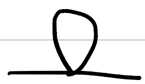
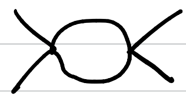
Computation of the one-loop integrals

We would like to compute regularized versions of

$$I = \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2+m^2}$$

$$V = \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2+m^2} \frac{1}{(k-p)^2+m^2}$$

that appear in  and , for

① momentum cut-off

$$\frac{1}{k^2+m^2} = \int_0^\infty d\alpha\, e^{-\alpha(k^2+m^2)}$$

$$\longrightarrow \int_{1/\Lambda^2}^\infty d\alpha\, e^{-\alpha(k^2+m^2)} = \frac{e^{-\frac{k^2+m^2}{\Lambda^2}}}{k^2+m^2}$$

and

③ dimensional regularization : $4 \rightsquigarrow d = 4-\epsilon$.

Exercise : Do the computation.

(Option 1) Explain the steps marked ! in the following.

(Option 2) Do it in your own way.

$$I_{①} = \int \frac{d^4k}{(2\pi)^4} \int_{1/\Lambda^2}^{\infty} d\alpha \; e^{-\alpha(k^2+m^2)}$$

$$\overset{!}{=} \int_{1/\Lambda^2}^{\infty} d\alpha \; \frac{e^{-\alpha m^2}}{(4\pi)^2 \alpha^2}$$

$$\overset{!}{=} \frac{1}{(4\pi)^2} \left[\Lambda^2 - m^2 \left(\log\left(\frac{\Lambda^2}{m^2}\right) + 1 - \gamma \right) + m^2 O\left(\frac{m^2}{\Lambda^2}\right) \right].$$

$$I_{③} = \mu_{DR}^{4-d} \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2+m^2} = \mu_{DR}^{4-d} \frac{\mathrm{Vol}(S^{d-1})}{2(2\pi)^d} \int_0^{\infty} (k^2)^{d/2-1} \, dk^2 \, \frac{1}{k^2+m^2}$$

$$\overset{!}{=} \frac{\mu_{DR}^{4-d} \, m^{d-2}}{(4\pi)^{d/2} \, \Gamma(d/2)} \; B\left(\frac{d}{2}, 1-\frac{d}{2}\right) = \Gamma\left(\frac{d}{2}\right) \Gamma\left(1-\frac{d}{2}\right)$$

$$= m^2 \left(\frac{\mu_{DR}}{m}\right)^{4-d} \frac{1}{(4\pi)^{d/2}} \Gamma\left(1-\frac{d}{2}\right)$$

$d = 4 - \epsilon$

$$\overset{!}{=} -\frac{m^2}{(4\pi)^2} \left[\frac{2}{\epsilon} + \log\left(\frac{4\pi \mu_{DR}^2}{m^2}\right) + 1 - \gamma + O(\epsilon) \right].$$

$$V_{①} = \int \frac{d^4k}{(2\pi)^4} \int_{1/\Lambda^2}^{\infty} \int_{1/\Lambda^2}^{\infty} d\alpha\, d\beta\; e^{-\alpha(k^2+m^2) - \beta((k-p)^2+m^2)}$$

$$\overset{!}{=} \frac{1}{(4\pi)^2} \int_{1/\Lambda^2}^{\infty} \int_{1/\Lambda^2}^{\infty} \frac{d\alpha\, d\beta}{(\alpha+\beta)^2}\; e^{-\frac{\alpha\beta}{\alpha+\beta} p^2 - (\alpha+\beta)m^2}$$

insert $1 = \int_{2/\Lambda^2}^{\infty} d\lambda\, \delta(\lambda-\alpha-\beta)$ & substitute $\alpha \to \lambda x,\ \beta \to \lambda y$

$$= \frac{1}{(4\pi)^2} \int_{2/\Lambda^2}^{\infty} d\lambda \int_{1/\lambda\Lambda^2}^{\infty} \int_{1/\lambda\Lambda^2}^{\infty} \frac{dx\, dy}{(x+y)^2}\; e^{-\lambda\left(\frac{xy}{x+y} p^2 + (x+y)m^2\right)} \underbrace{\delta(\lambda(1-x-y))}_{\frac{1}{\lambda}\delta(1-x-y)}$$

$$= \frac{1}{(4\pi)^2} \int_{2/\Lambda^2}^{\infty} \frac{d\lambda}{\lambda} \underbrace{\int_{1/\lambda\Lambda^2}^{1-1/\lambda\Lambda^2}}_{= \underbrace{\int_0^1}_{X} - 2 \underbrace{\int_0^{1/\lambda\Lambda^2}}_{Y}} dx\; e^{-\lambda(x(1-x)p^2+m^2)}$$

$$= X - 2Y.$$

$$X = \frac{1}{(4\pi)^2} \int_{2/\Lambda^2}^{\infty} \frac{d\lambda}{\lambda} \int_0^1 dx\; e^{-\lambda(x(1-x)p^2+m^2)}$$

$$\overset{!}{=} \frac{1}{(4\pi)^2} \left[\log\left(\frac{\Lambda^2}{2m^2}\right) - \gamma - \int_0^1 dx\, \log\left(1 + x(1-x)\frac{p^2}{m^2}\right) + O\left(\frac{p^2}{\Lambda^2}, \frac{m^2}{\Lambda^2}\right) \right].$$

$$Y = \frac{1}{(4\pi)^2}\int_{2/\Lambda^2}^{\infty}\frac{d\lambda}{\lambda}\int_0^{1/\lambda\Lambda^2} dx\, e^{-\lambda(x(1-x)p^2+m^2)}$$

$$\left[\ 0 \leq \lambda x \leq \frac{1}{\Lambda^2} \Rightarrow e^{-\lambda(x(1-x)p^2+m^2)} = e^{-\lambda m^2}\left(1+O\left(\frac{p^2}{\Lambda^2}\right)\right)\right.$$

$$= \frac{1}{(4\pi)^2}\int_{2/\Lambda^2}^{\infty}\frac{d\lambda}{\lambda}\,\frac{1}{\lambda\Lambda^2}\,e^{-\lambda m^2}\left(1+O\left(\frac{p^2}{\Lambda^2}\right)\right)$$

$$= \frac{1}{(4\pi)^2}\left(\frac{1}{2}+O\left(\frac{m^2}{\Lambda^2},\frac{p^2}{\Lambda^2}\right)\right).$$

$$\therefore\ V_{①} = X - 2Y$$

$$= \frac{1}{(4\pi)^2}\left[\log\left(\frac{\Lambda^2}{2m^2}\right) - \gamma - 1 - \int_0^1 dx\,\log\left(1+x(1-x)\frac{p^2}{m^2}\right) + O\left(\frac{p^2}{\Lambda^2},\frac{m^2}{\Lambda^2}\right)\right].$$

$$V_{③} = \mu_{DR}^{4-d}\int\frac{d^dk}{(2\pi)^d}\,\frac{1}{k^2+m^2}\,\frac{1}{(k-p)^2+m^2}$$

$$\overset{!}{=} \mu_{DR}^{4-d}\int\frac{d^dk'}{(2\pi)^d}\int_0^1\frac{dx}{(k'^2+\underbrace{x(1-x)p^2+m^2}_{=:\Delta})^2}$$

$$\overset{!}{=} \frac{\mu_{DR}^{4-d}}{(4\pi)^{d/2}\Gamma(d/2)}\int_0^1 dx\,\Delta^{\frac{d}{2}-2}\,B\left(\frac{d}{2},2-\frac{d}{2}\right) = \Gamma\left(\frac{d}{2}\right)\Gamma\left(2-\frac{d}{2}\right)$$

$$= \frac{\mu_{DR}^{4-d}}{(4\pi)^{d/2}} \Gamma\left(2-\frac{d}{2}\right) \int_0^1 dx\, \Delta^{\frac{d}{2}-2}$$

$d=4-\epsilon$

$$\doteq \frac{1}{(4\pi)^2} \left[\frac{2}{\epsilon} - \gamma - \int_0^1 dx \log\left(\frac{\Delta}{4\pi \mu_{DR}^2}\right) + O(\epsilon) \right]$$

$$= \frac{1}{(4\pi)^2} \left[\frac{2}{\epsilon} + \log\left(\frac{4\pi \mu_{DR}^2}{m^2}\right) - \gamma - \int_0^1 dx \log\left(1 + x(1-x)\frac{p^2}{m^2}\right) + O(\epsilon) \right]$$

You may use

- $$\int_\epsilon^\infty \frac{dt}{t} e^{-t} = -\log \epsilon - \gamma + O(\epsilon)$$

- $$B(p,q) = \int_0^\infty \frac{y^{p-1} dy}{(1+y)^{p+q}} = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

- $$\Gamma(n+1) = n! \quad \text{for } n = 0, 1, 2, \cdots$$

- $$\Gamma(z) = \frac{1}{z} - \gamma + O(z) \quad \text{as } z \to 0$$

- $$\Gamma(-1+z) = -\frac{1}{z} + \gamma - 1 + O(z) \quad \text{as } z \to 0$$

- $$\frac{1}{AB} = \int_0^1 \frac{dx}{(xA + (1-x)B)^2}$$