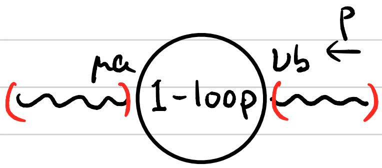
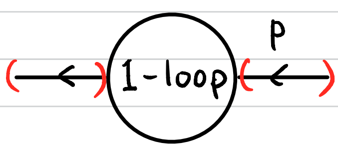
The result



$$= -\left(\delta^{\mu\nu}p^2 - p^\mu p^\nu\right)\delta^{ab}\frac{1}{(4\pi)^2}\int_0^1 dx\,\Gamma\!\left(2-\frac{d}{2}\right)\times$$

$$\underbrace{\boxed{\begin{aligned}\Big\{ & \tilde{\Delta}_{m_f}^{\frac{d}{2}-2}\, T_{V_f}\, 8x(1-x) \\ & + \tilde{\Delta}_{m_b}^{\frac{d}{2}-2}\, T_{V_b}\, (2x-1)^2 \\ & - \tilde{\Delta}_{0}^{\frac{d}{2}-2}\, h^\vee \left(\left(1-\frac{d}{2}\right)(2x-1)^2+2\right)\Big\}\end{aligned}}}_{\star A}$$

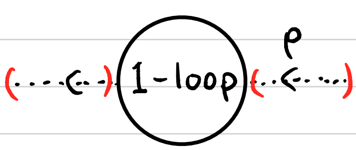
where $\tilde{\Delta}_m := \left(x(1-x)p^2 + m^2\right)/4\pi\mu_{DR}^2$



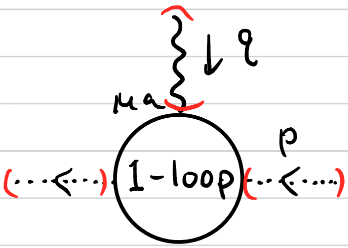
$$= C_2(V_f)\,\frac{e^2}{(4\pi)^2}\int_0^1 dx\,\Gamma\!\left(2-\frac{d}{2}\right)\tilde{\Delta}_f^{\frac{d}{2}-2}$$

$$\left\{(d-2)\,x\not{p} - d\,m_f\right\}$$

where $\tilde{\Delta}_f := \left(x(1-x)p^2 + (1-x)m_f^2\right)/4\pi\mu_{DR}^2$



$$= -h^\vee\,\frac{e^2}{(4\pi)^2}\int_0^1 dx\,\Gamma\!\left(2-\frac{d}{2}\right)\tilde{\Delta}_0^{\frac{d}{2}-2}\,x\,p^2$$



$$= i h^{\nu} a d e^{a} \cdot \frac{e^2}{(4\pi)^2} \int_{\triangle} dy\, dz\, \Gamma\left(2-\frac{d}{2}\right) \tilde{\Delta}_v^{\frac{d}{2}-2}$$

$$\left\{ (p+q)^\mu \frac{d}{4} + (4-d) \frac{p^\mu R + q^\mu S}{\Delta_v} \right\}$$

where $\triangle := \left\{ (y,z) \in \mathbb{R}^2 \;\middle|\; \begin{matrix} y, z \geq 0 \\ y+z \leq 1 \end{matrix} \right\}$

$$\Delta_v := y(1-y)q^2 + z(1-z)p^2 + 2yz\, qp$$

$$\tilde{\Delta}_v = \Delta_v / 4\pi\mu_{OR}^2$$

R, S : linear in q^2, p^2 & $q{\cdot}p$,

and at most linear in y & z.

$\therefore$ The functions that appear in the renormalization condition are, at 1-loop,

$$\Pi(p^2) = \frac{1}{e^2} + \frac{1}{(4\pi)^2}\int_0^1 dx\,\Gamma\left(2-\frac{d}{2}\right)\star_A + \left(\frac{Z_A}{e_0^2}\right)^{(1)}$$

$$A(p^2) = -1 - C_2(V_f)\frac{e^2}{(4\pi)^2}\int_0^1 dx\,\Gamma\left(2-\frac{d}{2}\right)\tilde{\Delta}_f^{\frac{d}{2}-2}(d-2)x - Z_\psi^{(1)}$$

$$B(p^2) = m_f + C_2(V_f)\frac{e^2}{(4\pi)^2}\int_0^1 dx\,\Gamma\left(2-\frac{d}{2}\right)\tilde{\Delta}_f^{\frac{d}{2}-2}\,d\,m_f + (Z_\psi m_0)^{(1)}$$

$$\Pi_{gh}(p^2) = -1 + h^\vee\frac{e^2}{(4\pi)^2}\int_0^1 dx\,\Gamma\left(2-\frac{d}{2}\right)\tilde{\Delta}_0^{\frac{d}{2}-2}x - Z_c^{(1)}$$

$$C_0(p,q) = -i - i h^\vee\frac{e^2}{(4\pi)^2}\int_{\triangle} dy\,dz\,\Gamma\left(2-\frac{d}{2}\right)\tilde{\Delta}_v^{\frac{d}{2}-2}\left(\frac{d}{4} + (4-d)\frac{R}{\Delta_v}\right) - i\left(Z_c^{(1)} + \sqrt{Z_A}^{(1)}\right).$$

The last terms are the contributions by the one-loop counter terms. The renormalization condition

$$\Pi(\mu^2) = \frac{1}{e^2},\quad A(\mu^2) = -1,\quad B(\mu^2) = m_f,\quad \Pi_{gh}(\mu^2) = -1$$

$$C_0(p,q)\big|_{p^2=q^2=(p+q)^2=\mu} = -i$$

determines them:

$$\left(\frac{Z_A}{e_0^2}\right)^{(1)} = -\frac{1}{(4\pi)^2}\int_0^1 dx\, \Gamma\left(2-\frac{d}{2}\right) \star_A \Big|_{p^2=\mu^2}$$

$$Z_\psi^{(1)} = -C_2(V_f)\frac{e^2}{(4\pi)^2}\int_0^1 dx\, \Gamma\left(2-\frac{d}{2}\right) \tilde{\Delta}_f^{\frac{d}{2}-2} (d-2)\, x \Big|_{p^2=\mu^2}$$

$$(Z_\psi m_0)^{(1)} = -C_2(V_f)\frac{e^2}{(4\pi)^2}\int_0^1 dx\, \Gamma\left(2-\frac{d}{2}\right) \tilde{\Delta}_f^{\frac{d}{2}-2}\, d\, m_f \Big|_{p^2=\mu^2}$$

$$Z_c^{(1)} = h^v \frac{e^2}{(4\pi)^2}\int_0^1 dx\, \Gamma\left(2-\frac{d}{2}\right) \tilde{\Delta}_0^{\frac{d}{2}-2}\, x \Big|_{p^2=\mu^2}$$

$$Z_c^{(1)} + \sqrt{Z_A}^{(1)}$$

$$= -h^v \frac{e^2}{(4\pi)^2}\int_{\triangle} dy\, dz\, \Gamma\left(2-\frac{d}{2}\right) \tilde{\Delta}_v^{\frac{d}{2}-2}\left(\frac{d}{4} + (4-d)\frac{R}{\Delta_v}\right)\Bigg|_{\substack{p^2=q^2\\ =(p+q)^2=\mu^2}}$$

Note: $\Gamma\left(2-\frac{d}{2}\right)\tilde{\Delta}^{\frac{d}{2}-2}\left(\alpha + (d-4)\beta\right)$ for $d=4-\epsilon$

$$= \Gamma\left(\frac{\epsilon}{2}\right)\tilde{\Delta}^{-\frac{\epsilon}{2}}(\alpha + \epsilon\beta)$$

$$= \left(\frac{2}{\epsilon} - \gamma + O(\epsilon)\right)\left(1 - \frac{\epsilon}{2}\log\tilde{\Delta} + O(\epsilon)\right)(\alpha + \epsilon\beta)$$

$$= \left(\frac{2}{\epsilon} - \gamma - \log\tilde{\Delta}\right)\alpha + 2\beta + O(\epsilon)$$

Let us first consider the case $m_f, m_b \ll \mu$ and ignore $O(m_f^2/\mu^2, m_b^2/\mu^2)$ terms. Then, all of $\log\tilde{\Delta}$'s at the renormalization point $= \log\mu^2 + \mu$-independent.

Thus, with $X := \frac{2}{\epsilon} - \log\mu^2$,

$$\left(\frac{Z_A}{e_0^2}\right)^{(1)} = -\frac{1}{(4\pi)^2}\int_0^1 dx\, X \left\{ T_{V_f}\, 8x(1-x) + T_{V_b}(2x-1)^2 - h^v\left(-(2x-1)^2+2\right)\right\}$$

$$+ \cdots$$

μ independent terms $+ O(\epsilon)$

$$= -\frac{1}{(4\pi)^2} X \left\{ \frac{4}{3} T_{V_f} + \frac{1}{3} T_{V_b} - \frac{5}{3} h^v \right\} + \cdots$$

$$Z_\psi^{(1)} = -C_2(V_f)\frac{e^2}{(4\pi)^2}\underbrace{\int_0^1 dx\, X\, 2x}_{=1} + \cdots$$

$$(Z_\psi m_0)^{(1)} = -C_2(V_f)\frac{e^2}{(4\pi)^2}\underbrace{\int_0^1 dx}_{=1} X\, 4m_f + \cdots$$

$$Z_c^{(1)} = h^v \frac{e^2}{(4\pi)^2}\underbrace{\int_0^1 dx\, X\, x}_{\frac{1}{2}} + \cdots$$

$$Z_c^{(1)} + \sqrt{Z_A}^{(1)} = -h^v \frac{e^2}{(4\pi)^2}\underbrace{\int_{\triangle} dy\, dz}_{\frac{1}{2}}\, X + \cdots$$

$$\Rightarrow \sqrt{Z_A}^{(1)} = -h^\vee \frac{e^2}{(4\pi)^2} X + \cdots, \qquad Z_A^{(1)} = -2h^\vee \frac{e^2}{(4\pi)^2} X + \cdots$$

$$\left(\frac{Z_A}{e_0^2}\right)^{(1)} = \left(\frac{1}{e_0^2}\right)^{(1)} + \frac{Z_A^{(1)}}{e^2} \Rightarrow$$

$$\left(\frac{1}{e_0^2}\right)^{(1)} = -\frac{1}{(4\pi)^2} X \left(\frac{4}{3} T_{V_f} + \frac{1}{3} T_{V_b} \overbrace{- \frac{5}{3} h^\vee - 2h^\vee}^{-\frac{11}{3} h^\vee} \right) + \cdots$$

$$\therefore \frac{1}{e_0^2} = \frac{1}{e^2} - \frac{1}{(4\pi)^2}\left(\frac{2}{\epsilon} - \log\mu^2\right)\left(\frac{4}{3} T_{V_f} + \frac{1}{3} T_{V_b} - \frac{11}{3} h^\vee\right) + \cdots$$

$\uparrow$ μ-indep & higher loop

$$(Z_\psi m_0)^{(1)} = Z_\psi^{(1)} m_f + m_0^{(1)} \Rightarrow$$

$$m_0^{(1)} = -3\, C_2(V_f) \frac{e^2}{(4\pi)^2} X m_f + \cdots$$

$$\therefore m_0 = m_f - 3\, C_2(V_f) \frac{e^2}{(4\pi)^2}\left(\frac{2}{\epsilon} - \log\mu^2\right) m_f + \cdots$$

Using these, we can determine the β & γ functions of the renormalization group, at 1-loop:

$$0 = \mu \frac{d}{d\mu}\left(\frac{1}{e_0^2}\right) = \underbrace{\mu \frac{d}{d\mu}\left(\frac{1}{e^2}\right)}_{-2e^{-3}\,\mu \frac{d}{d\mu} e} + \frac{2}{(4\pi)^2}\left(\frac{4}{3} T_{V_f} + \frac{1}{3} T_{V_b} - \frac{11}{3} h^\vee\right) + \cdots \text{(higher loop)}$$

$$0 = \mu \frac{d}{d\mu} m_0 = \underbrace{\mu \frac{d}{d\mu} m_f}_{-\gamma_{m_f} m_f} + 6\, C_2(V_f) \frac{e^2}{(4\pi)^2} + \cdots$$

$$0 = \mu \frac{d}{d\mu} \phi_0 = \mu \frac{d}{d\mu} Z_\phi^{\frac{1}{2}} \cdot \phi + Z_\phi^{\frac{1}{2}} \underbrace{\mu \frac{d\phi}{d\mu}}_{-\gamma_\phi \phi} + \cdots$$

$$\beta = \mu \frac{d}{d\mu} e = \frac{e^3}{(4\pi)^2}\left(\frac{4}{3} T_{V_f} + \frac{1}{3} T_{V_b} - \frac{11}{3} h^\vee\right) + \cdots$$

$$\gamma_{m_f} = -\mu \frac{d}{d\mu} \log m_f = 6\, C_2(V_f) \frac{e^2}{(4\pi)^2} + \cdots$$

$$\gamma_A = \mu \frac{d}{d\mu} \log \sqrt{Z_A} = 2 h^\vee \frac{e^2}{(4\pi)^2} + \cdots$$

$$\gamma_\psi = \mu \frac{d}{d\mu} \log \sqrt{Z_\psi} = C_2(V_f) \frac{e^2}{(4\pi)^2} + \cdots$$

$$\gamma_c = \mu \frac{d}{d\mu} \log \sqrt{Z_c} = -\frac{1}{2} h^\vee \frac{e^2}{(4\pi)^2} + \cdots$$

For general m_f & m_b,

$$\beta = \frac{e^3}{(4\pi)^2}\left(T_{V_f} F_f(m_f,\mu) + T_{V_b} F_b(m_b,\mu) - \frac{11}{3} h^v \right) + \cdots$$

$$F_f(m_f,\mu) = \int_0^1 dx \, \frac{x(1-x)\mu^2}{x(1-x)\mu^2 + m_f^2} \, 8x(1-x) \sim \begin{cases} 4/3 & m_f \ll \mu \\ 0 & m_f \gg \mu \end{cases}$$

$$F_b(m_f,\mu) = \int_0^1 dx \, \frac{x(1-x)\mu^2}{x(1-x)\mu^2 + m_b^2} \, (2x-1)^2 \sim \begin{cases} 1/3 & m_b \ll \mu \\ 0 & m_b \gg \mu \end{cases}$$

$$\gamma_{m_f} = C_2(V_f)\frac{e^2}{(4\pi)^2}\int_0^1 dx \, \frac{2x\mu^2}{x\mu^2 + m_f^2}(4-2x) \sim \begin{cases} 6C_2(V_f)\frac{e^2}{(4\pi)^2} & m_f \ll \mu \\ 0 & m_f \gg \mu \end{cases}$$

$$\gamma_\psi = C_2(V_f)\frac{e^2}{(4\pi)^2}\int_0^1 dx \, \frac{2x\mu^2}{x\mu^2 + m_f^2}\, x \sim \begin{cases} C_2(V_f)\frac{e^2}{4\pi^2} & m_f \ll \mu \\ 0 & m_f \gg \mu \end{cases}$$

γ_A, γ_c remain the same.

Heavy fields ($m \gg \mu$) do not contribute to the running of the gauge coupling constant at scale μ.

(They must be integrated out and are absent in the effective theory at scale μ.)

Only the light fields ($m \ll \mu$) contribute.

Closer look at

$$\beta = -\frac{e^3}{(4\pi)^2}\underbrace{\left(\frac{11}{3}h^\vee - \frac{4}{3}T_{V_f} - \frac{1}{3}T_{V_b}\right)}_{b_1} + \cdots$$

$$\mu\frac{d}{d\mu}\left(\frac{1}{e^2}\right) = \frac{2b_1}{(4\pi)^2} + O(e^2)$$

$$\frac{1}{e^2(\mu)} - \frac{1}{e^2(\mu_0)} = \frac{b_1}{8\pi^2}\log\left(\mu/\mu_0\right) + \cdots$$

$$e^2(\mu) = \frac{e^2(\mu_0)}{1 + \frac{e^2(\mu_0)}{8\pi^2} b_1 \log\left(\mu/\mu_0\right) + \cdots}$$

Suppose $b_1 > 0$

$\mu \to \infty$: $e(\mu) \to 0$

UV free ...... "asymptotic freedom"

Or equivalently,

μ smaller : $e(\mu)$ larger

gauge coupling is stronger at lower energies.

At some $\mu = \Lambda$, it diverges :

$$1 + \frac{e^2(\mu_0)}{8\pi^2} b_1 \log(\Lambda/\mu_0) = 0 \quad \text{i.e.} \quad \Lambda = \mu_0 \exp\left(-\frac{8\pi^2}{b_1 e^2(\mu_0)}\right)$$

... characteristic energy of the system, the "dynamical scale".

(NOT to be confused with the UV cut-off !)

It determines how the gauge coupling runs :

$$\exp\left(-\frac{8\pi^2}{e^2(\mu)}\right) = \left(\frac{\Lambda}{\mu}\right)^{b_1} + \cdots$$

Λ, not e, is the appropriate parameter of the theory.

.... "dimensional transmutation"

At $\mu \ll \Lambda$, $e^2(\mu) \gg 1$: "gauge theory" is NOT a good description !

$\boxed{\text{If } b_1 > 0, \text{ finding the low energy behaviour of the system is a difficult problem. "Interesting theory"}}$

When is $b_1 > 0$?

More general systems

- General gauge group G:

$\{e^a\} \subset \mathfrak{g}$ an arbitrary basis, $F_{\mu\nu} = \sum_a e^a F_{a\mu\nu}$,

$$\mathcal{L}_E = \frac{1}{4} \sum_{a,b} \left(\frac{1}{e^2}\right)^{ab} F_a^{\mu\nu} F_{b\mu\nu} + \cdots$$

$$\mu \frac{d}{d\mu} \left(\frac{1}{e^2}\right)^{ab} = \frac{1}{8\pi^2} \left(\frac{11}{3} T_{\mathfrak{g}}^{ab} - \frac{4}{3} T_{V_f}^{ab} - \frac{1}{3} T_{V_b}^{ab} \right)$$

$$T_V^{ab} = -\mathrm{tr}_V (e^a e^b)$$

For $G = \prod_\alpha U(1)_\alpha \times \prod_I G_I$ ← simple factors, with

$$\mathcal{L}_E = \sum_{\alpha,\beta} \frac{1}{4 e^2_{\alpha,\beta}} F_\alpha^{\mu\nu} F_{\beta\mu\nu} + \sum_I \frac{1}{4 e_I^2} F_I^{\mu\nu} \cdot F_{I\mu\nu} + \cdots$$

- $b_1^I = \frac{11}{3} h_I^\vee - \frac{4}{3} T_{V_f^I} - \frac{1}{3} T_{V_b^I}$

$V^I = V$ as a representation of G_I

- $\mu \frac{d}{d\mu} \frac{8\pi^2}{e^2_{\alpha,\beta}} = -\frac{4}{3} T_{V_f}^{\alpha\beta} - \frac{1}{3} T_{V_b}^{\alpha\beta}$

$T_V^{\alpha\beta} = \sum_i Q_i^\alpha Q_i^\beta$ ($\{Q_i^\alpha\}$ $U(1)_\alpha$ charges of V)

(If there is just a single $U(1)$ factor,

$$b_1^{U(1)} = -\frac{4}{3} T_{V_f}^{U(1)} - \frac{1}{3} T_{V_b}^{U(1)},$$

$$T_V^{U(1)} = \sum_i Q_i^2 .$$
)

- Dirac fermion ψ

  ⇝ Majorana fermion ($\psi^c = \psi$):

  V_f must be a representation$/\mathbb{R}$

  or ⇝ chiral fermion ($\psi = \psi_L$ or ψ_R):

  ※ anomaly must be absent in the full system

  Contribution to T_{V_f} is halved.

- Complex scalar ϕ

  ⇝ real scalar ($\bar{\phi} = \phi$):

  V_b must be a representation$/\mathbb{R}$.

  Contribution to T_{V_b} is halved.

Examples

- QED

charge Q_i electron

$$G = U(1), \quad V_f = \bigoplus_i \mathbb{C}(Q_i), \quad V_b = 0$$

$$b_1 = -\frac{4}{3}\sum_i Q_i^2 < 0$$

$\therefore$ Infra-red free.

- Pure Yang-Mills theory

$$G = \text{simple}, \quad V_f = V_b = 0.$$

$$b_1 = \frac{11}{3}h^\vee > 0$$

$\therefore$ Asymptotically free.

- $SU(N_c)$ QCD with N_f flavors

$$G = SU(N_c), \quad V_f = (\mathbb{C}^{N_c})^{\oplus N_f}, \quad V_b = 0$$

$$h^\vee = N_c, \quad T_{\mathbb{C}^N} = \frac{1}{2} \quad \therefore T_{V_f} = \frac{1}{2}\times N_f.$$

$$b_1 = \frac{11}{3}N_c - \frac{4}{3}\cdot\frac{N_f}{2} = \frac{1}{3}(11N_c - 2N_f)$$

$\therefore$ Asymptotically free if $N_f < \frac{11}{2}N_c$.

• The standard model $G = SU(3)\times SU(2)\times U(1)$

Leptons

$$V_{f,L} = \big[(\mathbb{1},\mathbb{2},-\tfrac{1}{2})\oplus(\mathbb{3},\mathbb{2},\tfrac{1}{6})\big]^{\oplus 3}$$

Quarks

generations

$$V_{f,R} = \big[(\mathbb{1},\mathbb{1},-1)\oplus(\mathbb{3},\mathbb{1},\tfrac{2}{3})\oplus(\mathbb{3},\mathbb{1},-\tfrac{1}{3})\big]^{\oplus 3}$$

$$V_b = (\mathbb{1},\mathbb{2},\tfrac{1}{2})$$

Higgs

$$SU(3):\quad T_{V_f} = \overset{L}{\frac{1}{2}}\Big[\overset{T_{C_3}}{\frac{1}{2}}\times\overset{\mathbb{2}}{2}\Big]\times\overset{\text{gen}}{3} + \overset{R}{\frac{1}{2}}\Big[\overset{T_{C_3}}{\frac{1}{2}}\times\overset{\mathbb{1}\oplus\mathbb{1}}{(1+1)}\Big]\times\overset{\text{gen}}{3} = 3\,,\quad T_{V_b} = 0$$

$$b_1^{SU(3)} = \frac{11}{3}\cdot\overset{h^\vee}{3} - \frac{4}{3}\cdot 3 = 11-4 = 7 > 0 \qquad \underline{\text{A.F.}}$$

$$SU(2):\quad T_{V_f} = \overset{L}{\frac{1}{2}}\Big[\overset{T_{C_2}}{\frac{1}{2}}\times\overset{\mathbb{1}\oplus\mathbb{3}}{(1+3)}\Big]\times\overset{\text{gen}}{3} = 3\,,\quad T_{V_b} = \overset{T_{C_2}}{\frac{1}{2}}\times\overset{\mathbb{1}}{1} = \frac{1}{2}$$

$$b_1^{SU(2)} = \frac{11}{3}\cdot\overset{h^\vee}{2} - \frac{4}{3}\cdot 3 - \frac{1}{3}\cdot\frac{1}{2} = \frac{19}{6} = 3.166.. > 0 \qquad \underline{\text{A.F.}}$$

$$U(1):\quad T_{V_f} = \overset{L}{\frac{1}{2}}\Big[\big(-\tfrac{1}{2}\big)^2\times\overset{(\mathbb{1},\mathbb{2})}{2} + \big(\tfrac{1}{6}\big)^2\times\overset{(\mathbb{3},\mathbb{2})}{3\cdot 2}\Big]\times\overset{\text{gen}}{3}$$

$$+\overset{R}{\frac{1}{2}}\Big[(-1)^2\times\overset{(\mathbb{1},\mathbb{1})}{1} + \big(\tfrac{2}{3}\big)^2\times\overset{(\mathbb{3},\mathbb{1})}{3} + \big(-\tfrac{1}{3}\big)^2\times\overset{(\mathbb{3},\mathbb{1})}{3}\Big]\times\overset{\text{gen}}{3} = 5$$

$$T_{V_b} = \big(\tfrac{1}{2}\big)^2\times\overset{(\mathbb{1},\mathbb{2})}{2} = \frac{1}{2}$$

$$b_1^{U(1)} = -\frac{4}{3}\cdot 5 - \frac{1}{3}\cdot\frac{1}{2} = -\frac{41}{6} = -6.833.. < 0 \qquad \underline{\text{IR free}}$$

- $SO(N_c)$ QCD with N_f flavors

$$G = SO(N_c),\quad V_f = (\mathbb{R}^{N_c})^{\oplus N_f}\ (\text{Majorana}),\quad V_b = 0$$

$$h^\vee = N_c - 2,\quad T_{\mathbb{C}^N} = 1 \quad \therefore\ T_{V_f} = 1 \times N_f \times \frac{1}{2} = \frac{N_f}{2}$$

$$b_1 = \frac{11}{3}(N_c - 2) - \frac{4}{3}\cdot\frac{N_f}{2} = \frac{1}{3}(11N_c - 22 - 2N_f)$$

$\therefore$ Asymptotically free if $N_f < \frac{11}{2}N_c - 11$

- $USp(N_c)$ QCD with N_f flavors ($N_c = 2\nu$, N_f both even)

$$G = USp(N_c) = \left\{ g \in U(N_c) \,\middle|\, g^T \begin{pmatrix} 0 & -\mathbb{1}_\nu \\ \mathbb{1}_\nu & 0 \end{pmatrix} g = \begin{pmatrix} 0 & -\mathbb{1}_\nu \\ \mathbb{1}_\nu & 0 \end{pmatrix} \right\}$$

$$h^\vee = \nu + 1 = \frac{1}{2}(N_c + 2),\quad T_{\mathbb{C}^{N_c}} = \frac{1}{2}$$

$$V_f = (\mathbb{C}^{N_c})^{\oplus N_f/2},\quad V_b = 0$$

$$T_{V_f} = \frac{1}{2} \times \frac{N_f}{2} = \frac{N_f}{4}$$

$$b_1 = \frac{11}{3}\cdot\frac{N_c+2}{2} - \frac{4}{3}\cdot\frac{N_f}{4} = \frac{1}{6}(11N_c + 22 - 2N_f)$$

$\therefore$ Asymptotically free if $N_f < \frac{11}{2}N_c + 11$

- $\mathcal{N}=1$ supersymmetric Yang-Mills theory (SYM)

G = simple, $V_f = \mathfrak{g}$ (Majorana), $V_b = 0$

$$T_{V_f} = h^\vee \times \frac{1}{2}$$

$$b_1 = \frac{11}{3}h^\vee - \frac{4}{3}\cdot\frac{h^\vee}{2} = 3h^\vee > 0$$

$\therefore$ Asymptotically free.

- $\mathcal{N}=1$ $SU(N_c)$ supersymmetric QCD (SQCD) with N_f flavors

$G = SU(N_c)$, $V_f = \mathfrak{g}$ (Majorana) $\oplus (\mathbb{C}^{N_c})^{\oplus N_f}$

$$V_b = (\mathbb{C}^{N_c} \oplus \mathbb{C}^{N_c *})^{\oplus N_f}$$

$$T_{V_f} = N_c \times \frac{1}{2} + \frac{1}{2}\times N_f, \quad T_{V_b} = \left(\frac{1}{2}+\frac{1}{2}\right)\times N_f$$

$$b_1 = \frac{11}{3}N_c - \frac{4}{3}\left(\frac{N_c}{2}+\frac{N_f}{2}\right) - \frac{1}{3}N_f = 3N_c - N_f$$

$\therefore$ Asymptotically free if $N_f < 3N_c$.

- $\mathcal{N}=1$ $SO(N_c)$ SQCD with N_f flavours

$$G = SO(N_c), \quad V_f = \mathfrak{g} \oplus (\mathbb{R}^{N_c})^{\oplus N_f} \quad (\text{all Majorana})$$

$$V_b = (\mathbb{C}^{N_c})^{\oplus N_f}$$

$$T_{V_f} = (N_c - 2 + 1 \times N_f) \times \frac{1}{2}, \quad T_{V_b} = 1 \times N_f$$

$$b_1 = \frac{11}{3}(N_c - 2) - \frac{4}{3} \frac{N_c - 2 + N_f}{2} - \frac{1}{3} N_f = 3(N_c - 2) - N_f$$

$\therefore$ Asymptotically free if $N_f < 3(N_c - 2)$.

- $\mathcal{N}=1$ $USp(N_c)$ SQCD with N_f flavours (N_c, N_f both even)

$$G = USp(N_c), \quad V_f = \mathfrak{g}\ (\text{Majorana}) \oplus (\mathbb{C}^{N_c})^{\oplus \frac{N_f}{2}}$$

$$V_b = (\mathbb{C}^{N_c})^{\oplus N_f}$$

$$T_{V_f} = \frac{N_c + 2}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{N_f}{2}, \quad T_{V_b} = \frac{1}{2} \times N_f$$

$$b_1 = \frac{11}{3} \frac{N_c + 2}{2} - \frac{4}{3} \cdot \frac{N_c + 2 + N_f}{4} - \frac{1}{3} \cdot \frac{N_f}{2} = \frac{3(N_c + 2) - N_f}{2}$$

$\therefore$ Asymptotically free if $N_f < 3(N_c + 2)$.

• $\mathcal{N}=2$ SYM

$G =$ simple, $V_f = \mathfrak{g}_{\mathbb{C}}$, $V_b = \mathfrak{g}_{\mathbb{C}}$

$$T_{V_f} = T_{V_b} = h^\vee$$

$$b_1 = \frac{11}{3} h^\vee - \frac{4}{3} h^\vee - \frac{1}{3} h^\vee = 2h^\vee > 0$$

$\therefore$ Asymptotically free

• $\mathcal{N}=2$ $SU(N_c)$ SQCD with N_f flavors

$G = SU(N_c)$, $V_f = \mathfrak{g}_{\mathbb{C}} \oplus (\mathbb{C}^{N_c})^{\oplus N_f}$

$V_b = \mathfrak{g}_{\mathbb{C}} \oplus (\mathbb{C}^{N_c} \oplus (\mathbb{C}^{N_c})^*)^{\oplus N_f}$

$T_{V_f} = N_c + \frac{1}{2} \times N_f$, $T_{V_b} = N_c + \left(\frac{1}{2} + \frac{1}{2}\right) \times N_f$

$$b_1 = \frac{11}{3} N_c - \frac{4}{3}\left(N_c + \frac{N_f}{2}\right) - \frac{1}{3}(N_c + N_f) = 2N_c - N_f$$

$\therefore$ Asymptotically free if $N_f < 2N_c$

- $\mathcal{N}=2$ $SO(N_c)$ SQCD with N_f flavors

$$G = SO(N_c), \quad V_f = \mathfrak{g}_{\mathbb{C}} \oplus (\mathbb{R}^{N_c})^{\oplus N_f} \text{ (Majorana)}$$

$$V_b = \mathfrak{g}_{\mathbb{C}} \oplus (\mathbb{C}^{N_c})^{\oplus N_f}$$

$$T_{V_f} = N_c - 2 + 1 \times N_f \times \frac{1}{2}, \quad T_{V_b} = N_c - 2 + 1 \times N_f$$

$$b_1 = \frac{11}{3}(N_c-2) - \frac{4}{3}\left(N_c - 2 + \frac{N_f}{2}\right) - \frac{1}{3}(N_c - 2 + N_f)$$

$$= 2(N_c-2) - N_f$$

$\therefore$ Asymptotically free if $N_f < 2(N_c-2)$.

- $\mathcal{N}=2$ $USp(N_c)$ SQCD with N_f flavors (N_c, N_f both even)

$$G = USp(N_c), \quad V_f = \mathfrak{g}_{\mathbb{C}} \oplus (\mathbb{C}^{N_c})^{\oplus \frac{N_f}{2}}$$

$$V_b = \mathfrak{g}_{\mathbb{C}} \oplus (\mathbb{C}^{N_c})^{\oplus N_f}$$

$$T_{V_f} = \frac{N_c+2}{2} + \frac{1}{2}\cdot\frac{N_f}{2}, \quad T_{V_b} = \frac{N_c+2}{2} + \frac{1}{2}N_f$$

$$b_1 = \frac{11}{3}\cdot\frac{N_c+2}{2} - \frac{4}{3}\left(\frac{N_c+2}{2} + \frac{N_f}{4}\right) - \frac{1}{3}\,\frac{N_c+2+N_f}{2} = N_c + 2 - \frac{N_f}{2}$$

$\therefore$ Asymptotically free if $N_f < 2(N_c+2)$.

· $\mathcal{N}=4$ SYM

$G=$ simple, $V_f = \mathfrak{g}^{\oplus 4}$ (Majorana) $= \mathfrak{g}_{\mathbb{C}}^{\oplus 2}$

$V_b = \mathfrak{g}^{\oplus 6}$ (real) $= \mathfrak{g}_{\mathbb{C}}^{\oplus 3}$

$T_{V_f} = h^\vee \times 2$, $T_{V_b} = h^\vee \times 3$

$$b_1 = \frac{11}{3} h^\vee - \frac{4}{3} \cdot 2h^\vee - \frac{1}{3} 3h^\vee = 0.$$

Note : $b_1 = 0$ for

$\mathcal{N}=1$ SQCD, $SU(N_c)$, $N_f = 3N_c$
$SO(N_c)$, $N_f = 3(N_c - 2)$
$USp(N_c)$, $N_f = 3(N_c + 2)$

$\beta > 0$ from higher loop (IR free)

$\mathcal{N}=2$ SQCD, $SU(N_c)$, $N_f = 2N_c$
$SO(N_c)$, $N_f = 2(N_c - 2)$
$USp(N_c)$, $N_f = 2(N_c + 2)$

$\mathcal{N}=4$ SYM

$\boxed{\beta \equiv 0 \text{ exactly all loop \& non-perturbatively}}$

finite, scale invariant.

They are belived to be invariant also under (super) conformal transformations.

"(super) conformal field theories"