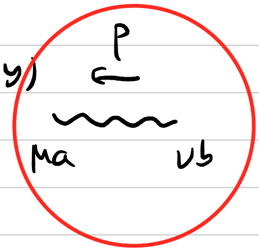
Computation of the 1-loop diagrams

Propagators

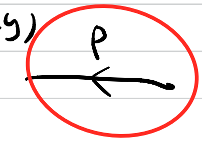
We shall work with $\xi = 1$

$$\overbrace{A_{\mu a}(x)\, A_{\nu b}}(y) = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip(x-y)}$$



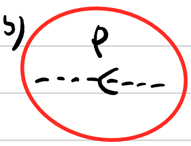
$$= e^2 \delta_{ab}\left(\frac{\delta_{\mu\nu}}{p^2} + (\xi-1)\frac{p_\mu p_\nu}{p^4}\right)$$

$$\overbrace{\psi(x)\,\bar\psi}(y) = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip(x-y)}$$



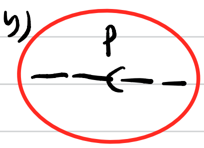
$$= \frac{1}{-\not{p} + m_f} = \frac{\not{p} + m_f}{p^2 + m_f^2}$$

$$\overbrace{C(x)\,\bar C}(y) = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip(x-y)}$$



$$= \frac{1}{-p^2}$$

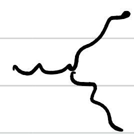
$$\overbrace{\phi(x)\,\bar\phi}(y) = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip(x-y)}$$



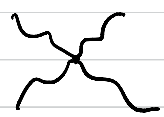
$$= \frac{1}{p^2 + m_b^2}$$

Vertices

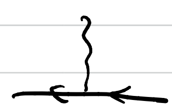
$$-S_{E,int} = \int d^4x \Big\{ -\frac{1}{2e^2}(\partial^\mu A^\nu - \partial^\nu A^\mu)\cdot[A_\mu, A_\nu]$$



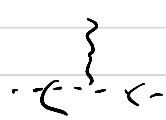
$$-\frac{1}{4e^2}[A^\mu, A^\nu]\cdot[A_\mu, A_\nu]$$



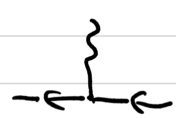
$$+\, i\bar\psi \not{A} \psi$$



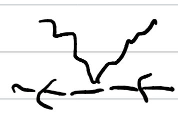
$$+\, \partial^\mu \bar C \cdot [A_\mu, C]$$



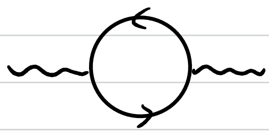
$$+\, \bar\phi A^\mu \partial_\mu \phi - \partial^\mu \bar\phi A_\mu \phi$$



$$+\, \bar\phi A^\mu A_\mu \phi \quad \Big\}$$



• Contributions to $\langle A_{\mu_0 a_0}(x)\, A_{\nu_0 b_0}(y) \rangle$



$z_1 \leftrightarrow z_2$

$$= A_{\mu_0 a_0}(x)\ \frac{1}{2}\int d^4z_1 d^4z_2\ i\bar{\psi}\not{A}\psi(z_1)\ i\bar{\psi}\not{A}\psi(z_2)\ A_{\nu_0 b_0}(y) \times 2$$

move w/ (-1)

$$= i^2(-1)\int d^4z_1 d^4z_2\ A_{\mu_0 a_0}(x) A_{\mu a}(z_1) \qquad \int \frac{d^4p_1}{(2\pi)^4} e^{-ip_1(x-z_1)}\ \overset{p_1}{\leftarrow}\ (\mu_0 a_0 \sim\sim \mu a)$$

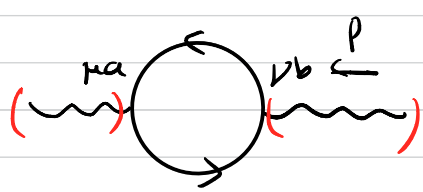
$$\mathrm{tr}_{V_f \otimes S}\left(\gamma^\mu e^a\, \psi(z_1)\bar{\psi}(z_2)\, \gamma^\nu e^b\, \psi(z_2)\bar{\psi}(z_1)\right)\ A_{\nu b}(z_2) A_{\nu_0 b_0}(y)$$

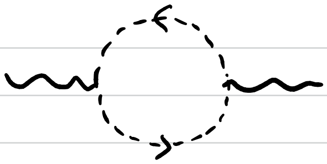
$$\int \frac{d^4k_1}{(2\pi)^4} e^{-ik_1(z_1-z_2)}\ \overset{k_1}{\longleftarrow} \qquad \int \frac{d^4k_2}{(2\pi)^4} e^{-ik_2(z_2-z_1)}\ \overset{k_2}{\longleftarrow} \qquad \int \frac{d^4p_2}{(2\pi)^4} e^{-ip_2(z_2-y)}\ \overset{p_2}{\leftarrow}\ (\nu b \sim\sim \nu_0 b_0)$$

$$\int d^4z_1 d^4z_2 \rightsquigarrow (2\pi)^4 \delta(p_1 - k_1 + k_2)\cdot(2\pi)^4 \delta(k_1 - k_2 - p_2)$$

$$= (2\pi)^8\, \delta(p_1 - p_2)\, \delta(k_2 - (k_1 - p_1)) \rightsquigarrow \quad p_1 = p_2 = p, \quad k_1 = k,\ k_2 = k - p$$

$$= \int \frac{d^4p}{(2\pi)^4} e^{-ipx}\ \overset{p}{\leftarrow}\ (\mu_0 a_0 \sim\sim \mu a)\ \boxed{\int \frac{d^4k}{(2\pi)^4}\, \mathrm{tr}_{V_f \otimes S}\left(\gamma^\mu e^a \overset{k}{\leftarrow} \gamma^\nu e^b \overset{k-p}{\leftarrow}\right)}\ \overset{p}{\leftarrow}\ (\nu b \sim\sim \nu_0 b_0)\ e^{ipy}$$





$z_1 \leftrightarrow z_2$

$$= A_{\mu_0 a_0}(x)\, \frac{1}{2} \int d^4z_1\, d^4z_2\; \partial^\mu \bar{C}\cdot [A_\mu, C](z_1)\; \partial^\nu \bar{C}\cdot [A_\nu, C](z_2)\; A_{\nu_0 b_0}(y) \times 2$$

move ~ (-1)

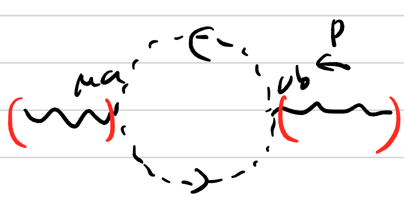
$$= (-1) \int d^4z_1\, d^4z_2\;\; A_{\mu_0 a_0}(x)\, A_{\mu a}(z_1)$$

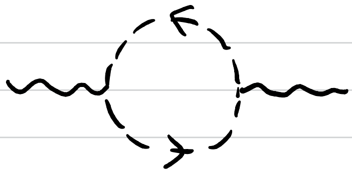
$$\mathrm{tr}_g \left(\mathrm{ad}\, e^a\; C(z_1)\, \partial^\nu \bar{C}(z_2)\; \mathrm{ad}\, e^b\; C(z_2)\, \partial^\mu \bar{C}(z_1) \right) A_{\nu b}(z_2)\, A_{\nu_0 b_0}(y)$$

$$\int \frac{d^4k_1}{(2\pi)^4}\, e^{-ik_1(z_1 - z_2)} \quad \overset{k_1}{\dashleftarrow}\; ik_1^\nu \qquad\qquad \int \frac{d^4k_2}{(2\pi)^4}\, e^{-ik_2(z_2 - z_1)} \quad \overset{k_2}{\dashleftarrow}\; ik_2^\mu$$

$$= \int \frac{d^4p}{(2\pi)^4}\, e^{-ipx} \;\overset{p}{\underset{\mu_0 a_0 \qquad \mu a}{\sim\!\sim\!\sim}}$$

$$(-1) \int \frac{d^4k}{(2\pi)^4}\, \mathrm{tr}_g \left(\mathrm{ad}\, e^a \;\overset{k}{\dashleftarrow}\; ik^\nu\, \mathrm{ad}\, e^b \;\overset{k-p}{\dashleftarrow}\; i(k-p)^\mu \right) \quad \overset{p}{\underset{\nu b \qquad \nu_0 b_0}{\sim\!\sim\!\sim}}\; e^{ipy}$$





$$= A_{\mu_0 a_0}(x)\,\frac{1}{2}\int d^4z_1\, d^4z_2\; V_3^B(z_1)\, V_3^B(z_2)\, A_{\nu_0 b_0}(y) \times 2$$

$z_1 \leftrightarrow z_2$

$$V_3^B = \bar{\phi} A^\mu \partial_\mu \phi - \partial^\mu \bar{\phi} A_\mu \phi = \bar{\phi}\,(\overrightarrow{\partial}^\mu - \overleftarrow{\partial}^\mu)\, e^a \phi \cdot A_{\mu a}$$

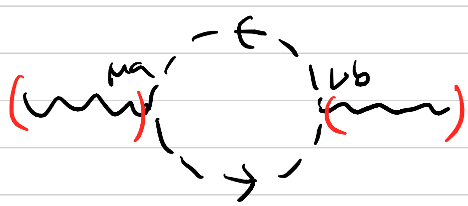
$$= \int d^4z_1\, d^4z_2\; A_{\mu_0 a_0}(x)\, A_{\mu a}(z_1)$$

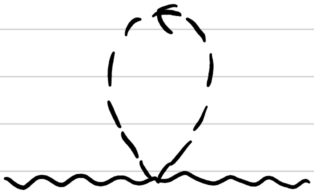
$$\bar{\phi}(z_1)(\overrightarrow{\partial}^\mu - \overleftarrow{\partial}^\mu) e^a \phi(z_1)\; \bar{\phi}(z_2)(\overrightarrow{\partial}^\nu - \overleftarrow{\partial}^\nu) e^b \phi(z_2)\; A_{\nu b}(z_2)\, A_{\nu_0 b_0}(y)$$

move

$$= \int \frac{d^4p}{(2\pi)^4}\, e^{-ipx} \underset{\mu_0 a_0 \qquad \mu a}{\overset{p}{\longleftarrow}} \times \underset{\nu b \qquad \nu_0 b_0}{\overset{p}{\longleftarrow}} e^{ipy} \times$$

$$\boxed{\int \frac{d^4k}{(2\pi)^4}\, \mathrm{tr}_{V_b}\left(\left(-ik^\mu - i(k-p)^\mu\right) e^a \overset{k}{\longleftarrow} \left(-i(k-p)^\nu - ik^\nu\right) e^b \overset{k-p}{\longleftarrow} \right)}$$



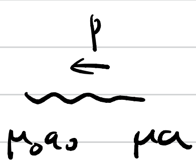


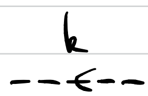
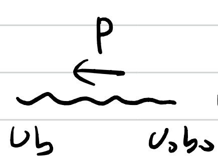
$$= A_{\mu_0 a_0}(x) \int d^4z \, \bar{\phi} A_\mu \delta^{\mu\nu} A_\nu \phi(z) \, A_{\nu_0 b_0}(y) \times 2$$

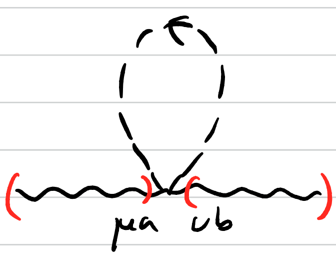
move

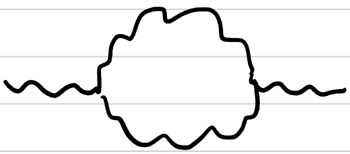
$A_\mu \leftrightarrow A_\nu$

$$= 2 \int d^4z \, A_{\mu_0 a_0}(x) A_{\mu a}(z) \, \delta^{\mu\nu} \, \mathrm{tr}_{V_b}\left(e^a e^b \phi(z) \bar{\phi}(z)\right) A_{\nu b}(z) A_{\nu_0 b_0}(y)$$

$$= \int \frac{d^4p}{(2\pi)^4} e^{-ipx}$$ 

$$2 \int \frac{d^4k}{(2\pi)^4} \delta^{\mu\nu} \, \mathrm{tr}_{V_b}\left(e^a e^b \right.$$  $$\left.\right)$$  $$e^{ipy}$$





3·3·2 terms

$z_1 \leftrightarrow z_2$

$$= A_{\mu_0 a_0}(x)\, \frac{1}{2}\int d^4z_1\, d^4z_2\; V_3^G(z_1)\, V_3^G(z_2)\; A_{\nu_0 b_0}(y) \times 2 \;;$$

$$V_3^G := -\frac{1}{2e^2}\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right)\cdot [A_\mu, A_\nu] = \frac{1}{e^2}\,\partial^\nu A^\mu \cdot [A_\mu, A_\nu]$$

$$= \frac{1}{2}\int d^4z_1\, d^4z_2\; A_{\mu_0 a_0}(x)\, V_3[A,A,A](z_1)\, V_3[A,A,A](z_2)\, A_{\nu_0 b_0}(y) \;;$$

$$V_3[A_1, A_2, A_3] := \frac{1}{e^2}\,\partial^\nu A_1^\mu \cdot [A_{2\mu}, A_{3\nu}] + \text{permutations} \quad (3!\ \text{terms})$$

If we put $A_I = e^a dx^\mu e^{ik_I x} A_{\mu a}(k_I)$ $\quad I = 1, 2, 3$, then

$$V_3[A_1, A_2, A_3]$$

$$= \frac{1}{e^2}\, i k_1^\nu\, e^{a_1} A^\mu_{a_1}(k_1) \cdot \left[e^{a_2} A_{\mu a_2}(k_2),\, e^{a_3} A_{\nu a_3}(k_3)\right] e^{i(k_1+k_2+k_3)x}$$

$$+ \text{permutations}$$

$$= \frac{i}{e^2}\, e^{a_1}\cdot\left[e^{a_2}, e^{a_3}\right] k_1^\nu A^\mu_{a_1}(k_1)\, A_{\mu a_2}(k_2)\, A_{\nu a_3}(k_3)\, e^{i(k_1+k_2+k_3)x}$$

$$+ \text{permutations}$$

$$= V^{\mu_1 a_1\, \mu_2 a_2\, \mu_3 a_3}_{k_1,\, k_2,\, k_3}\; A_{\mu_1 a_1}(k_1)\, A_{\mu_2 a_2}(k_2)\, A_{\mu_3 a_3}(k_3)\, e^{i(k_1+k_2+k_3)x} \;;$$

$$V^{\mu_1 a_1\, \mu_2 a_2\, \mu_3 a_3}_{k_1,\, k_2,\, k_3} := \frac{i}{e^2}\, e^{a_1}\cdot[e^{a_2}, e^{a_3}]\, \delta^{\mu_1\mu_2}\, k_1^{\mu_3} \quad + \text{permutations}$$

<u>Note</u>: The inner product $\cdot$ in $\mathfrak{g}$ is adjoint invariant

$$[X,Y]\cdot Z + Y\cdot[X,Z] = 0$$

and we are assuming $\{e^a\}\subset\mathfrak{g}$ is an orthonormal basis,

$e^a\cdot e^b = \delta^{ab}$. Suppose $[e^a, e^b] = \sum_c e^c f^{cab}$.

the structure constant.

Then, $e^a\cdot[e^b, e^c] = f^{abc}$

It is antisymmetric in $b\leftrightarrow c$ exchange, $f^{abc} = -f^{acb}$.

By the adjoint invariance of $\cdot$, we have

$[e^b, e^a]\cdot e^c + e^a\cdot[e^b, e^c] = 0$. i.e. $f^{cba} + f^{abc} = 0$.

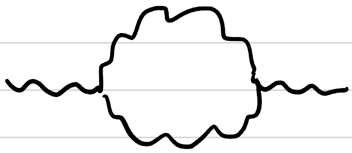
$\Rightarrow f^{bac} = -f^{bca} = f^{acb} = -f^{abc}$.

I.e. f^{abc} is totaly antisymmetric in exchanges of a, b, c.

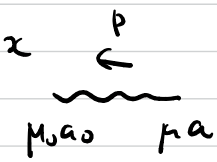
$\Rightarrow$ it is also cyclic invariant, $f^{abc} = f^{bca} = f^{cab}$.

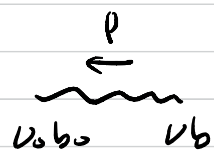
Using this property of $f^{a_1a_2a_3} = e^{a_1}\cdot[e^{a_2}, e^{a_3}]$, we find

$$V^{\mu_1 a_1\, \mu_2 a_2\, \mu_3 a_3}_{k_1,\, k_2,\, k_3}$$
$$= \frac{i}{e^2} f^{a_1a_2a_3}\left\{\delta^{\mu_1\mu_2}(k_1-k_2)^{\mu_3} + \delta^{\mu_2\mu_3}(k_2-k_3)^{\mu_1} + \delta^{\mu_3\mu_1}(k_3-k_1)^{\mu_2}\right\}$$

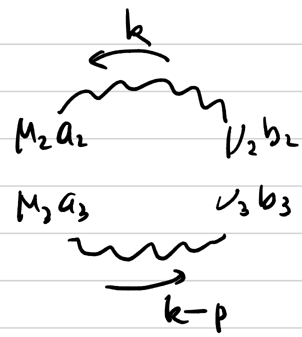
Back to 

$$= \frac{1}{2}\int d^4z_1\, d^4z_2\, A_{\mu_0 a_0}(x)\, V_3[A, A, A](z_1)\, V_3[A, A, A](z_2)\, A_{\nu_0 b_0}(y)$$

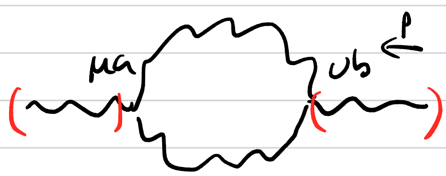
$$= \int \frac{d^4p}{(2\pi)^4}\, e^{-ipx}$$ 

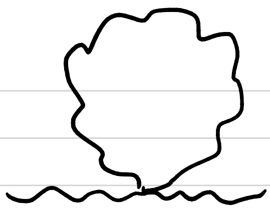
$\times$ 

$e^{ipy}\ \times$

$$\frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\, V^{\mu a\ \mu_2 a_2\ \mu_3 a_3}_{p,\ -k,\ k-p}$$ 

$$V^{\nu_2 b_2\ \nu_3 b_3\ \nu b}_{k,\ -(k-p),\ -p}$$





$$= A_{\mu_0 a_0}(x) \int d^4z \, V_4^G(z) \, A_{\nu_0 b_0}(y) \qquad (4 \cdot 3 \text{ terms}) \; ;$$

$$V_4^G := -\frac{1}{4e^2} [A^\mu, A^\nu] \cdot [A_\mu, A_\nu]$$

$V_4[A, A, A, A]$

$$= \int d^4z \, A_{\mu_0 a_0}(x) \, V_4[A, A, A, A](z) \, A_{\nu_0 b_0}(y) \times \frac{1}{2} \; ;$$

$$V_4[A_1, A_2, A_3, A_4] := -\frac{1}{4e^2} [A_1^\mu, A_2^\nu] \cdot [A_{3\mu}, A_{4\nu}] + \text{permutations}$$

$$(4! = 24 \text{ terms})$$

If we put $A_I = e^a dx^\mu e^{ik_I x} A_{\mu a}(k_I) \quad I = 1, 2, 3, 4$, then

$$V_4[A_1, A_2, A_3, A_4]$$

$$= -\frac{1}{4e^2} (e^{a_1}, e^{a_2}] \cdot [e^{a_3}, e^{a_4}] \, A^\mu_{a_1}(k_1) A^\nu_{a_2}(k_2) A_{\mu a_3}(k_3) A_{\nu a_4}(k_4) \, e^{i(k_1 + \cdots + k_4)x}$$

$$+ \text{permutations}$$

$$= -\frac{1}{4e^2} [e^{a_1}, e^{a_2}] \cdot [e^{a_3}, e^{a_4}] \, \delta^{\mu_1 \mu_3} \, \delta^{\mu_2 \mu_4}$$

$$\cdot A_{\mu_1 a_1}(k_1) \cdots A_{\mu_4 a_4}(k_4) \, e^{i(k_1 + \cdots + k_4)x}$$

$$+ \text{permutations}$$

$(12)\leftrightarrow(34)$ $\quad$ $\overset{\frown}{12}\ \overset{\frown}{34}$

$$= -\frac{1}{4e^2} A_{\mu_1 a_1}(k_1)\cdots A_{\mu_4 a_4}(k_4)\, e^{i(k_1+\cdots+k_4)x} \times 2 \times 2$$

$$\{ [e^{a_1}, e^{a_2}]\cdot[e^{a_3}, e^{a_4}] (\delta^{\mu_1\mu_3}\delta^{\mu_2\mu_4} - \delta^{\mu_1\mu_4}\delta^{\mu_2\mu_3})$$

$$+ [e^{a_1}, e^{a_3}]\cdot[e^{a_2}, e^{a_4}] (\delta^{\mu_1\mu_2}\delta^{\mu_3\mu_4} - \delta^{\mu_1\mu_4}\delta^{\mu_3\mu_2})$$

$$+ [e^{a_1}, e^{a_4}]\cdot[e^{a_3}, a^{a_2}] (\delta^{\mu_1\mu_3}\delta^{\mu_4\mu_2} - \delta^{\mu_1\mu_2}\delta^{\mu_4\mu_3}) \}$$

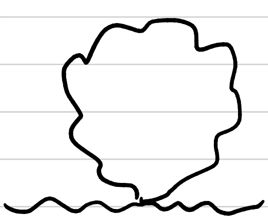
$$= V^{\mu_1 a_1, \cdots, \mu_4 a_4} A_{\mu_1 a_1}(k_1)\cdots A_{\mu_4 a_4}(k_4)\, e^{i(k_1+\cdots+k_4)x} \ ;$$

$$V^{\mu_1 a_1, \cdots, \mu_4 a_4}$$

$$:= -\frac{1}{e^2} \Big\{ \sum_b f^{b a_1 a_2} f^{b a_3 a_4} (\delta^{\mu_1\mu_3}\delta^{\mu_2\mu_4} - \delta^{\mu_1\mu_4}\delta^{\mu_2\mu_3})$$

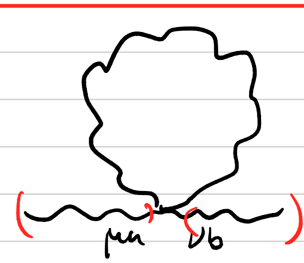
$$+ \sum_b f^{b a_1 a_3} f^{b a_2 a_4} (\delta^{\mu_1\mu_2}\delta^{\mu_3\mu_4} - \delta^{\mu_1\mu_4}\delta^{\mu_3\mu_2})$$

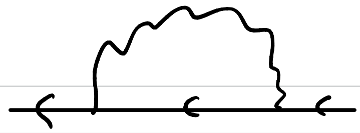
$$+ \sum_b f^{b a_1 a_4} f^{b a_3 a_2} (\delta^{\mu_1\mu_3}\delta^{\mu_4\mu_2} - \delta^{\mu_1\mu_2}\delta^{\mu_4\mu_3}) \Big\}$$



$$= \int \frac{d^4p}{(2\pi)^4} e^{-ipx} \underset{\mu_0 a_0 \quad \mu a}{\overset{\leftarrow p}{\sim\sim\sim}} \cdot \underset{\nu b \quad \nu_0 b_0}{\overset{\leftarrow p}{\sim\sim\sim}} e^{ipy}$$

$$\times \boxed{\frac{1}{2}\int \frac{d^4k}{(2\pi)^4} V^{\mu a, \mu_2 a_2, \mu_3 a_3, \nu b} \underset{\mu_2 a_2 \quad \mu_3 a_3}{\overset{\leftarrow k}{\sim\sim\sim}}}$$



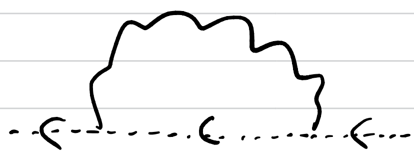
- **Contribution to $\langle \psi(x)\bar{\psi}(y)\rangle$ of** 

$$= \psi(x)\,\frac{1}{2}\int d^4z_1\, d^4z_2\; i\bar{\psi}\not{A}\psi(z_1)\; i\bar{\psi}\not{A}\psi(z_2)\;\bar{\psi}(y)\times 2$$

($\times 2$: $z_1 \leftrightarrow z_2$)

$$= \int\frac{d^4p}{(2\pi)^4}\, e^{-ipx} \xleftarrow{p} \left[\int\frac{d^4k}{(2\pi)^4}\, i\gamma^\mu e^a \xleftarrow{k} i\gamma^\nu e^b\right] \xleftarrow{p} e^{ipy}$$

(boxed part: wavy line from μa to νb, momentum $k-p$)

- **Contribution to $\langle c(x)\bar{c}(y)\rangle$ of** 

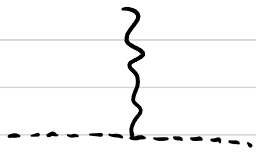
$$= c(x)\,\frac{1}{2}\int d^4z_1\, d^4z_2\; \partial^\mu\bar{c}\,[A_\mu, c](z_1)\; \partial^\nu\bar{c}\,[A_\nu, c](z_2)\;\bar{c}(y)\times 2$$

($\times 2$: $z_1 \leftrightarrow z_2$)

$$= \int\frac{d^4p}{(2\pi)^4}\, e^{-ipx} \xleftarrow{p} \left[\int\frac{d^4k}{(2\pi)^4}\, ip^\mu\, \mathrm{ad}\, e^a \xleftarrow{k} ik^\nu\, \mathrm{ad}\, e^b\right] \xleftarrow{p} e^{ipy}$$

(boxed part: wavy line from μa to νb, momentum $k-p$; ghost propagators dotted)

• Contributions to $\langle C(x)\, A_{\mu_0 b_0}(w)\, \bar{C}(y) \rangle$



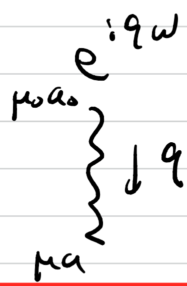
(tree diagram)

$$= C(x) \int d^4z \; \partial^\mu \bar{C}\, [A_\mu, C](z)\; A_{\mu_0 a_0}(w)\; \bar{C}(y)$$

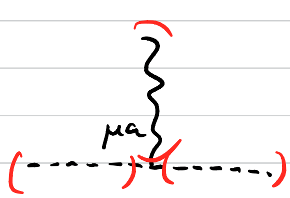
$$= \int d^4z \; C(x)\, \partial^\mu \bar{C}(z)\; \mathrm{ad}\, e^a\; A_{\mu a}(z)\, A_{\mu_0 a_0}(w)\; C(z)\, \bar{C}(y)$$

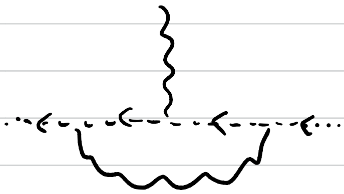
$\int \frac{d^4p_1}{(2\pi)^4} e^{-ip_1(x-z)}$ (p_1, ghost line) $\; i p_1^\mu$ $\qquad$ $\int \frac{d^4q}{(2\pi)^4} e^{-iq(z-w)}$ (q, gluon line μa — $\mu_0 a_0$) $\qquad$ $\int \frac{d^4p_2}{(2\pi)^4} e^{-ip_2(z-y)}$ (p_2, ghost line)

$$\int d^4z \;\rightarrow\; (2\pi)^4\, \delta(p_1 - q - p_2) \quad \leadsto \quad p_2 = p,\; p_1 = p + q$$



$$= \int \frac{d^4p}{(2\pi)^4} \frac{d^4q}{(2\pi)^4}\; e^{-i(p+q)x} \;\overset{p+q}{\longleftarrow}\; \boxed{i(p+q)^\mu\, \mathrm{ad}\, e^a} \;\overset{p}{\longleftarrow}\; e^{iqy}$$





$$= \frac{1}{3!}\int \prod_{i=1}^{3} d^4 z_i$$

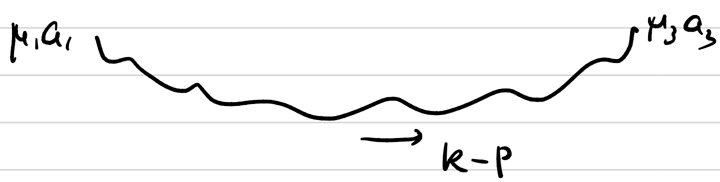
$$C(x)\ \partial^{\mu_1}\bar{C}\,[A_{\mu_1}, C](1)\ \partial^{\mu_2}\bar{C}\,[A_{\mu_2}, C](2)\ \partial^{\mu_3}\bar{C}\,[A_{\mu_3}, C](3)\ A_{\mu_0 a_0}(w)\ \bar{C}(y)$$

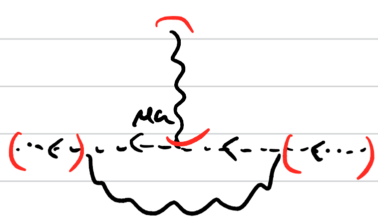
$$\times 3!$$

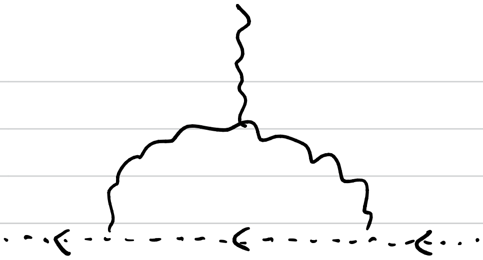
↖ permutation of z_1, z_2, z_3

$$= \int \frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\, e^{-i(p+q)x} \ \overset{p+q}{\cdots\leftarrow\cdots} \circ \qquad e^{iqw}$$

$$\int \frac{d^4k}{(2\pi)^4}\ i(p+q)^{\mu_1}\, ad\, e^{a_1} \overset{k+q}{\cdots\leftarrow\cdots}\ i(k+q)^{\mu}\, ad\, e^{a} \overset{k}{\cdots\leftarrow\cdots}\ ik^{\mu_3}\, ad\, e^{a_3} \ \circ \overset{p}{\cdots\leftarrow\cdots}\ e^{ipy}$$





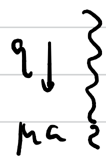


$$= \frac{1}{2!}\int d^4z_1\, d^4z_2\, d^4z_3$$

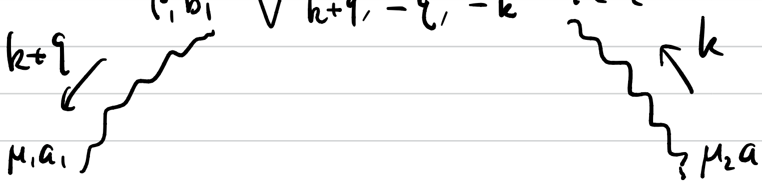
$$C(x)\ \partial^{\mu_1}\bar{C}\cdot[A_{\mu_1}, C](1)\ V_3[A,A,A](3)\ \partial^{\mu_2}\bar{C}\cdot[A_{\mu_2}, C](2)\ A_{\mu_0 a_0}(w)\ \bar{C}(y)$$

$\times 2$ ↖ $z_1 \leftrightarrow z_2$

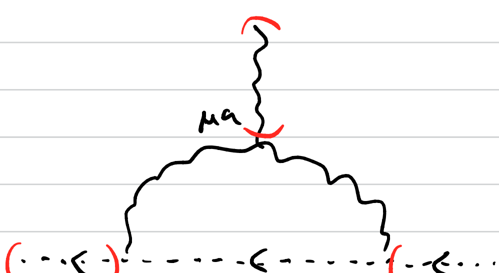
$$= \int \frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\ e^{-i(p+q)x}\ \overset{p+q}{\cdots\leftarrow\cdots\circ}\qquad \overset{\mu_0 a_0}{}\ e^{iqw}$$



$$V^{\rho_1 b_1\ \mu a\ \rho_2 b_2}_{k+q,\ -q,\ -k}$$



$$\int\frac{d^4k}{(2\pi)^4}\ i(p+q)^{\mu_1}\mathrm{ad}\,e^{a_1}\ \overset{p-k}{\cdots\leftarrow\cdots}\ i(p-k)^{\mu_2}\mathrm{ad}\,e^{a_2}\qquad \overset{p}{\circ\cdots\leftarrow\cdots}\ e^{ipy}$$



Some useful facts on simple Lie algebra $\mathfrak{g}$

$\{e^a\} \subset \mathfrak{g}$ as before $\left(\begin{array}{l}\text{an orthonormal basis wrt an adjoint inv} \\ \text{inner product } e^a \cdot e^b = \delta^{ab},\ [e^a, e^b] = \sum_c e^c f^{cab} \\ \rightsquigarrow f^{abc} \text{ totally antisymmetric}\end{array}\right)$

For any representation V of $\mathfrak{g}$

$$\mathrm{tr}_V(e^a e^b) = -T_V \delta^{ab}, \quad T_V \in \mathbb{R}_{\geq 0}$$

$$\sum_a e^a e^a = -C_2(V): \quad \text{a scalar (quadratic Casimir) on each irreducible component}$$

- If V is irreducible, $T_V \dim G = C_2(V) \dim V$
- $V = \mathfrak{g}$ (adjoint rep), $T_{\mathfrak{g}} = C_2(\mathfrak{g}) =: h^\vee$ is called the dual Coxeter number of $\mathfrak{g}$ for a suitable normalization of "$\cdot$".

- $\sum_b e^b e^a e^b = \sum_b e^b [e^a, e^b] + \sum_b e^b e^b e^a$ (with $\sum_b e^b e^b = -C_2(V)$)

$$\sum_{bc} e^b e^c f^{cab} = \frac{1}{2}\sum_{b,c} [e^b, e^c] f^{cab} = \frac{1}{2}\sum_{b,c,d} e^d f^{dbc} f^{cab}$$

($f^{dbc} f^{cab}$ summed: $-f^{bdc}$)

$$= -\frac{1}{2}\sum_d e^d\, \mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}\, e^d\, \mathrm{ad}\, e^a) = \frac{1}{2} h^\vee e^a$$

(where $\mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}\, e^d\, \mathrm{ad}\, e^a) = -T_{\mathfrak{g}} \delta^{da} = -h^\vee \delta^{da}$)

$$= \left(\frac{h^\vee}{2} - C_2(V)\right) e^a$$

$\overset{V=\mathfrak{g}}{\rightsquigarrow}$ $\sum_b \mathrm{ad}\, e^b\, \mathrm{ad}\, e^a\, \mathrm{ad}\, e^b = -\frac{h^\vee}{2} \mathrm{ad}\, e^a$

We need to evaluate the integrals of the form

$$I(f) = \int \frac{d^4k}{(2\pi)^4} \frac{f(k)}{(k^2+m^2)((k-p)^2+\mu^2)},$$

$$J(g) = \int \frac{d^4k}{(2\pi)^4} \frac{g(k)}{(k+q)^2 k^2 (k-p)^2},$$

polynomials of k^μ's

which are often divergent. We shall employ the dimensional regularization in which these are replaced by

$$I_{DR}(f) = M_{DR}^{4-d} \int \frac{d^dk}{(2\pi)^d} \frac{f(k)}{(k^2+m^2)((k-p)^2+\mu^2)},$$

$$J_{DR}(g) = M_{DR}^{4-d} \int \frac{d^dk}{(2\pi)^d} \frac{g(k)}{(k+q)^2 k^2 (k-p)^2},$$

with $d = 4-\epsilon$ (eventually we take $\epsilon \to 0$). We use

$$\frac{1}{AB} = \int_0^1 \frac{dx}{[(1-x)A + xB]^2}$$

$$\frac{1}{ABC} = \int_{\triangle} \frac{2\,dy\,dz}{((1-y-z)A + yB + zC)^3}$$

where $\triangle := \{ (y,z) \in \mathbb{R}^2 \mid y \geq 0,\ z \geq 0,\ y+z \leq 1 \}$

Computation goes as follows

$$I_{DR}(f) = \mu_{DR}^{4-d} \int \frac{d^d k}{(2\pi)^d} \int_0^1 \frac{dx\, f(k)}{\underbrace{\left((1-x)(k^2+m^2) + x((k-p)^2+\mu^2)\right)}^2}$$

$$k^2 - 2xpk + xp^2 + (1-x)m^2 + x\mu^2$$

$$= (\underbrace{k - xp}_{\ell})^2 + \underbrace{x(1-x)p^2 + (1-x)m^2 + x\mu^2}_{\Delta}$$

$$= \int_0^1 dx\, \mu_{DR}^{4-d} \int \frac{d^d \ell}{(2\pi)^d} \frac{f(\ell + xp)}{(\ell^2 + \Delta)^2}$$

- We expand $f(\ell + xp)$ in ℓ^μ's, drop odd power terms and replace even power terms by a function of ℓ^2

$f(\ell + xp) \to \tilde{f}(\ell^2, xp)$. (e.g. $\ell^\mu \ell^\nu \to \frac{1}{d} \delta^{\mu\nu} \ell^2$)

- Use $\displaystyle\int \frac{d^d \ell}{(2\pi)^d} F(\ell^2) = \frac{\mathrm{Vol}(S^{d-1})}{(2\pi)^d} \int_0^\infty \ell^{d-1} d\ell\, F(\ell^2)$

$$= \frac{\mathrm{Vol}(S^{d-1})}{2(2\pi)^d} \int_0^\infty \ell^{d-2} d\ell^2\, F(\ell^2)$$

$$= \frac{1}{(4\pi)^{d/2} \Gamma(d/2)} \int_0^\infty t^{\frac{d}{2}-1} dt\, F(t)$$

$$I_{DR}(f) = \frac{\mu_{DR}^{4-d}}{(4\pi)^{d/2}\,\Gamma(d/2)} \int_0^1 dx \int_0^\infty \frac{t^{\frac{d}{2}-1} dt\, \tilde{f}(t, xp)}{(t+\Delta)^2}$$

We may use $\displaystyle\int_0^\infty \frac{t^{p-1}\,dt}{(t+\Delta)^{p+q}} = \frac{B(p,q)}{\Delta^q} = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)\Delta^q}$

In this way we find

$$I_{DR}(f) = \frac{1}{(4\pi)^2} \int_0^1 dx \left(\frac{4\pi\mu_{DR}^2}{\Delta}\right)^{2-\frac{d}{2}} \Gamma\left(2-\frac{d}{2}\right) \hat{f}$$

where $\Delta = x(1-x)p^2 + (1-x)m^2 + x\mu^2$ and

$$\hat{1} = 1, \quad \widehat{k^\mu} = x p^\mu$$

$$\widehat{k^\mu k^\nu} = \delta^{\mu\nu} \frac{\Delta}{2-d} + x^2 p^\mu p^\nu$$

$$\widehat{k^\mu (k-p)^\nu} = \delta^{\mu\nu} \frac{\Delta}{2-d} - x(1-x) p^\mu p^\nu$$

$$\widehat{(k-p)^\mu (k-p)^\nu} = \delta^{\mu\nu} \frac{\Delta}{2-d} + (1-x)^2 p^\mu p^\nu$$

Similarly (exercise):

$$J_{DR}(g) = \frac{1}{(4\pi)^2}\int_{\triangle} dy\,dz \left(\frac{4\pi\mu_{DR}^2}{\Delta}\right)^{2-\frac{d}{2}} \frac{\Gamma(3-\frac{d}{2})}{\Delta}\,\hat{g}$$

where $\Delta = y(1-y)q^2 + z(1-z)p^2 + 2yzqp$ and

$$\hat{1} = 1, \quad \hat{k^\mu} = -yq^\mu + zp^\mu$$

$$\widehat{k^\mu k^\nu} = \delta^{\mu\nu}\frac{\Delta}{4-d} + (-yq^\mu + zp^\mu)(-yq^\nu + zp^\nu)$$

Note: $\Gamma(3-\frac{d}{2}) = (2-\frac{d}{2})\Gamma(2-\frac{d}{2}) = \frac{1}{2}(4-d)\,\Gamma(2-\frac{d}{2})$

$$\therefore J_{DR}(g) = \frac{1}{(4\pi)^2}\int_{\triangle} dy\,dz \left(\frac{4\pi\mu_{DR}^2}{\Delta}\right)^{2-\frac{d}{2}} \Gamma(2-\tfrac{d}{2})\,\hat{\hat{g}}$$

$$\hat{\hat{1}} = \frac{4-d}{2\Delta}, \quad \hat{\hat{k^\mu}} = \frac{4-d}{2\Delta}(-yq^\mu + zp^\mu)$$

$$\widehat{\widehat{k^\mu k^\nu}} = \frac{1}{2}\delta^{\mu\nu} + \frac{4-d}{2\Delta}(-yq^\mu + zp^\mu)(-yq^\nu + zp^\nu)$$