Possible form of $\delta_{N,\infty} X$ for $X = A_\mu, \psi, c$

We would like to constrain the possible form of $\delta_{N,\infty} X$ by

$$\{\delta_B, \delta_{N,\infty}\} X \doteq 0,$$

$$\frac{\delta}{\delta \bar{c}} \delta_{N,\infty} X \doteq 0,$$

and symmetries and dimension.

By $\bar{c}$-independence, dimension & symmetries, $\delta_{N,\infty} X$ must be of the form of

$$\delta_{N,\infty} A_\mu = \xi_N \partial_\mu c + f_N (A_\mu, c)$$

$$\delta_{N,\infty} \psi = -\rho_N (c) \psi$$

$$\delta_{N,\infty} c = -\frac{1}{2} \varphi_N (c, c)$$

where ξ_N is a constant,

$$\left.\begin{array}{ll} f_N : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} & \text{bilinear} \\ \rho_N : \mathfrak{g} \to \mathrm{End}(V) & \text{linear} \\ \varphi_N : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} & \text{bilinear antisymmetric} \end{array}\right\} \begin{array}{c} G\text{-equivariant} \\ \Uparrow \\ \text{rigid gauge symmetry} \end{array}$$

G-equiv $\Leftrightarrow$ $g \in G$, $X, Y \in \mathfrak{g}$,

$$f_N(\mathrm{Ad}_g X, \mathrm{Ad}_g Y) = \mathrm{Ad}_g f_N(X,Y)$$

$$\rho_N(\mathrm{Ad}_g X) = g\,\rho_N(X)\,g^{-1}$$

$$\varphi_N(\mathrm{Ad}_g X, \mathrm{Ad}_g Y) = \mathrm{Ad}_g \varphi_N(X,Y)$$

infinitesimally, $X, Y, Z \in \mathfrak{g}$,

$$f_N([Z,X],Y) + f_N(X,[Z,Y]) = [Z, f_N(X,Y)]$$

$$\rho_N([Z,X]) = [Z, \rho_N(X)]$$

$$\varphi_N([Z,X],Y) + \varphi_N(X,[Z,Y]) = [Z, \varphi_N(X,Y)]$$

f_N, ρ_N, φ_N are constrained by $\{\delta_B, \delta_{N,\infty}\} X \doteq 0$.

To simplify the notation drop "N" for a while.

$$0 \doteq \delta_B \delta_\infty A_\mu + \delta_\infty \delta_B A_\mu \quad = D_\mu C$$

$$= \xi \partial_\mu \delta_B C + f(\delta_B A_\mu, C) + f(A_\mu, \delta_B C)$$
$$+ D_\mu \delta_\infty C + [\delta_\infty A_\mu, C]$$

$$= \xi \partial_\mu \cancel{\left(-\tfrac{1}{2}[C,C]\right)} + f(D_\mu C, C) + f\left(A_\mu, -\tfrac{1}{2}[C,C]\right)$$
$$+ D_\mu\left(-\tfrac{1}{2}\varphi(C,C)\right) + [\cancel{\xi \partial_\mu C}, C] + [f(A_\mu, C), C]$$

By G-equivariance of f,

$$[f(A_\mu, C), C] = -f([A_\mu, C], C) + f(A_\mu, [C, C]).$$

$$\therefore\ f(A_\mu, -\tfrac{1}{2}[C, C]) + [f(A_\mu, C), C]$$

$$= -f([A_\mu, C], C) + \tfrac{1}{2} f(A_\mu, [C, C]).$$

By bilinearity and antisymmetry of φ,

$$D_\mu\left(-\tfrac{1}{2}\varphi(C, C)\right) = -\varphi(\partial_\mu C, C) - \tfrac{1}{2}[A_\mu, \varphi(C, C)].$$

$$\therefore\ 0 \doteq \delta_B \delta_\infty A_\mu + \delta_\infty \delta_B A_\mu$$

$$= f(D_\mu C, C) - \varphi(\partial_\mu C, C) - \tfrac{1}{2}[A_\mu, \varphi(C, C)]$$

$$- f([A_\mu, C], C) + \tfrac{1}{2} f(A_\mu, [C, C])$$

$$= f(\partial_\mu C, C) - \varphi(\partial_\mu C, C)$$

$$- \tfrac{1}{2}[A_\mu, \varphi(C, C)] + \tfrac{1}{2} f(A_\mu, [C, C])$$

$$\therefore\ \left.\begin{aligned} & f(X, Y) \doteq \varphi(X, Y) \\ & f(X, [Y, Z]) \doteq [X, \varphi(Y, Z)] \\ & \quad \text{for } \forall X, Y, Z \in \mathfrak{g} \end{aligned}\right\} \star$$

$$0 \doteq \delta_B \delta_\infty \psi + \delta_\infty \delta_B \psi$$

$$= -\rho(\delta_B c)\psi + \rho(c)\,\delta_B \psi - \delta_\infty c\,\psi + c\,\delta_\infty \psi$$

$$= \frac{1}{2}\rho([c,c])\psi \underbrace{- \rho(c)\,c\,\psi + \frac{1}{2}\varphi(c,c)\psi - c\,\rho(c)\psi}_{G\text{-equiv} \;\to\; -\rho([c,c])\psi}$$

$$= \frac{1}{2}\left(\varphi(c,c) - \rho([c,c])\right)\psi$$

$$\therefore \quad \varphi(X,Y) \doteq \rho([X,Y]) \qquad \star\star$$

$$0 \doteq \delta_B \delta_\infty c + \delta_\infty \delta_B c$$

$$= -\frac{1}{2}\varphi(\delta_B c, c)\times 2 - \frac{1}{2}[\delta_\infty c, c]\times 2$$

$$= \frac{1}{2}\varphi([c,c],c) + \frac{1}{2}[\varphi(c,c),c] \overset{G\text{-equivariance}}{=} -\varphi([c,c],c)$$

$$= -\frac{1}{2}\varphi([c,c],c)$$

$c = \sum_a c_a e^a, \quad \{e^a\} \subset \mathfrak{g}$ basis

$$= -\frac{1}{2}\sum_{a,b,c} c_a c_b c_c\, \varphi([e^a,e^b],e^c)$$

$$= -\sum_{a<b<c} c_a c_b c_c \left\{ \varphi([e^a,e^b],e^c) + \varphi([e^b,e^c],e^a) + \varphi([e^c,e^a],e^b) \right\}$$

$$\therefore \quad \varphi([X,Y],Z) + \text{cyclic} \doteq 0 \qquad \star\star\star$$

Mathematical fact (see an additional note)

When G is a simple compact Lie group, a G-equivariant antisymmetric bilinear map $\varphi : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying

$$\varphi([X,Y],Z) + \text{cyclic} = 0$$

is proportional to the Lie algebra bracket

$$\varphi(X,Y) \propto [X,Y].$$

Thus, when G is a simple compact Lie group,

$$☆☆☆ \iff \varphi(X,Y) \doteq \eta[X,Y] \text{ for some constant } \eta.$$

Then, $☆ \iff f(X,Y) \doteq \eta[X,Y]$

$$☆☆ \iff \rho(X) \doteq \eta X \quad \text{as } \mathfrak{g} = [\mathfrak{g},\mathfrak{g}] \text{ if } \mathfrak{g} \text{ is simple.}$$

Recovering "N",

$$\delta_{N,\infty} A_\mu \doteq \xi_N \partial_\mu C + \eta_N [A_\mu, C]$$

$$\delta_{N,\infty} \psi \doteq -\eta_N C \psi$$

$$d_{N,\infty} C \doteq -\frac{1}{2}\eta_N [C,C].$$