Anomalies

A symmetry of the classical action is said to be <u>anomalous</u> when it is not a symmetry of the path-integral measure. E.g. when it is broken by regularization procedure.

<u>Anomaly of global symmetry</u> is harmless.

Nothing is wrong with the theory itself. It is just that the quantum system loses the classical symmetry.

<u>Anomaly of gauge symmetry</u> is unacceptable!

The theory does not make sense as a quantum theory.
Absence of anomaly is an important criterion of consistency.

<u>t' Hooft anomaly</u>

A global symmetry may become anomalous when you try to gauge it. Nothing is wrong with the theory itself, as long as you do not really gauge it. Rather, the anomaly is an important and useful information of the theory.
It is invariant under the RG flow:

anomaly of high energy (or elementary) theory
= anomaly of low energy (or effective) theory.

Suppose we have a classical system $(\phi, S[\phi])$ with a group G of global symmetries : $S[\phi^g] = S[\phi]$, $g \in G$.

This may be made invariant under position dependent $g(x)$'s by coupling it to a gauge field A : $S[A^g, \phi^g] = S[A, \phi]$.

The partition (or correlation function) is a functional of A :

$$Z[A] = \int \mathcal{D}_A \phi \, e^{-S[A,\phi]}.$$

Is it invariant under $A \mapsto A^g$?

$$Z[A^g] = \int \mathcal{D}_{A^g} \phi \, e^{-S[A^g,\phi]} = \int \mathcal{D}_{A^g} \phi^g \, e^{-S[A^g,\phi^g]}$$

$$S[A^g,\phi^g] = S[A,\phi]$$

If $\mathcal{D}_{A^g}\phi^g = \mathcal{D}_A \phi$, then yes, $Z[A^g] = Z[A]$.

Then we can make A dynamical and consider the gauge theory with variable (A, ϕ) :

$$Z = \int_{\mathcal{A}/\mathcal{G}} \text{measure} \; e^{-S_{YM}[A]} Z[A]$$

This is the gauging. It is possible if $Z[A^g] = Z[A]$.

However, this may not be so

$$\mathcal{D}_{A^g}\phi^g = \mathcal{D}_A\phi \, e^{i a_g[A,\phi]} .$$

This is the anomaly. The infinitesimal form is

$$\delta_\epsilon \mathcal{D}_A\phi = \mathcal{D}_A\phi \, i a_\epsilon[A,\phi] .$$

Sometimes, $a_g[A,\phi]$ is field-independent, $= a_g[A]$.

In such a case,

$$Z[A^g] = Z[A] e^{i a_g[A]}$$

$$\rightsquigarrow \delta_\epsilon Z[A] = Z[A] \, i a_\epsilon[A] .$$

This is the usual form of t'Hooft anomaly.

Remarks

- If $a_\epsilon[A,\phi]$ is variation of some local expression

$$a_\epsilon[A,\phi] = \delta_\epsilon \int \Delta\mathcal{L}(A,\partial A,\cdots,\phi,\partial\phi,\cdots)\, d^4x,$$

then we can modify the system by adding the local counter term $i\Delta\mathcal{L}(A,\partial A,\cdots,\phi,\partial\phi,\cdots)$ to the Lagrangian:

$$S'[A,\phi] = S[A,\phi] + i\int \Delta\mathcal{L}(A,\cdots,\phi,\cdots)\, d^4x$$

So that $\delta_\epsilon S'[A,\phi] = i\, a_\epsilon[A,\phi]$

and the modified system has no anomaly

$$\delta_\epsilon\left(\mathcal{D}_A\phi\, e^{-S'[A,\phi]}\right) = 0.$$

Finite version:

$$S'[A^g,\phi^g] = S'[A,\phi] + i\, a_g[A,\phi], \quad \text{and}$$

$$\mathcal{D}_{A^g}\phi^g\, e^{-S'[A^g,\phi^g]} = \mathcal{D}_A\phi\, e^{i\cancel{a_g[A,\phi]}} \cdot e^{-S'[A,\phi] - i\cancel{a_g[A,\phi]}}$$

$$= \mathcal{D}_A\phi\, e^{-S'[A,\phi]}.$$

After this modification, the symmetry can be gauged.

Therefore, anomaly is defined only up to variation of local expression

$$a_\epsilon[A,\phi] \sim a_\epsilon[A,\phi] + \delta_\epsilon\int \Delta\mathcal{L}(A,\cdots,\phi,\cdots)\, d^4x$$

- The <u>symmetry current</u> J is defined by

$$\delta\left(\mathcal{D}_A\phi\, e^{-S[A,\phi]}\right) = \mathcal{D}_A\phi\, e^{-S[A,\phi]} \int d^4x\, \delta A \cdot J$$

for an arbitrary variation $A \to A + \delta A$.

In particular, for the gauge transformation $\delta_\epsilon A_\mu = D_\mu \epsilon$

$$\delta_\epsilon\left(\mathcal{D}_A\phi\, e^{-S[A,\phi]}\right) = \mathcal{D}_A\phi\, e^{-S[A,\phi]} \int d^4x\, D_\mu\epsilon \cdot J^\mu .$$

Thus, the anomaly can be expressed in terms of the current

$$i\mathcal{A}_\epsilon[A,\phi] = \int d^4x\, D_\mu\epsilon \cdot J^\mu = -\int d^4x\, \epsilon \cdot D_\mu J^\mu$$

- The anomaly of the original global symmetry is simply the anomaly at the trivial gauge field,

$$i\mathcal{A}_\epsilon[\phi] := i\mathcal{A}_\epsilon[A=0,\phi]$$

$$= \int d^4x\, \partial_\mu\epsilon \cdot J^\mu = -\int d^4x\, \epsilon\, \partial_\mu J^\mu$$

$\therefore$ anomaly $\Longleftrightarrow$ current-non-conservation.

Two basic examples : axial anomaly (global)

chiral anomaly (gauge or 't Hooft)

Axial anomaly

Theory : gauge theory with gauge group G, massless Dirac fermion ψ in a representation V of G :

$$S[A,\psi] = S_{YM}[A] + \int d^4x \left(-i\bar{\psi}\not{D}_A\psi\right)$$

Symmetry : $U(1)_5$ = axial phase rotations of fermion

$$\psi \to e^{i\epsilon\gamma_5}\psi, \quad \bar{\psi} \to \bar{\psi}e^{i\epsilon\gamma_5}$$

recall $\gamma_5 = \gamma^4\gamma^1\gamma^2\gamma^3$, $\gamma_5\gamma^\mu = -\gamma^\mu\gamma_5$, $\gamma_5 = \begin{cases} +1 & \psi_R \\ -1 & \psi_L \end{cases}$

indeed $\bar{\psi}\not{D}_A\psi \to \bar{\psi}\underbrace{e^{i\epsilon\gamma_5}\not{D}_A}_{\not{D}_A \cancel{e^{-i\epsilon\gamma_5}}}\cancel{e^{i\epsilon\gamma_5}}\psi = \bar{\psi}\not{D}_A\psi$

$\epsilon \rightsquigarrow \epsilon(x)$: $i\bar{\psi}\not{D}_A\psi \rightsquigarrow i\bar{\psi}\gamma^\mu\partial_\mu(i\epsilon\gamma_5)\psi = -\partial_\mu\epsilon\,\bar{\psi}\gamma^\mu\gamma_5\psi$

$\therefore$ The Noether current is

$$j_5^\mu = -\bar{\psi}\gamma^\mu\gamma_5\psi = \bar{\psi}\gamma_5\gamma^\mu\psi$$

anomaly :

$$\partial_\mu \hat{j}_5^\mu = \frac{i}{16\pi^2} \epsilon^{\mu\nu\rho\lambda} \mathrm{tr}_V (F_{\mu\nu} F_{\rho\lambda})$$

i.e. $$a^5_\epsilon [A] := a^5_\epsilon [A_5 = 0, A, \psi]$$

$$= \int d^4x \, \epsilon(x) \frac{-1}{16\pi^2} \epsilon^{\mu\nu\rho\lambda} \mathrm{tr}_V (F_{\mu\nu} F_{\rho\lambda})$$

$$= \int \frac{-1}{4\pi^2} \epsilon \, \mathrm{tr}_V (F_A \wedge F_A)$$

Here we used differential forms :

$$A = A_\mu dx^\mu, \quad F_A = dA + \frac{1}{2}[A, A] = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$$

$$F_A \wedge F_A = \frac{1}{4} F_{\mu\nu} F_{\rho\lambda} \underbrace{dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\lambda}_{\epsilon^{\mu\nu\rho\lambda} d^4x} \quad ; \quad \epsilon^{1234} = 1$$

<u>Remark</u> We may ungauge G.

Original : external internal

$$(\overset{\frown}{A_5}, \overset{\frown}{A, \psi}) : \quad a^5_\epsilon \text{ is } \begin{cases} \text{field-dependent} \\ \text{field-independent} \end{cases}$$

ungauged : external internal

Chiral anomaly

Theory : free massless fermion ψ_R and ψ_L in representations V_R and V_L of G

$$S = \int d^4x \left(-i \overline{\psi}_R \not{\partial} \psi_R - i \overline{\psi}_L \not{\partial} \psi_L \right)$$

Symmetry : $\psi_R \to g^{-1}\psi_R$, $\overline{\psi}_R \to \overline{\psi}_R\, g$

$\psi_L \to g^{-1}\psi_L$, $\overline{\psi}_L \to \overline{\psi}_L\, g$

$$S[A, \psi_R, \psi_L] = \int d^4x \left(-i \overline{\psi}_R \not{D}_A \psi_R - i \overline{\psi}_L \not{D}_A \psi_L \right)$$

Current $\quad J^{\mu a} = i \overline{\psi}_R \gamma^\mu e^a \psi_R + i \overline{\psi}_L \gamma^\mu e^a \psi_L$

anomaly :

$$D_\mu J^{\mu a} = \frac{1}{24\pi^2} \epsilon^{\mu\nu\rho\lambda} \Big\{ \mathrm{tr}_{V_R}\left[e^a \partial_\mu \left(A_\nu \partial_\rho A_\lambda + \tfrac{1}{2} A_\nu A_\rho A_\lambda \right) \right] - \mathrm{tr}_{V_L}\left[e^a \partial_\mu \left(A_\nu \partial_\rho A_\lambda + \tfrac{1}{2} A_\nu A_\rho A_\lambda \right) \right] \Big\}$$

field independent

i.e.

$$a_\epsilon[A] = \int d^4x\, \epsilon_a(x)\, i D_\mu J^{\mu a}$$

$$= \int \frac{i}{24\pi^2} \Big\{ \mathrm{tr}_{V_R}\left[\epsilon\, d\left(A dA + \tfrac{1}{2} A^2 \right) \right] - \mathrm{tr}_{V_L}\left[\epsilon\, d\left(A dA + \tfrac{1}{2} A^2 \right) \right] \Big\}$$

<u>Remark</u> Axial anomaly (in which G-gauge potential A is regarded as external) can be regarded as a special case of chiral anomaly:

$$G_{tot} = U(1)_5 \times G$$

$V_{tot\,R} = V(1)$ ... V as a representation of G

$U(1)_5$ charge $+1$

$V_{tot\,L} = V(-1)$ ... V as a representation of G

$U(1)_5$ charge -1

Then, the axial anomaly (where A is external) can be regarded as the chiral anomaly for

the variation $\mathcal{E}_{tot} = (\epsilon, 0)$

the background $A_{tot} = (0, A)$

<u>NB</u> We need modification by local counter term for the match.

Computation of anomalies

We compute the axial & chiral anomalies and verify the above statements.

These anomalies are field-independent (if G-gauge potential is regarded external for axial anomaly). Thus

$$\delta_\epsilon Z[A] = Z[A]\, i\,a_\epsilon[A].$$

If we write $Z[A] = \int \mathcal{D}_A\phi\, e^{-S[A,\phi]} = e^{-W[A]}$, then

$$\delta Z[A] = \int \mathcal{D}_A\phi\, e^{-S[A,\phi]} \int d^4x\, \delta A\cdot J$$

$$\| $$

$$Z[A](-\delta W[A])$$

$$\therefore -\delta W[A] = \left\langle \int d^4x\, \delta A\cdot J \right\rangle_A$$

In particular

$$i\,a_\epsilon[A] = \delta_\epsilon Z[A]/Z[A]$$

$$= -\delta_\epsilon W[A] = \left\langle \int d^4x\, D_\mu\epsilon\cdot J^\mu \right\rangle_A$$

$$\delta_2 \delta_1 Z[A] = \int \mathcal{D}_A \phi \, e^{-S[A,\phi]} \int d^4x_1 \, \delta_1 A \cdot J \int d^4x_2 \, \delta_2 A \cdot J$$

$$\rotatebox{90}{$=$}$$

$$\delta_2 \left(Z[A] \, \delta_1 (-W[A]) \right) = Z[A] \left(-\delta_2 \delta_1 W[A] + \delta_2 W[A] \, \delta_1 W[A] \right)$$

$$\therefore \; -\delta_2 \delta_1 W[A] = \left\langle \int d^4x_1 \, \delta_1 A \cdot J \int d^4x_2 \, \delta_2 A \cdot J \right\rangle_A$$

$$- \left\langle \int d^4x_1 \, \delta_1 A \cdot J \right\rangle_A \left\langle \int d^4x_2 \, \delta_2 A \cdot J \right\rangle_A$$

$$= \left\langle \int d^4x_1 \, \delta_1 A \cdot J \int d^4x_2 \, \delta_2 A \cdot J \right\rangle_{A,\,\mathrm{conn}}$$

$$\vdots$$

$$-\delta_1 \cdots \delta_n W[A] = \left\langle \prod_{i=1}^{n} \int d^4x_i \, \delta_i A \cdot J \right\rangle_{A,\,\mathrm{conn}}$$

$$i \, \delta_1 \cdots \delta_n \, a_\epsilon [A] = - \delta_1 \cdots \delta_n \delta_\epsilon W[A]$$

$$= \left\langle \int d^4x \, D_\mu \epsilon \cdot J^\mu \prod_{i=1}^{n} \int d^4x_i \, \delta_i A \cdot J \right\rangle_{A,\,\mathrm{conn}}$$

$$+ \sum_{i=1}^{n} \left\langle \int d^4x \, [\delta_i A_\mu, \epsilon] \cdot J^\mu \prod_{j \neq i} \int d^4x_j \, \delta_j A \cdot J \right\rangle_{A,\,\mathrm{conn}}$$

Axial anomaly

$$i\,a^5_\epsilon[A] = -\delta^5_\epsilon W[A_5, A]\Big|_{A_5=0} = \left\langle \int d^4x\, \partial_\mu\epsilon\, j_5^\mu \right\rangle_A$$

$$i\,a^5_\epsilon[0] = \left\langle \int d^4x\, \partial_\mu\epsilon\, j_5^\mu \right\rangle = 0 \quad \text{by ``Lorentz'' inv.}$$

$$i\,\delta a^5_\epsilon[A]\Big|_{A=0} = \left\langle \int d^4x\, \partial_\mu\epsilon\, j_5^\mu \int d^4y\, \delta A\cdot J(y) \right\rangle_{\text{conn}}$$

$$\begin{cases} \partial_\mu\epsilon\, j_5^\mu = \partial_\mu\epsilon\, \bar\psi\gamma_5\gamma^\mu\psi \\ \delta A\cdot J = i\bar\psi\, \cancel{\delta A}\, \psi = \delta A_{\nu a}\, i\bar\psi\gamma^\nu e^a\psi \end{cases}$$

$$= \int d^4x\, \partial_\mu\epsilon(x) \int d^4y\, \delta A_{\nu a}(y)$$

$$\times(-1)\,\mathrm{tr}_{V\otimes S}\left[\gamma_5\gamma^\mu \overbrace{\psi(x)\bar\psi(y)}\, i\gamma^\nu e^a \overbrace{\psi(y)\bar\psi(x)}\right]$$

$$\overset{!}{=} 0$$

$$i\,\delta_1\delta_2 a^5_\epsilon[A]\Big|_{A=0} = \left\langle \int d^4x\, \partial_\mu\epsilon\, j_5^\mu \prod_{i=1}^{2}\int d^4x_i\, \delta_i A\cdot J(x_i) \right\rangle_{\text{conn}}$$

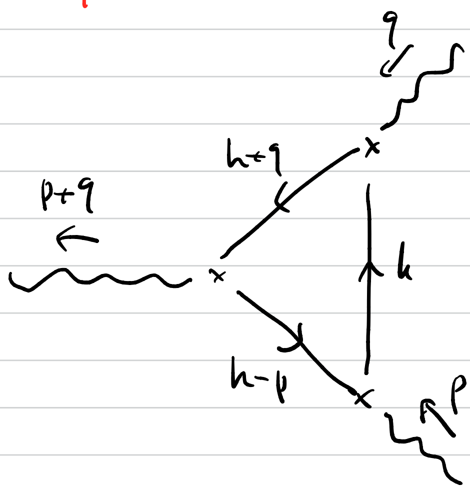
Do this for $\delta_1 A = dx^\nu e^a e^{-iqx}$, $\delta_2 A = dx^\rho e^b e^{-ipx}$

$$= \int d^4x\, \partial_\mu\epsilon(x) \int d^4x_1\, e^{-iqx_1} \int d^4x_2\, e^{-ipx_2}$$

$$(-1)\,\mathrm{tr}_{V\otimes S}\left[\gamma_5\gamma^\mu \overbrace{\psi(x)\bar\psi(x_1)}\, i\gamma^\nu e^a \overbrace{\psi(x_1)\bar\psi(x_2)}\, i\gamma^\rho e^b \overbrace{\psi(x_2)\bar\psi(x)}\right] + (1\leftrightarrow 2)$$

$$= \int d^4x\, \underbrace{\partial_\mu \epsilon(x)\, e^{-i(p+q)x}}_{\epsilon(x)\, i(p+q)_\mu e^{-i(p+q)x}} \int \frac{d^4k}{(2\pi)^4}\, \mathrm{tr}_{V\otimes S}\left(\gamma_5 \gamma^\mu \frac{1}{-(\cancel{k+q})} \gamma^\nu e^a \frac{1}{-\cancel{k}} \gamma^\rho e^b \frac{1}{-(\cancel{k-p})}\right)$$

$$+ (q, a, \nu) \leftrightarrow (p, b, \rho)$$



$\uparrow$ Logarithmically divergent

Pauli-Villars regularization

Introduce a regulator : a spinor, mass Λ, opposite statistics (bose)

$$f(k,m) = \mathrm{tr}_{V\otimes S}\left(\gamma_5 (\cancel{p+q}) \frac{1}{-(\cancel{k+q})+m} \gamma^\nu e^a \frac{1}{-\cancel{k}+m} \gamma^\rho e^b \frac{1}{-(\cancel{k-p})+m}\right)$$

$$+ (q, a, \nu) \leftrightarrow (p, b, \rho)$$

$$i\, \delta_1 \delta_2\, a^5_\epsilon[A]\Big|_{A=0}$$

$$= i \int d^4x\, \epsilon(x)\, e^{-i(p+q)x} \int \frac{d^4k}{(2\pi)^4} \Big(\underbrace{f(k,0)}_{\text{original}} - \underbrace{f(k,\Lambda)}_{\text{regulator}}\Big)$$

Compute $f(k,m)$

$$\gamma_5(\not p+\not q) = \gamma_5 \not q - \not p\gamma_5$$

$$= -\gamma_5\left(-(\not k+\not q)+m\right) - \left(-(\not k-\not p)+m\right)\gamma_5 + 2m\gamma_5$$

$$f(k,m) = \mathrm{tr}_V(e^a e^b)\,\mathrm{tr}_S\Big(-\gamma_5\gamma^\nu\frac{1}{-\not k+m}\gamma^\rho\frac{1}{-(\not k-\not p)+m}$$

$$-\gamma_5\frac{1}{-(\not k+\not q)+m}\gamma^\nu\frac{1}{-\not k+m}\gamma^\rho$$

$$+2m\gamma_5\frac{1}{-(\not k+\not q)+m}\gamma^\nu\frac{1}{-\not k+m}\gamma^\rho\frac{1}{-(\not k-\not p)+m}\Big) + \text{exchange}$$

$$= \mathrm{tr}_V(e^a e^b)\Bigg\{-\frac{\mathrm{tr}_S\left(\gamma_5\gamma^\nu(\not k+m)\gamma^\rho((\not k-\not p)+m)\right)}{(k^2+m^2)((k-p)^2+m^2)}$$

$$-\frac{\mathrm{tr}_S\left(\gamma_5((\not k+\not q)+m)\gamma^\nu(\not k+m)\gamma^\rho\right)}{((k+q)^2+m^2)(k^2+m^2)}$$

$$+2m\frac{\mathrm{tr}_S\left(\gamma_5((\not k+\not q)+m)\gamma^\nu(\not k+m)\gamma^\rho((\not k-\not p)+m)\right)}{((k+q)^2+m^2)(k^2+m^2)((k-p)^2+m^2)}\Bigg\}$$

$$+ (q,\nu)\leftrightarrow(p,\rho)$$

Use $\gamma_5 = -\gamma^1\gamma^2\gamma^3\gamma^4$

$$\mathrm{tr}_S(\gamma_5\gamma^{\mu_1}\cdots\gamma^{\mu_s}) = 0 \qquad s\leq 3,\ s=5$$

$$\mathrm{tr}_S(\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\lambda) = -4\epsilon^{\mu\nu\rho\lambda} \qquad (\epsilon^{1234}=1)$$

- numerator of the 1st term $= \mathrm{tr}_S(\gamma_5\gamma^\nu \not{k}\gamma^\rho(\not{k}-\not{p}))$

$$= -4\epsilon^{\nu\lambda\rho\sigma}k_\lambda(k-p)_\sigma = 4\epsilon^{\nu\lambda\rho\sigma}k_\lambda p_\sigma$$

after k-integration of the regularized system: $k_\lambda \propto p_\lambda$

$$\rightarrow \quad 4\epsilon^{\nu\lambda\rho\sigma}p_\lambda p_\sigma = 0. \qquad \underline{\text{No contribution}}$$

- Similarly the 2nd term has <u>no contribution</u> to the integral.

- numerator of the 3rd term

$$\mathrm{tr}_S(\gamma_5((\not{k}+\not{q})+m)\gamma^\nu(\not{k}+m)\gamma^\rho((\not{k}-\not{p})+m))$$

$$= m\,\mathrm{tr}_S(\gamma_5\{\gamma^\nu\not{k}\gamma^\rho(\not{k}-\not{p}) + (\not{k}+\not{q})\gamma^\nu\gamma^\rho(\not{k}-\not{p}) + (\not{k}+\not{q})\gamma^\nu\not{k}\gamma^\rho\})$$

$$= m\,\mathrm{tr}_S(\gamma_5\{\cancel{\gamma^\nu\not{k}\gamma^\rho(-\not{p})} + \cancel{\not{k}\gamma^\nu\gamma^\rho(-\not{p})} + \cancel{\not{q}\gamma^\nu\gamma^\rho\not{k}} + \not{q}\gamma^\nu\gamma^\rho(-\not{p})$$
$$+ \cancel{\not{q}\gamma^\nu\not{k}\gamma^\rho}\})$$

$$= 4m\,\epsilon^{\lambda\nu\rho\sigma}q_\lambda p_\sigma$$

$$\therefore \int \frac{d^4k}{(2\pi)^4}\left(f(k,0) - f(k,\Lambda)\right)$$

$$= \mathrm{tr}_V(e^a e^b)\int \frac{d^4k}{(2\pi)^4}(-2\Lambda)\frac{4\Lambda\,\epsilon^{\lambda\nu\rho\sigma} q_\lambda p_\sigma}{((k+q)^2+\Lambda^2)(k^2+\Lambda^2)((k-p)^2+\Lambda^2)} + \underbrace{(q,\nu)\leftrightarrow(p,\rho)}_{\text{doubles}}$$

$k = \Lambda k'$, $\Lambda \to \infty$ Omit $\frac{q}{\Lambda}$, $\frac{p}{\Lambda}$

$$= -16\,\mathrm{tr}_V(e^a e^b)\int \frac{d^4k'}{(2\pi)^4}\frac{\epsilon^{\lambda\nu\rho\sigma} q_\lambda p_\sigma}{(k'^2+1)^3}$$

$$= \frac{\mathrm{Vol}(S^3)}{2(2\pi)^4}\int_0^\infty \frac{k'^2\,dk'^2}{(k'^2+1)^3} = \frac{1}{2(4\pi)^2}$$

$$= -\frac{1}{2\pi^2}\,\mathrm{tr}_V(e^a e^b)\,\epsilon^{\lambda\nu\rho\sigma} q_\lambda p_\sigma$$

$$\therefore\ \delta_1\,\delta_2\,a^5_\epsilon[A]\,\big|_{A=0} = \int d^4x\,\epsilon(x)\,e^{-i(p+q)x}\,\frac{-1}{2\pi^2}\,\mathrm{tr}_V(e^a e^b)\,\epsilon^{\lambda\nu\rho\sigma} q_\lambda p_\sigma$$

for $\delta_1 A = dx^\nu e^a e^{-iqx}$, $\delta_2 A = dx^\rho e^b e^{-ipx}$

These are to be compared with

$$\delta_1 \cdots \delta_n \underbrace{\int d^4x\,\epsilon(x)\,\frac{-1}{16\pi^2}\,\epsilon^{\mu\nu\rho\lambda}\,\mathrm{tr}_V F_{\mu\nu}F_{\rho\lambda}}_{\times}\ \Big|_{A=0}$$

$$X\big|_{A=0} = 0 = a^5_\epsilon[0] .$$

$$\delta X\big|_{A=0} = 0 = \delta a^5_\epsilon[A]\big|_{A=0} .$$

$$\delta_1 \delta_2 X\big|_{A=0}$$

$$= \int d^4x\, \epsilon(x) \frac{-1}{16\pi^2} \epsilon^{\mu_1\mu_2\mu_3\mu_4} \mathrm{tr}_V\, \delta_1 F_{\mu_1\mu_2}\, \delta_2 F_{\mu_3\mu_4} \times 2$$

$\delta_i A$ as above: $\left(-iq_{\mu_1}\delta^\nu_{\mu_2} + iq_{\mu_2}\delta^\nu_{\mu_1}\right)e^a e^{-iqx}$ $\quad \left(-ip_{\mu_3}\delta^\rho_{\mu_4} + ip_{\mu_4}\delta^\rho_{\mu_3}\right)e^b e^{-ipx}$

$$= \int d^4x\, \epsilon(x)\, e^{-i(p+q)x} \frac{1}{2\pi^2} \mathrm{tr}_V(e^a e^b)\, \underbrace{\epsilon^{\mu_1\mu_2\mu_3\mu_4} q_{\mu_1}\delta^\nu_{\mu_2} p_{\mu_3}\delta^\rho_{\mu_4}}_{\epsilon^{\mu_1\nu\mu_3\rho} q_{\mu_1} p_{\mu_3} = -\epsilon^{\lambda\nu\rho\sigma} q_\lambda p_\sigma}$$

$$= \delta_1 \delta_2 a^5_\epsilon[A]\big|_{A=0} .$$

$$\boxed{\therefore\ a^5_\epsilon[A] = \int d^4x\, \epsilon(x) \frac{-1}{16\pi^2} \epsilon^{\mu\nu\rho\lambda} \mathrm{tr}_V F_{\mu\nu} F_{\rho\lambda} \quad \text{at } O(A^2)}$$

Chiral anomaly

Consider the case $V_R = V$, $V_L = \{0\}$ for simplicity.

$$Z[A] = \int \mathcal{D}\bar{\psi}_R \mathcal{D}\psi_R \, e^{\int i\bar{\psi}_R \not{D}_A \psi_R \, d^4x}$$

$$= \text{Const} \cdot \int \mathcal{D}\bar{\psi}_R \mathcal{D}\psi_R \mathcal{D}\bar{\psi}_L \mathcal{D}\psi_L \, e^{\int (i\bar{\psi}_R \not{D}_A \psi_R + i\bar{\psi}_L \not{\partial} \psi_L) d^4x}$$

$$= \text{Const} \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \, e^{\int (i\bar{\psi}\not{\partial}\psi + i\bar{\psi}\not{A} P_R \psi) d^4x}$$

where $P_R = \frac{1+\gamma_5}{2}$ projection to R-components.

For the purpose of computation of anomaly, we can consider the Dirac fermion ψ with values in V where A is coupled to $\psi_R = P_R \psi$ only. I.e.

$$J = i\bar{\psi}_R \not{A} \psi_R = i\bar{\psi} \not{A} P_R \psi.$$

Now let us compute $-\delta_1 \cdots \delta_n \delta_\epsilon W[A]\big|_{A=0}$.

$$-\delta_\epsilon W[A=0] = \left\langle \int d^4x \, D_\mu \epsilon \cdot J^\mu \right\rangle = 0 \quad \text{by "Lorentz" inv.}$$

$$-\delta\,\delta_\epsilon W[A]\Big|_{A=0} = \left\langle \int d^4x\, \partial_\mu \epsilon\, J^\mu \int d^4y\, \delta A \cdot J \right\rangle_{conn}$$

$$+ \left\langle \int d^4x\, [\delta A_\mu, \epsilon] \cdot J^\mu \right\rangle \to 0 \text{ by "Lorentz"}$$

$$= \int d^4x\, \partial_\mu \epsilon_a(x) \int d^4y$$

$$(-1)\, tr_{V\otimes S}\left(i\gamma^\mu e^a P_R \psi(x)\bar\psi(y)\, i\,\cancel{\delta A}\, P_R \psi(y)\bar\psi(x) \right)$$

$$\delta A = dx^\nu e^b e^{-iqx}$$

$$= \int d^4x\, \partial_\mu \epsilon_a(x) \int d^4y\, e^{-iqy}\, tr_{V\otimes S}\left(\gamma^\mu e^a P_R \psi(x)\bar\psi(y)\gamma^\nu e^b P_R \psi(y)\bar\psi(x) \right)$$

$$= \int d^4x\, \underbrace{\partial_\mu \epsilon_a(x) e^{-iqx}}_{\sim\, i q_\mu \epsilon_a(x) e^{-iqx}}\, tr_V(e^a e^b) \underbrace{\int \frac{d^4k}{(2\pi)^4} Tr_S\left(\gamma^\mu P_R \frac{1}{-\cancel{k}} \gamma^\nu P_R \frac{1}{-(\cancel{k}-\cancel{q})} \right)}_{=\ \star}$$

quadratically divergent.

Pauli-Villars regularization

name	0	1	2	3
mass	$\Lambda_0 = 0$	Λ_1	Λ_2	Λ_3
statistics	fermi ($\epsilon_0 = 1$)	bose ($\epsilon_1 = -1$)	fermi ($\epsilon_2 = 1$)	bose ($\epsilon_3 = -1$)
	original	regulators		

Replace ✪ by

$$I^{\mu\nu}(q) := \int \frac{d^4k}{(2\pi)^4} \sum_{i=0}^{3} \epsilon_i \, \mathrm{tr}_S \left(\gamma^\mu P_R \frac{1}{-\not{k}+\Lambda_i} \gamma^\nu P_R \frac{1}{-(\not{k}-\not{q})+\Lambda_i} \right)$$

It turns out that $\sum_{i=1}^{3} \epsilon_i \Lambda_i^2 = 0$ is sufficient to make it convergent.

After some computation, we find $I^{\mu\nu}(q)$

$$= -\frac{1}{3(4\pi)^2} \Big\{ (\delta^{\mu\nu} q^2 - q^\mu q^\nu)\, 2\Big(\log q^2 - \frac{5}{3} + \sum_{i=1}^{3} \epsilon_i \log \Lambda_i^2\Big) + \delta^{\mu\nu}\Big(6 \sum_{i=1}^{3} \epsilon_i \Lambda_i^2 \log \Lambda_i^2 - q^2\Big) \Big\}$$

$$-\delta\,\delta_\epsilon W[A]\Big|_{A=0}$$

$$= i \int d^4x\, \epsilon_a(x)\, e^{iqx}\, \mathrm{tr}_V(e^a e^b)\, \underbrace{q_\mu I^{\mu\nu}(q)}_{-\frac{q^\nu}{3(4\pi)^2}\left(6\sum_{i=1}^{3}\epsilon_i\Lambda_i^2\log\Lambda_i^2 - q^2\right)} \neq 0$$

This does not look to match with the claimed anomay.

In fact, this can be cancelled by adding a local counter term to W[A].