lambda_0(m', \lambda'; \mu', \Lambda)$$

$$\begin{array}{ccc} & \Gamma_0(\phi_0, m_0, \lambda_0; \Lambda) & \\ & \parallel & \parallel \\ \Gamma(\phi, m, \lambda; \mu, \Lambda) & \star & \Gamma(\phi', m', \lambda'; \mu', \Lambda) \end{array}$$

The change $(\phi, m, \lambda) \rightarrow (\phi', m', \lambda')$ for $\mu \rightarrow \mu'$ is called the renormalization group (RG) transformation, and the equality \star is called the RG equation.

The relation between the renormalized fields/couplings has a limit as $\Lambda \rightarrow \infty$, and

$$\Gamma(\phi, m, \lambda; \mu) := \lim_{\Lambda \rightarrow \infty} \Gamma(\phi, m, \lambda; \mu, \Lambda) \text{ satisfies}$$

$$\Gamma(\phi, m, \lambda; \mu) = \Gamma(\phi', m', \lambda'; \mu').$$

The RG transformation may be written as

$$\phi' = Z^{-\frac{1}{2}}(m, \lambda; \mu', \mu) \phi,$$

$$m' = R^m(m, \lambda; \mu', \mu),$$

$$\lambda' = R^\lambda(m, \lambda; \mu', \mu).$$

Put

$$\mu' \frac{\partial}{\partial \mu'} Z^{-\frac{1}{2}}(m, \lambda; \mu', \mu) \Big|_{\mu'=\mu} =: \gamma(m, \lambda; \mu),$$

$$\mu' \frac{\partial}{\partial \mu'} R^m(m, \lambda; \mu', \mu) \Big|_{\mu'=\mu} =: -\gamma_m(m, \lambda; \mu) m,$$

$$\mu' \frac{\partial}{\partial \mu'} R^\lambda(m, \lambda; \mu', \mu) \Big|_{\mu'=\mu} =: \beta(m, \lambda; \mu).$$

Then, the infinitesimal RG transformation (RG flow) is

$$\mu \frac{d}{d\mu} \phi = -\gamma(m, \lambda; \mu) \phi,$$

$$\mu \frac{d}{d\mu} m = -\gamma_m(m, \lambda; \mu) m,$$

$$\mu \frac{d}{d\mu} \lambda = \beta(m, \lambda; \mu),$$

and the infinitesimal RG equation is

$$\left[-\gamma \phi \cdot \frac{\delta}{\delta \phi} - \gamma_m m \frac{\partial}{\partial m} + \beta \frac{\partial}{\partial \lambda} + \mu \frac{\partial}{\partial \mu} \right] \Gamma(\phi, m, \lambda; \mu) = 0.$$

Computation in 4d ϕ^4 theory

$$\text{Recall } Z_0 = 1 + \hbar \lambda a_1 + \hbar^2 \lambda^2 a_2 + \dots$$

$$Z_0 m_0^2 = m^2 + \hbar \lambda b_1 + \hbar^2 \lambda^2 b_2 + \dots$$

$$Z_0^2 \lambda_0 = \lambda + \hbar \lambda^2 c_1 + \hbar^2 \lambda^3 c_2 + \dots$$

$$a_1 = 0$$

$$b_1 = -\frac{1}{2(4\pi)^2} \left[\Lambda^2 - m^2 \left(\log\left(\frac{\Lambda^2}{m^2}\right) + 1 - \gamma \right) + m^2 O\left(\frac{m^2}{\Lambda^2}\right) \right]$$

$$c_1 = \frac{3}{2(4\pi)^2} \left[\log\left(\frac{\Lambda^2}{2m^2}\right) - \gamma - 1 - \int_0^1 dx \log\left(1 + x(1-x) \frac{4\mu^2}{3m^2}\right) + O\left(\frac{m^2}{\Lambda^2}\right) \right]$$

Thus, to 1-loop, $Z_0 = 1 + O(\hbar^2)$, and

$$0 = \mu \frac{d}{d\mu} \phi_0 = \mu \frac{d}{d\mu} \phi + O(\hbar^2)$$

$$0 = \mu \frac{d}{d\mu} m_0^2 = \mu \frac{d}{d\mu} m^2 + \hbar \left(\mu \frac{d}{d\mu} \lambda b_1 + \lambda \mu \frac{d}{d\mu} b_1 \right) + O(\hbar^2)$$

$$0 = \mu \frac{d}{d\mu} \lambda_0 = \mu \frac{d}{d\mu} \lambda + \hbar \left(\mu \frac{d}{d\mu} \lambda^2 c_1 + \lambda^2 \mu \frac{d}{d\mu} c_1 \right) + O(\hbar^2)$$

$$\Rightarrow \mu \frac{d}{d\mu} \phi = O(\hbar^2)$$

$$\mu \frac{d}{d\mu} \lambda = -\hbar \lambda^2 \mu \frac{d}{d\mu} c_1 + O(\hbar^2)$$

$$\mu \frac{d}{d\mu} m = O(\hbar^2)$$

$$\therefore \gamma(m, \lambda; \mu) = \mathcal{O}(\hbar^2)$$

$$\gamma_m(m, \lambda; \mu) = \mathcal{O}(\hbar^2)$$

$$\beta(m, \lambda; \mu) = -\hbar \lambda^2 \mu \frac{d}{d\mu} C_1$$

$$= \frac{3\hbar \lambda^2}{2(4\pi)^2} \int_0^1 dx \frac{x(1-x) \frac{8\mu^2}{3m^2}}{1 + x(1-x) \frac{4\mu^2}{3m^2}} + \mathcal{O}(\hbar^2)$$

$$= \begin{cases} \frac{3\hbar \lambda^2}{(4\pi)^2} + \mathcal{O}(\hbar^2) & \mu \gg m \\ \frac{2\hbar \lambda^2}{3(4\pi)^2} \frac{\mu^2}{m^2} + \mathcal{O}(\hbar^2) & \mu \ll m \end{cases}$$

Let us consider the limit $m \rightarrow 0$.

As λ & Z are dimensionless, β & γ are functions of λ only:

$$\beta = \beta(\lambda), \quad \gamma = \gamma(\lambda).$$

Indeed, in this limit,

$$\gamma = 0, \quad \beta = \frac{3\lambda^2}{(4\pi)^2} \quad \text{at 1-loop.}$$

The RG flow : $\mu \frac{d}{d\mu} \lambda = \beta(\lambda)$

$$\mu \frac{d}{d\mu} \phi = -\gamma(\lambda) \phi$$

The RG eqn :

$$\left[\mu \frac{\partial}{\partial \mu} + \beta(\lambda) \frac{\partial}{\partial \lambda} - \gamma(\lambda) \phi \cdot \frac{\delta}{\delta \phi} \right] \Gamma(\phi, \lambda; \mu) = 0.$$

Instead of μ , we $t = \log(\mu/\mu_0)$ or $\mu = e^t \mu_0$.

Then, the RG flow takes the form

$$\lambda = \bar{\lambda}(t) \quad \left(\leftarrow \text{a solution to } \frac{d}{dt} \lambda = \beta(\lambda) \right)$$

$$\phi = \bar{\phi}(t) = \bar{\phi}(0) \cdot e^{-\int_0^t dt' \gamma(\bar{\lambda}(t'))}$$

and the RGE :

$$\Gamma(\bar{\phi}(t), \bar{\lambda}(t); e^t \mu_0) \text{ is } t\text{-independent.}$$

Write

$$\Gamma = \sum_{n=0}^{\infty} \frac{1}{n!} \int \prod_{i=1}^n \frac{d^4 p_i}{(2\pi)^4} \cdot (2\pi)^4 \delta(p_1 + \dots + p_n)$$

$$\Gamma(p_1, \dots, p_n, \lambda; \mu) \phi(p_1) \dots \phi(p_n)$$

RGE:

$$e^{-n \int_0^t dt' \gamma(\bar{\lambda}(t'))} \Gamma(p_1, \dots, p_n, \bar{\lambda}(t); e^t \mu_0) \quad \text{is } t\text{-independent}$$

$$= \Gamma(p_1, \dots, p_n, \bar{\lambda}(0); \mu_0)$$

On the other hand, the canonical dimensions are

$$[\mu] = 1, [\phi] = 1, [\chi] = 0, [\Gamma] = 0$$

$$\therefore [\Gamma(p_1, \dots, p_n, \lambda; \mu)] = 4 - n \Rightarrow$$

$$\Gamma(e^t p_1, \dots, e^t p_n, \lambda; e^t \mu) = e^{(4-n)t} \Gamma(p_1, \dots, p_n, \lambda; \mu).$$

"dimensional analysis"

Combining,

$$\Gamma(e^t p_1, \dots, e^t p_n, \bar{\lambda}(0); \mu_0)$$

$$\stackrel{\text{RGE}}{=} e^{-n \int_0^t dt' \gamma(\bar{\lambda}(t'))} \Gamma(e^t p_1, \dots, e^t p_n, \bar{\lambda}(t); e^t \mu_0)$$

$$\stackrel{\text{dim. an.}}{=} e^{4t - n \int_0^t dt' (1 + \gamma(\bar{\lambda}(t')))} \Gamma(p_1, \dots, p_n, \bar{\lambda}(t); \mu_0)$$

This means

① If we uniformly rescale the momenta as

$$p_i \rightarrow e^\tau p_i,$$

the coupling λ effectively changes as

$$\bar{\lambda}(0) \rightarrow \bar{\lambda}(\tau).$$

$\bar{\lambda}(\tau)$ is the "effective coupling constant".

② The dimension of ϕ has also changed as

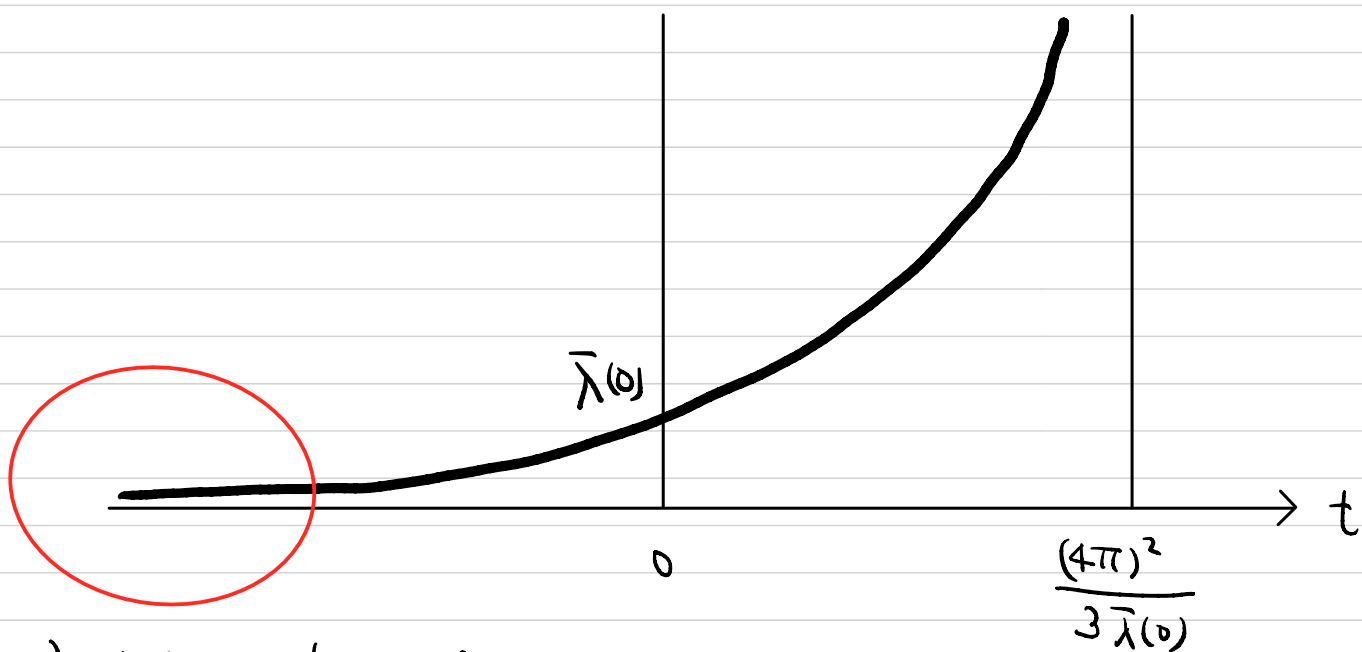
$$l \rightarrow l + \gamma(\bar{\lambda}(\tau))$$

$\gamma(\lambda)$ is the "anomalous dimension" of ϕ .

At 1-loop, $\frac{d\lambda}{dt} = \frac{3\lambda^2}{(4\pi)^2}$.

$$\int \frac{d\lambda}{\lambda^2} = \int \frac{3dt}{(4\pi)^2} \sim -\frac{1}{\lambda(t)} + \frac{1}{\lambda(0)} = \frac{3t}{(4\pi)^2}$$

$$\bar{\lambda}(t) = \frac{\bar{\lambda}(0)}{1 - \frac{3t}{(4\pi)^2} \bar{\lambda}(0)}$$



$\lambda \rightarrow 0$ as $t \rightarrow -\infty$

$\mu = e^t \mu_0 \rightarrow 0$

The coupling is weaker at lower energies

or stronger at higher energies.

- $$\lambda(\mu) = \frac{\lambda(\mu_0)}{1 - \frac{3\lambda(\mu_0)}{(4\pi)^2} \log(\mu/\mu_0)}$$

is valid for $\mu \ll \mu_0$ even if $|\log(\mu/\mu_0)|$ may be large.

- The series expansion

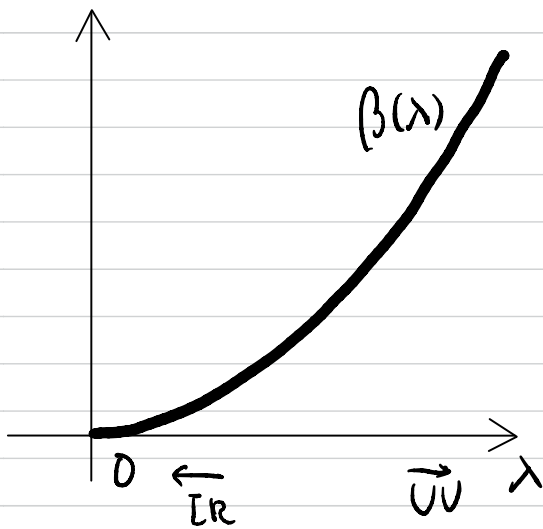
$$\lambda(\mu) = \sum_{n=0}^{\infty} \lambda(\mu_0) \left(\frac{3\lambda(\mu_0)}{(4\pi)^2} \log(\mu/\mu_0) \right)^n$$

has a Feynman diagram interpretation:

$$\begin{aligned} \text{Diagram} &\sim \lambda^{n+1} \int_{\mu}^{\mu_0} \frac{d^4 k_1}{(k_1^2)^2} \dots \int_{\mu}^{\mu_0} \frac{d^4 k_n}{(k_n^2)^2} \\ &\sim \lambda^{n+1} (\log \mu_0/\mu)^n \end{aligned}$$

"RG sums up a series of Feynman diagrams"

Various possibilities



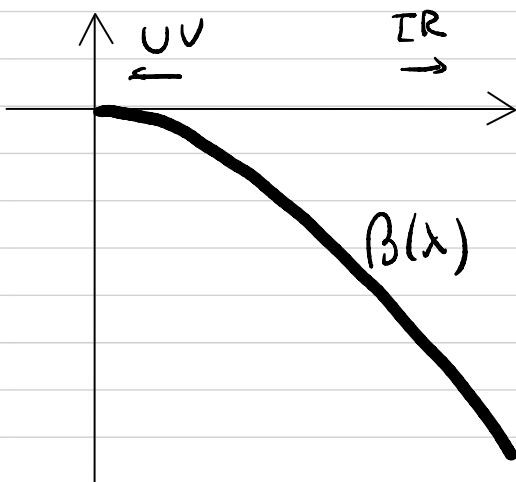
$\lambda \rightarrow 0$ in the IR limit

infra-red free theory

We've just seen
e.g. $4d \phi^4$ theory, QED_4 ^{QFT II}

QCD_4 with large # of flavors

next

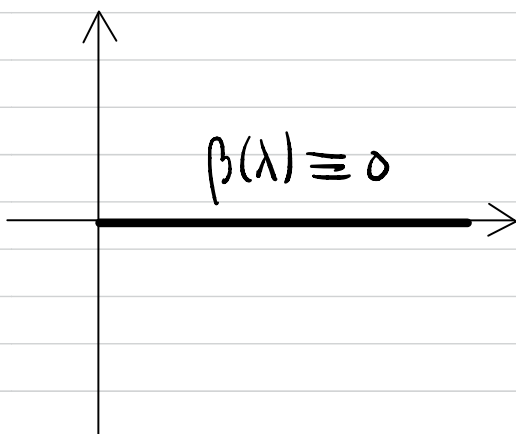


$\lambda \rightarrow 0$ in the UV limit

asymptotically free theory

e.g. $4d$ Yang-Mills theory
 QCD_4 with small # of flavors

next



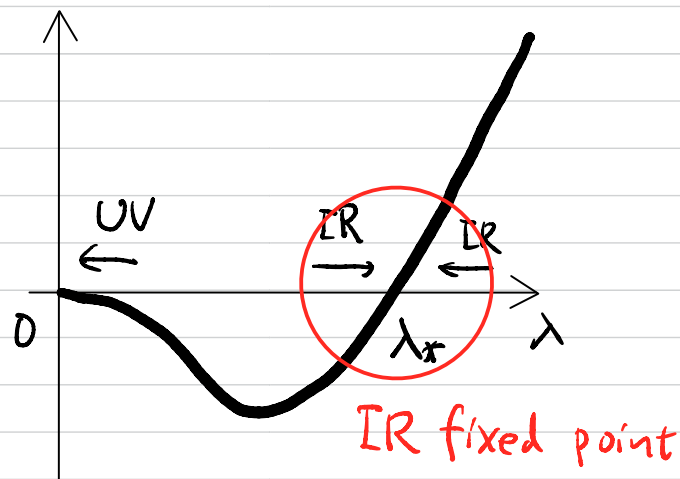
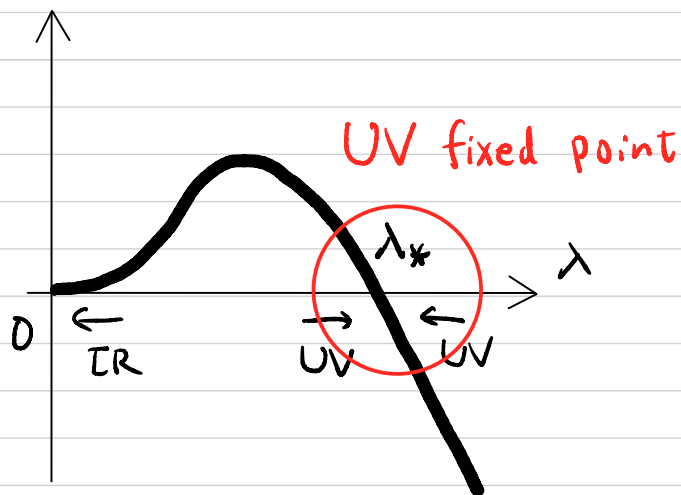
λ does not run!

finite theory

e.g. $4d \mathcal{N}=4$ supersymmetric Yang-Mills

next

Other possibilities:



... \exists non-trivial fixed point of RG flow.

At such a point λ_* , with $\gamma_* := \gamma(\lambda_*)$,

$$\begin{aligned} & \Gamma(e^t p_1, \dots, e^t p_n, \lambda_*; \mu_0) \\ &= e^{(4-n(1+\gamma_*))t} \Gamma(p_1, \dots, p_n, \lambda_*; \mu_0) \end{aligned}$$

Correlation functions scales in a simple way.

“scale invariant theory”

e.g. $\Gamma(-p, p) = \text{const} \cdot (p^2)^{1-\gamma_*}$

$$\Gamma(p) = \frac{1}{2} \phi_i A_{ij} \phi_j - \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{i_1 \dots i_n} \lambda_{1PI}^{i_1 \dots i_n} \phi_{i_1} \dots \phi_{i_n}$$

$$\Gamma(-p, p) = p^2 - \lambda_{1PI}^{(2)}(p^2)$$

$$\text{---} \textcircled{\text{---}} \text{---} = \text{---} + \text{---} \textcircled{1PI} \text{---} + \text{---} \textcircled{1PI} \textcircled{1PI} \text{---} + \dots$$

$$= \text{---} \left(1 - \textcircled{1PI} \text{---} \right)^{-1}$$

$$= \frac{1}{p^2} \left(1 - \lambda_{1PI}^{(2)}(p^2) \frac{1}{p^2} \right)^{-1} = \frac{1}{p^2 - \lambda_{1PI}^{(2)}(p^2)} = \frac{1}{\Gamma(-p, p)}$$

$$\langle \phi(x) \phi(0) \rangle = \int \frac{d^4 p}{(2\pi)^4} \frac{e^{-ipx}}{\Gamma(-p, p)} \propto (p^2)^{1-\gamma_4}$$

$$\propto \frac{1}{|x|^{2+2\gamma_4}}$$

RG of 4d non-Abelian gauge theories

Consider a 4d gauge theory with gauge group G and a Dirac fermion Ψ in a \mathbb{C} -representation V_f (and/or a scalar ϕ in a \mathbb{C} -representation V_b).

Aim Compute the functions β, γ that determines the RG flow of couplings and fields.

In particular, the β -function for the gauge coupling constant e .

For this, we choose a renormalization condition and find out

$$Z_A, Z_\Psi, Z_C, e_0, m_0$$

as a function of renormalized couplings e, m , renormalization point μ , and cut-off Λ .

What kind of renormalization condition can we impose?

We may impose a condition on the 1PI effective action $\Gamma[A, \psi, c, \bar{c}, B; e, m, \xi; \Lambda]$ so that $Z_A, Z_\psi, Z_c, e_0, m_0$ are fixed at each order in the loop expansion.

Recall

$$\begin{aligned}
 S[X_0, K_0, \overset{=0}{e_0}, m_0, \xi_0] &= O(1, \hbar, \dots, \hbar^{N-1}, \hbar^N) \text{ from old} \\
 &+ \hbar^N \int d^d x \left\{ \frac{1}{4} \left(\frac{Z_A^{(N)}}{e_0^2} \right)_{\text{new}} F^{\mu\nu} \cdot F_{\mu\nu} + \frac{1}{2} \frac{\sqrt{Z_A^{(N)}}_{\text{new}}}{e^2} F^{\mu\nu} \cdot [A_\mu, A_\nu] \right. \\
 &\quad - i Z_\psi^{(N)}_{\text{new}} \bar{\psi} \not{\partial} \psi - i \sqrt{Z_A^{(N)}}_{\text{new}} \bar{\psi} \not{A} \psi + \bar{\psi} (Z_\psi m_0)_{\text{new}}^{(N)} \psi \\
 &\quad \left. + \bar{c} \not{\partial}^\mu (Z_c^{(N)}_{\text{new}})_{\mu} c + (Z_c^{(N)}_{\text{new}} + \sqrt{Z_A^{(N)}}_{\text{new}}) [A_\mu, c] \right\} \\
 &+ O(\hbar^{N+1}).
 \end{aligned}$$

~

A renormalization condition that constrains the terms

$$\frac{1}{4e^2} (\partial_\mu A_\nu - \partial_\nu A_\mu)^2, -i \bar{\psi} \not{\partial} \psi, \bar{\psi} m \psi, \bar{c} \not{\partial}^2 c, \bar{c} \not{\partial}^\mu [A_\mu, c]$$

in Γ , for example.

Let us write

$$\begin{aligned}
 \Gamma = & \int \frac{d^4 p}{(2\pi)^4} \left\{ \frac{1}{2} A_{\mu a}(-p) \Gamma^{\mu\nu, ab}(-p, p) A_{\nu b}(p) \right. \\
 & + \bar{\Psi}(-p) \Gamma_{\psi}(-p, p) \Psi(p) \\
 & \left. + \bar{C}(-p) \Gamma_{gh}(-p, p) C(p) \right\} \\
 & + \int \frac{d^4 p}{(2\pi)^4} \frac{d^4 q}{(2\pi)^4} \bar{C}(-p-q) \Gamma_{gh}^{\mu a}(-p-q, q, p) A_{\mu a}(q) C(p) \\
 & + \dots
 \end{aligned}$$

$$\Gamma_{\psi}(-p, p) \in \text{End}(V_f \otimes S),$$

$$\Gamma_{gh}(-p, p), \Gamma_{gh}^{\mu a}(-p-q, q, p) \in \text{End}(\mathfrak{g})$$

By Ward identities associated with symmetries (Lorentz, rigid G , ghost #, BRST, ...) and additional arguments, these coefficient functions are constrained as:

$$\Gamma^{\mu\nu, ab}(-p, p) = (d^{\mu\nu} p^2 - p^\mu p^\nu) d^{ab} \Pi(p^2)$$

$$\Gamma_\psi(-p, p) = A(p^2) \not{p} + B(p^2)$$

$$\Gamma_{gh}(-p, p) = p^2 \Pi_{gh}(p^2) \text{id}_{\mathcal{G}}$$

$$\Gamma_{gh}^{M_a}(-p-q, q, p) = (p+q)^M \sum_I X_I^a C_I(p, q) + q^M \sum_I X_I^c D_I(p, q)$$

• $A(p^2), B(p^2) \in \text{End}(V_f) \otimes \text{id}_{\mathcal{G}}$ commuting with G .

• $\Pi(p^2), \Pi_{gh}(p^2), C_I(p, q), D_I(p, q)$ are scalars.

$C_I(p, q), D_I(p, q)$ depends on p, q via $p^2, q^2, p \cdot q$.

• $\{X_I^a\}$ basis of subspaces of $\text{End } \mathcal{G}$ isomorphic to \mathcal{G} , obeying

$$\text{Ad}_g \circ X_I^a \circ \text{Ad}_g^{-1} = X_I^b (\text{Ad}_g)_b^a$$

where $((\text{Ad}_g)_b^a)$ is the matrix representing $\text{Ad}_g : \mathcal{G} \rightarrow \mathcal{G}$

with respect to a basis $\{e^a\} \subset \mathcal{G}$.

We assume $I=0$ corresponds to the adjoint action of \mathcal{G} on \mathcal{G} ,

so that

$$X_0^a = \text{ad } e^a.$$

(See the additional note for derivation.)

As a renormalization condition, we may take

$$\Pi(\mu^2) = \frac{1}{e^2},$$

$$A(\mu^2) = -id_{V_f \otimes S}, \quad B(\mu^2) = m \otimes id_S,$$

$$\Pi_{gh}(\mu^2) = -1,$$

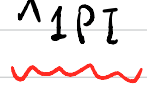
$$C_0(p, q) \Big|_{p^2 = q^2 = (p+q)^2 = \mu^2} = -i.$$

Rmk We do not impose any condition on

$$C_I(p, q) \text{ for } I \neq 0, \quad D_I(p, q) \text{ for } \forall I.$$

Recall

$$\Gamma = S_{\text{free}} - \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{i_1 \dots i_n} \lambda_{1PI}^{i_1 \dots i_n} \phi_{i_1} \dots \phi_{i_n}$$



 \uparrow 1PI vertex

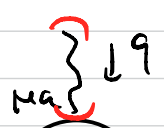
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$$\Gamma^{\mu\nu, ab}(-p, p) = \frac{1}{e^2} (\delta^{\mu\nu} p^2 - p^\mu p^\nu) \delta^{ab} - \left(\text{---} \left(\text{---} \right)_{\mu\nu} \text{---} \right) \text{1PI} \left(\text{---} \right)_{ab} \left(\leftarrow p \right)$$

$$\Gamma_{\psi}(-p, p) = -\not{p} + m^2 - \left(\leftarrow \right) \text{1PI} \left(\leftarrow \right) \left(\leftarrow p \right)$$

$$\Gamma_{gh}(-p, p) = -p^2 - \left(\text{---} \leftarrow \text{---} \right) \text{1PI} \left(\text{---} \leftarrow \text{---} \right) \left(\leftarrow p \right)$$

$$\Gamma_{gh}^{\mu a}(-p-q, q, p) = - \left(\text{---} \leftarrow \text{---} \right) \text{1PI} \left(\text{---} \leftarrow \text{---} \right) \left(\leftarrow p \right)$$



 \uparrow g

$$= -i(p+q)^\mu a d e^a - \dots$$

\uparrow
 $L \geq 1$ loops

Tree and one-loop contributions to 1PI vertices:

$$\begin{aligned}
 \text{wavy} \text{---} \text{circle} \text{---} \text{wavy} \text{ (1PI)} &= \text{wavy} \text{---} \text{cloud} \text{---} \text{wavy} + \text{wavy} \text{---} \text{cloud} \text{---} \text{wavy} \\
 &+ \text{wavy} \text{---} \text{circle} \text{---} \text{wavy} + \text{wavy} \text{---} \text{dashed circle} \text{---} \text{wavy} \\
 &\left(+ \text{wavy} \text{---} \text{dashed circle} \text{---} \text{wavy} + \text{wavy} \text{---} \text{dashed circle} \text{---} \text{wavy} \right) \leftarrow \text{scalar}
 \end{aligned}$$

$$\text{---} \text{circle} \text{---} \text{---} \text{ (1PI)} = \text{---} \text{cloud} \text{---} \text{---}$$

$$\text{---} \text{circle} \text{---} \text{---} \text{ (1PI)} = \text{---} \text{cloud} \text{---} \text{---}$$

$$\text{---} \text{circle} \text{---} \text{---} \text{ (1PI)} = \text{---} \text{dashed} \text{---} \text{dashed} \text{---} + \text{---} \text{dashed} \text{---} \text{cloud} \text{---} \text{dashed} \text{---} + \text{---} \text{dashed} \text{---} \text{cloud} \text{---} \text{dashed} \text{---}$$