Chiral anomaly (Outline : see the additional note for details)

Consider the case $V_R = V$, $V_L = \{0\}$ for simplicity.

$$Z[A] = \int \mathcal{D}\bar{\psi}_R \mathcal{D}\psi_R \, e^{\int i\bar{\psi}_R \not{D}_A \psi_R \, d^4x}$$

$$= \text{Const} \cdot \int \mathcal{D}\bar{\psi}_R \mathcal{D}\psi_R \mathcal{D}\bar{\psi}_L \mathcal{D}\psi_L \, e^{\int (i\bar{\psi}_R \not{D}_A \psi_R + i\bar{\psi}_L \not{\partial} \psi_L) d^4x}$$

$$= \text{Const} \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \, e^{\int (i\bar{\psi} \not{\partial} \psi + i\bar{\psi} \not{A} P_R \psi) d^4x}$$

where $P_R = \dfrac{1+\gamma_5}{2}$ projection to R-components.

For the purpose of computation of anomaly, we can consider the Dirac fermion ψ with values in V where A is coupled to $\psi_R = P_R \psi$ only.

$$J = i\bar{\psi}_R \delta\!\!\!/ A \psi_R = i\bar{\psi}\, \delta\!\!\!/ A \, P_R \psi .$$

Now let us compute $-\delta_1 \cdots \delta_n \delta_\epsilon W[A]\big|_{A=0}$.

$$-\delta_\epsilon W[A=0] = \left\langle \int d^4x \, D_\mu \epsilon \cdot J^\mu \right\rangle = 0 \text{ by "Lorentz" inv.}$$

$$-\delta\,\delta_\epsilon W[A]\big|_{A=0} = \left\langle \int d^4x\, \partial_\mu \epsilon\, J^\mu \int d^4y\, \delta A\cdot J \right\rangle_{conn}$$

$$+ \left\langle \int d^4x \left[\delta A_\mu, \epsilon\right]\cdot J^\mu \right\rangle \rightsquigarrow 0 \text{ by "Lorentz"}$$

$$\delta A = dx^\nu e^b e^{-iqx}$$

$$= \int d^4x\, \partial_\mu \epsilon_a(x) \int d^4y\, e^{-iqy}\, \mathrm{tr}_{V\otimes S}\left(\gamma^\mu e^a P_R \psi(x)\bar\psi(y)\gamma^\nu e^b P_R \psi(y)\bar\psi(x)\right)$$

$$= \int d^4x\, \underbrace{\partial_\mu \epsilon_a(x) e^{-iqx}}_{\sim\, i q_\mu \epsilon_a(x) e^{-iqx}}\, \mathrm{tr}_V(e^a e^b) \underbrace{\int \frac{d^4k}{(2\pi)^4} \mathrm{tr}_S\left(\gamma^\mu P_R \frac{1}{-\not k}\gamma^\nu P_R \frac{1}{-(\not k-\not q)}\right)}_{\text{!!}}$$

☆ quadratically divergent.

<u>Pauli-Villars regularization</u>   Introduce 3 regulators :

name	0	1	2	3
mass	$\Lambda_0 = 0$	Λ_1	Λ_2	Λ_3
statistics	fermi ($\epsilon_0 = 1$)	bose ($\epsilon_1 = -1$)	fermi ($\epsilon_2 = 1$)	bose ($\epsilon_3 = -1$)
	original	regulators		

This replaces ☆ by

$$I^{\mu\nu}(q) := \int \frac{d^4k}{(2\pi)^4} \sum_{i=0}^{3} \epsilon_i\, \mathrm{tr}_S\left(\gamma^\mu P_R \frac{1}{-\not k + \Lambda_i}\gamma^\nu P_R \frac{1}{-(\not k-\not q)+\Lambda_i}\right)$$

It turns out that the integral is convergent if $\sum_{i=1}^{3} \epsilon_i \Lambda_i^2 = 0$.
Then, after some computation, we find

$$I^{\mu\nu}(q) = \frac{-2}{(4\pi)^2}\Big[(\delta^{\mu\nu} q^2 - q^\mu q^\nu) \frac{1}{3}\Big(\log q^2 - \frac{5}{3} + \sum_{i=1}^{3} \epsilon_i \log \Lambda_i^2\Big) + \delta^{\mu\nu}\Big(\sum_{i=1}^{3} \epsilon_i \Lambda_i^2 \log \Lambda_i^2 - \frac{1}{6} q^2\Big)\Big]$$

and hence

$$-\delta \delta_\epsilon W[A]\Big|_{A=0}$$

$$= i\int d^4x\, \epsilon_a(x)\, e^{iqx}\, \mathrm{tr}_V(e^a e^b)\, q_\mu I^{\mu\nu}(q)$$

$$q_\mu I^{\mu\nu}(q) = \frac{-2}{(4\pi)^2} q^\nu \Big(\sum_{i=1}^{3} \epsilon_i \Lambda_i^2 \log \Lambda_i^2 - \frac{1}{6} q^2\Big)$$

$$\neq 0.$$

This does not match with $\delta \mathcal{Q}_\epsilon[A] = 0$ for the claimed formula.

If fact, this can be cancelled by adding a local counter term $\Delta W[A]$ to $W[A]$. In fact

$$\Delta W[A] = \int d^4x\, \mathrm{tr}_V\big(A_\mu \delta^{\mu\lambda}(C + D\partial^2) A_\lambda\big)$$

with $C = \frac{-1}{(4\pi)^2} \sum_{i=1}^{3} \epsilon_i \Lambda_i^2 \log \Lambda_i^2$ & $D = -\frac{1}{6(4\pi)^2}$

does the job.

We may also add

$$\Delta' W[A] = E \int d^4x \, \mathrm{tr}_V \left[A_\mu \left(\delta^{\mu\nu} \partial^2 - \partial^\mu \partial^\nu \right) A_\nu \right]$$

which has $\delta \delta_\epsilon \Delta' W[A] \big|_{A=0} = 0.$

We shall consider $W'[A] = W[A] + \Delta W[A] + \Delta' W[A]$

for the above C and D and for some $E = \hat{E}/3(4\pi)^2$

Next, let us compute $-\delta_2 \delta_1 \delta_\epsilon W'[A] \big|_{A=0}$

$$= \boxed{ - \int d^4x \, D_\mu \epsilon_a(x) \, \delta_2 \delta_1 \frac{\delta W'[A]}{\delta A_{\mu a}(x)} \bigg|_0 } =: ☆$$

$$\boxed{ - \int d^4x \, [\delta_2 A_\mu, \epsilon]_a(x) \, \delta_1 \frac{\delta W'[A]}{\delta A_{\mu a}(x)} \bigg|_0 } + (1 \leftrightarrow 2) \quad =: ⊛$$

for $\delta_1 A = dx^\nu e^b e^{-iqx}$ & $\delta_2 A = dx^\rho e^c e^{-ipx}$

⊛ is basically computed in $-\delta_1 \delta_\epsilon W[A] \big|_0$ and is

$$= - \int d^4x \, e^{-i(p+q)x} \, \mathrm{tr}_V \left(\epsilon(x) [e^b, e^c] \right) \left(\delta^{\mu\nu} q^2 - q^\mu q^\nu \right) \times$$

$$\left[\frac{2}{3(4\pi)^2} \left(\log q^2 - \frac{5}{3} + \sum_{i=1}^{3} \epsilon_i \log \Lambda_i^2 \right) - 2E \right]$$

$$\star = \sum_{i=0}^{3} \left\langle \int d^4x \, i\bar{\psi}_i \not{\partial}\epsilon P_R \psi_i \int d^4y \, i\bar{\psi}_i \not{\epsilon}_1 A P_R \psi_i \int d^4z \, i\bar{\psi}_i \not{\epsilon}_2 A P_R \psi_i \right\rangle_{\text{conn}}$$

$$= -\int d^4x \, \epsilon_a(x) \, e^{-i(p+q)x} \, \mathrm{tr}_V(e^a e^b e^c) \int \frac{d^4k}{(2\pi)^4} \sum_{i=0}^{3} \epsilon_i \, g(k, \Lambda_i)$$

$$+ \text{ exchange} \; ;$$

$$g(k,m) = \mathrm{tr}_S\left((\not{p}+\not{q}) P_R \frac{1}{-(\not{k}+\not{q})+m} \gamma^\nu P_R \frac{1}{-\not{k}+m} \gamma^\rho P_R \frac{1}{-(\not{k}-\not{p})+m} \right)$$

$$= \mathrm{tr}_S\Bigg(\gamma^\nu P_R \frac{1}{-\not{k}+m} \gamma^\rho P_R \frac{1}{-(\not{k}+\not{q})+m}$$

$$- \gamma^\nu P_R \frac{1}{-\not{k}+m} \gamma^\rho P_R \frac{1}{-(\not{k}-\not{p})+m}$$

$$- \frac{m^2 P_R (\not{p}+\not{q}) \gamma^\nu \not{k} \gamma^\rho}{((k+q)^2+m^2)(k^2+m^2)((k-p)^2+m^2)} \Bigg).$$

$$\int \frac{d^4k}{(2\pi)^4} \sum_{i=0}^{3} \epsilon_i \, g(k, \Lambda_i) = I^{\nu\rho}(-q) - I^{\nu\rho}(p) + J^{\nu\rho}(p,q) \; ;$$

$$J^{\nu\rho}(p,q) := \int \frac{d^4k}{(2\pi)^4} \sum_{i=1}^{3} \epsilon_i \frac{(-\Lambda_i^2) \, \mathrm{tr}_S(P_R (\not{p}+\not{q}) \gamma^\nu \not{k} \gamma^\rho)}{((k+q)^2+\Lambda_i^2)(k^2+\Lambda_i^2)((k-p)^2+\Lambda_i^2)}$$

$$= \frac{1}{3(4\pi)^2} \left\{ (\delta^{\nu\rho} q^2 - 2q^\nu q^\rho) - (\delta^{\nu\rho} p^2 - 2p^\nu p^\rho) - 2\epsilon^{\lambda\nu\sigma\rho} q_\lambda p_\sigma \right\}.$$

$+$ terms that vanish as $p/\Lambda_i \to 0$, $q/\Lambda_i \to 0$.

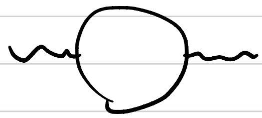
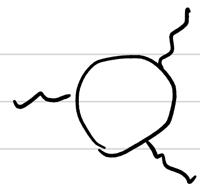
$$\text{✪} = -\int d^4x\, \epsilon_a(x)\, e^{-i(p+q)x}\, \mathrm{tr}_V(e^a e^b e^c)\left[I^{\nu\rho}(-q) - I^{\nu\rho}(p) + J^{\nu\rho}(p,q) \right]$$
$$+ (\nu, b, q) \leftrightarrow (\rho, c, p)$$

$$= \int d^4x\, \epsilon_a(x)\, e^{-i(p+q)x} \frac{2}{3(4\pi)^2}\Big[\mathrm{tr}_V(e^a\{e^b, e^c\})\, \epsilon^{\lambda\nu\sigma\rho} q_\lambda p_\sigma$$
$$+ \mathrm{tr}_V(e^a[e^b, e^c]) \Big\{ (\delta^{\nu\rho} q^2 - q^\nu q^\rho)\left(\log q^2 - \frac{8}{3} + \sum_{i=1}^{3} \epsilon_i \log \Lambda_i^2\right)$$
$$- (\delta^{\nu\rho} p^2 - p^\nu p^\rho)\left(\log p^2 - \frac{8}{3} + \sum_{i=1}^{3} \epsilon_i \log \Lambda_i^2\right) \Big\} \Big]$$

If we set $E = 1/3(4\pi)^2$, the $\delta^{\nu\rho} q^2 - q^\nu q^\rho$ & $\delta^{\nu\rho} p^2 - p^\nu p^\rho$ terms cancel in ✪ + ⊛ + $⊛_{1\leftrightarrow 2}$, and we find

$$-\delta_2 \delta_1 \delta_\epsilon W'[A]\big|_{A=0}$$
$$= \int d^4x\, e^{-i(p+q)x} \frac{2}{3(4\pi)^2} \mathrm{tr}_V(\epsilon(x)\{e^b, e^c\})\, \epsilon^{\lambda\nu\sigma\rho} q_\lambda p_\sigma .$$
$$= \delta_2 \delta_1\, i \int \frac{i}{24\pi^2} \mathrm{tr}_V(\epsilon\, dA \wedge dA).$$

$$\boxed{\therefore\quad a_\epsilon[A] = \int \frac{i}{24\pi^2} \mathrm{tr}_V\left(\epsilon\, d\left(A dA + \frac{1}{2} A^3\right)\right) \quad \text{modulo } O(A^3) \text{ terms}}$$

<u>Summary</u> By computing  &  ,

we have seen

$$a^5_\epsilon[A] = \int \frac{-1}{4\pi^2}\, \epsilon\, \mathrm{tr}_V(dA\,dA) + O(A^3)$$

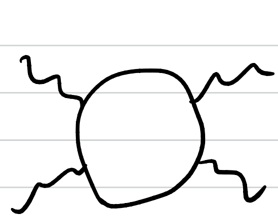
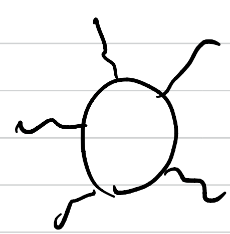
$$a^R_\epsilon[A] = \int \frac{i}{24\pi^2}\, \mathrm{tr}_{V_R}(\epsilon\, dA\,dA) + O(A^3)$$

modulo δ^R_ϵ local $[A]$.

This is consistent with

$$a^5_\epsilon[A] = \int \frac{-1}{4\pi^2}\, \epsilon\, \mathrm{tr}_V(F_A^2)$$

$$a^R_\epsilon[A] = \int \frac{i}{24\pi^2}\, \mathrm{tr}_{V_R}\left(\epsilon\, d\left(A\,dA + \tfrac{1}{2}A^3\right)\right).$$

We may compute  ,  , ...

to fix the higher order terms $O(A^3)$.

But that is not necessary if we use the general structure of anomalies.

The general structure of anomalies

① The anomaly $a_\epsilon[A,\phi]$ is local, i.e.

$$\int d^dx \text{ polynomial of derivatives of } (\epsilon, A, \phi),$$

because it comes from regularization procedure, which is non-trivial only for divergent diagrams $\leftrightarrow$ local.

② (Already discussed) There is a freedom to modify the action by local counter terms. Thus the anomaly $a_\epsilon[A,\phi]$ is defined only modulo δ_ϵ local functional of $[A,\phi]$.

③ $$\delta_{\epsilon_1}\delta_{\epsilon_2}\mathcal{D}_A\phi - \delta_{\epsilon_2}\delta_{\epsilon_1}\mathcal{D}_A\phi = \delta_{[\epsilon_1,\epsilon_2]}\mathcal{D}_A\phi$$

(we've been considering right action). Thus

$$\boxed{\delta_{\epsilon_1} a_{\epsilon_2}[A,\phi] - \delta_{\epsilon_2} a_{\epsilon_1}[A,\phi] = a_{[\epsilon_1,\epsilon_2]}[A,\phi]}$$

Wess-Zumino consistency condition

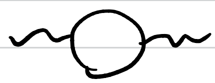
Let us consider the case of chiral anomaly with $V_L = \{0\}$.

(i) $$a^R_\epsilon[A] = \underset{\substack{\uparrow \\ \text{constant}}}{c} \int \mathrm{tr}_{V_R}\left(\epsilon\, d\left(AdA + \tfrac{1}{2}A^3\right)\right)$$

Satisfies the WZ consistency condition. (Exercise)

(ii) It is also the unique solution to the WZ condition (of course modulo δ_ϵ local $[A]$) with the "initial condition"

$$a^R_\epsilon[A] = c \int \mathrm{tr}_{V_R}(\epsilon\, dA dA) + O(A^3).$$

Thus, our computation from  &  is enough to prove

$$a^R_\epsilon[A] = \int \frac{i}{24\pi^2} \mathrm{tr}_{V_R}\left(\epsilon\, d\left(AdA + \tfrac{1}{2}A^3\right)\right).$$

In view of the relation between $a^5_\epsilon[A]$ & $a^{tot}_{(\epsilon,0)}[0,A]$ for $U(1)_5 \times G$, this also confirms

$$a^5_\epsilon[A] = \int \frac{-1}{4\pi^2}\, \epsilon\, \mathrm{tr}_V(F_A^2).$$

Fujikawa's method

... an alternative, direct method to compute anomalies.

Axial anomaly (G ungauged)

Dirac fermion in a representation V of G

$$S = \int d^4x\,(-i)\,\overline{\psi}\,\not{D}_A\,\psi .$$

Axial anomaly [Note: right action $\psi^\epsilon = (e^{i\epsilon\gamma_5})^{-1}\psi$]

$$\mathcal{D}(\overline{\psi}e^{-i\epsilon\gamma_5})\,\mathcal{D}(e^{-i\epsilon\gamma_5}\psi) = (\mathrm{Det}\, e^{i\epsilon\gamma_5})^2\,\mathcal{D}\overline{\psi}\,\mathcal{D}\psi$$

$$\therefore\ i a^5_\epsilon = \frac{d}{dt}\left(\mathrm{Det}\, e^{it\epsilon\gamma_5}\right)^2\Big|_{t=0} = 2\,\mathrm{Tr}(i\epsilon\gamma_5)$$

..... divergent.

A regularization:

$$a^5_\epsilon[A] = 2\,\mathrm{Tr}\left(\epsilon\gamma_5\, e^{-\not{D}_A^2/\Lambda^2}\right).$$

Note: $\not{D}_A^\dagger = \not{D}_A$ since $\gamma^{\mu\dagger} = -\gamma^\mu$ and $D_\mu^\dagger = -D_\mu$.

Thus $\not{D}_A$ has real eigenvalues and hence

$e^{-\not{D}_A^2/\Lambda^2}$ can provide a regularization.

Use the plane wave basis $\varphi_{k,i,\alpha}(x) = e^{ikx}\, e_i \otimes e_\alpha$

of the space of V-valued spinors

$(\{e_i\} \subset V,\ \{e_\alpha\} \subset S$ basis $)$

to evaluate the trace:

$$G^5_\epsilon[A] = 2\,\mathrm{Tr}\left(\epsilon\,\gamma_5\, e^{-\not{D}_A^2/\Lambda^2}\right)$$

$$= 2\int d^4x \int \frac{d^4k}{(2\pi)^4} \sum_{i,\alpha} \left(\varphi_{k,i,\alpha}(x),\ \epsilon(x)\,\gamma_5\, e^{-\not{D}_A^2/\Lambda^2}\, \varphi_{k,i,\alpha}(x)\right)$$

$$= 2\int d^4x\, \epsilon(x) \int \frac{d^4k}{(2\pi)^4}\, \mathrm{tr}_{V\otimes S}\left(\gamma_5\, e^{-ikx}\, e^{-\not{D}_A^2/\Lambda^2}\, e^{ikx}\right)$$

$$e^{-ikx}\,\not{D}_A\, e^{ikx} = \gamma^\mu(ik_\mu + \partial_\mu + A_\mu),$$

$$e^{-ikx}\,\not{D}_A^2\, e^{ikx} = \gamma^\mu\gamma^\nu(ik_\mu + \partial_\mu + A_\mu)(ik_\nu + \partial_\nu + A_\nu)$$

- $\gamma^\mu\gamma^\nu = \frac{1}{2}\{\gamma^\mu,\gamma^\nu\} + \frac{1}{2}[\gamma^\mu,\gamma^\nu] = -\delta^{\mu\nu} + \gamma^{\mu\nu}$
- $(ik_\mu + \partial_\mu + A_\mu)(ik_\nu + \partial_\nu + A_\nu) - (\mu \leftrightarrow \nu) = F_{\mu\nu}$

$$= -\delta^{\mu\nu}(ik_\mu + \partial_\mu + A_\mu)(ik_\nu + \partial_\nu + A_\nu) + \frac{1}{2}\gamma^{\mu\nu} F_{\mu\nu}$$

$$= k^2 \underbrace{- 2ik^\mu(\partial_\mu + A_\mu) + (\partial^\mu + A^\mu)(\partial_\mu + A_\mu)}_{=: X} + \frac{1}{2}\gamma^{\mu\nu}F_{\mu\nu}.$$

$$e^{-ikx}\, e^{-\not{D}_A^2/\Lambda^2}\, e^{ikx} = \exp\left(-e^{-ikx}\,\not{D}_A^2/\Lambda^2\, e^{ikx}\right)\cdot 1$$

$$= e^{-k^2/\Lambda^2 - (X + \frac{1}{2}\gamma^{\mu\nu}F_{\mu\nu})/\Lambda^2}\cdot 1$$

$$= e^{-k^2/\Lambda^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\left[\left(X + \frac{1}{2}\gamma^{\mu\nu}F_{\mu\nu}\right)/\Lambda^2\right]^n \cdot 1$$

- As $\mathrm{tr}_S(\gamma_5\gamma^{\mu_1}\cdots\gamma^{\mu_s}) = 0$ if $s \leq 3$, at least two powers of $\frac{1}{2}\gamma^{\mu\nu}F_{\mu\nu}/\Lambda^2$ is necessary to survive $\mathrm{tr}_S(\gamma_5 \text{ --- })$.

- $\displaystyle\int \frac{d^4k}{(2\pi)^4}\, e^{-k^2/\Lambda^2} (X/\Lambda^2)^\ell (\gamma^{\mu\nu}F_{\mu\nu}/\Lambda^2)^{m+2}\cdot 1$ or its reordering

$$\sim \Lambda^4 \frac{\Lambda^{p\leq \ell}}{(\Lambda^2)^{\ell+m+2}} \leq \Lambda^{-\ell-2m}$$

$\therefore$ Only the terms with $\ell = m = 0$ survives the limit $\Lambda \to \infty$.

$$\mathcal{A}_\epsilon^5[A] = 2\int d^4x\, \epsilon(x) \int \frac{d^4k}{(2\pi)^4}\, e^{-k^2/\Lambda^2}\, \frac{1}{2}\,\mathrm{tr}_{V\otimes S}\left[\gamma_5\left(\frac{1}{2}\gamma^{\mu\nu}F_{\mu\nu}/\Lambda^2\right)^2\right]$$

Use $\displaystyle\int \frac{d^4k}{(2\pi)^4}\, e^{-k^2/\Lambda^2} = \frac{\Lambda^4}{(4\pi)^2}$

$$= \int d^4x\, \epsilon(x) \frac{1}{(4\pi)^2} \mathrm{tr}_V(F_{\mu\nu} F_{\rho\lambda}) \underbrace{\frac{1}{4} \mathrm{Tr}_S(\gamma_5 \gamma^{\mu\nu} \gamma^{\rho\lambda})}_{-\epsilon^{\mu\nu\rho\lambda}}$$

$$= \int d^4x\, \epsilon(x) \frac{-1}{16\pi^2} \epsilon^{\mu\nu\rho\lambda} \mathrm{tr}_V(F_{\mu\nu} F_{\rho\lambda})$$

$$= \int \frac{-1}{4\pi^2} \epsilon\, \mathrm{tr}_V(F_A \wedge F_A).$$

Note: The axial rotation group is Abeian

$$e^{i\epsilon_1 \gamma_5} \circ e^{i\epsilon_2 \gamma_5} = e^{i(\epsilon_1+\epsilon_2)\gamma_5}$$

and the infinitesimal formula $a^5_\epsilon[A] = \int \frac{-1}{4\pi^2} \epsilon\, \mathrm{tr}_V F_A^2$ is additive

$$a^5_{\epsilon_1}[A] + a^5_{\epsilon_2}[A] = a^5_{\epsilon_1+\epsilon_2}[A].$$

Thus, it **integrates** to the anomaly formula for finite axial rotations,

$$\mathcal{D}_A(\bar\psi e^{-i\epsilon\gamma_5}) \mathcal{D}_A(e^{-i\epsilon\gamma_5}\psi) = \mathcal{D}_A\bar\psi\, \mathcal{D}_A\psi\, e^{i a^5_\epsilon[A]}$$

$$= \mathcal{D}_A\bar\psi\, \mathcal{D}_A\psi \exp\left[i \int \frac{-1}{4\pi^2} \epsilon\, \mathrm{tr}_V(F_A^2) \right]$$

Chiral anomaly

$V_R = V$, $V_L = \{0\}$

For the purpose of computing the anomaly, we may consider a Dirac fermion with values in V

$$S = \int d^4x\,(-i)\,\overline{\Psi}\,\not{D}_{A,R}\Psi \quad ; \quad \not{D}_{A,R} = \not{\partial} + \not{A}P_R$$

Chiral rotation :

$$\Psi^g = (gP_R + P_L)^{-1}\Psi, \quad \overline{\Psi}^g = \overline{\Psi}(P_R + gP_L)$$

$$\mathcal{D}\overline{\Psi}^g\mathcal{D}\Psi^g = \mathrm{Det}(P_R + gP_L)^{-1}\cdot\mathrm{Det}(gP_R + P_L)\,\mathcal{D}\overline{\Psi}\mathcal{D}\Psi$$

$$= \mathrm{Det}\left[(P_R + g^{-1}P_L)(gP_R + P_L)\right]\mathcal{D}\overline{\Psi}\mathcal{D}\Psi$$

$$= \mathrm{Det}(gP_R + g^{-1}P_L)\,\mathcal{D}\overline{\Psi}\mathcal{D}\Psi$$

$$\therefore\ i a^R_\epsilon = \mathrm{Tr}(\epsilon P_R - \epsilon P_L) = \mathrm{Tr}(\epsilon\gamma_5) \quad \cdots \text{divergent.}$$

A regularization :

$$i a^R_\epsilon[A] = \mathrm{Tr}\left(\epsilon\gamma_5\, e^{-\not{D}_{A,R}^2/\Lambda^2}\right)$$

⋮ similar computation

$$= \int \frac{-1}{24\pi^2}\,\mathrm{tr}_V\left[\epsilon\, d\left(AdA + \tfrac{1}{2}A^3\right)\right] + \delta_\epsilon\,\mathrm{loc}[A].$$

Axial anomaly in a general even dimension $d = 2n$

$$\{\gamma^\mu, \gamma^\nu\} = -2\delta^{\mu\nu} \qquad \mu,\nu = 1,\dots,d \quad \text{represented on } S = \mathbb{C}^{2^n}$$

$$(\gamma^1 \cdots \gamma^d)^2 = (-1)^{\frac{d(d-1)}{2}}$$

$$\gamma_{d+1} := i^{\frac{d(d-1)}{2}} \gamma^1 \cdots \gamma^d \; ; \quad \gamma_{d+1}^2 = 1, \quad \gamma_{d+1}\gamma^\mu = -\gamma^\mu \gamma_{d+1}$$

ψ a Dirac fermion on $\mathbb{R}^d$ with values in a rep V of G.

$$S = \int d^dx \, (-i)\, \bar\psi \not{D}_A \psi$$

Axial anomaly

$$\mathcal{D}_A(\bar\psi e^{-i\epsilon\gamma_{d+1}})\, \mathcal{D}_A(e^{-i\epsilon\gamma_{d+1}}\psi) = \mathcal{D}_A\bar\psi\, \mathcal{D}_A\psi \cdot e^{i a_\epsilon^{d+1}[A]} \; ;$$

$$a_\epsilon^{d+1}[A] = 2\,\mathrm{Tr}\left[\epsilon\, \gamma_{d+1}\, e^{-\not{D}_A^2/\Lambda^2}\right]$$

$$= 2\int d^dx\, \epsilon(x) \int \frac{d^dk}{(2\pi)^d}\, \mathrm{tr}_{V\otimes S}\left(\gamma_{d+1}\, e^{-ikx}\, e^{-\not{D}_A^2/\Lambda^2}\, e^{ikx}\right)$$

$\vdots$ same computation (Exercise)

$$= 2\int_{\mathbb{R}^d} \epsilon\, \frac{1}{n!}\, \mathrm{tr}_V \left(\frac{i}{2\pi} F_A\right)^n$$

Axial anomaly and index of Dirac operator

($d = 2n$ as above)

The axial anomaly for a constant ϵ

$$a_\epsilon^{d+1}[A] = 2\epsilon \, \mathrm{Tr}\left[\gamma_{d+1} \, e^{-\cancel{D}_A^2/\Lambda^2} \right]$$

has a topological meaning.

To avoid technical subtleties, it is better to replace $\mathbb{R}^d$ by a compact manifold X and consider a fermion ψ with values in a vector bundle E on X (X needs to be a spin manifold).

Instead of plane wave basis, we can use a basis consisting of eigenvectors of $\cancel{D}_A^{\,2}$.

By compactness of X, the spectrum of $\cancel{D}_A^2$ is discrete and the eigenvectors are square normalizable.

Suppose $\cancel{D}_A^2 \varphi = \lambda \varphi$. $\qquad \cancel{D}_A^\dagger = \cancel{D}_A$

$$\lambda \|\varphi\|^2 = (\varphi, \lambda\varphi) = (\varphi, \cancel{D}_A^2 \varphi) = (\cancel{D}_A \varphi, \cancel{D}_A \varphi) = \|\cancel{D}_A \varphi\|^2.$$

$\therefore \ \lambda \geq 0$ and $\lambda = 0 \Leftrightarrow \cancel{D}_A \varphi = 0$.

Also, as $\gamma_{d+1} \not{D}_A = -\not{D}_A \gamma_{d+1}$, γ_{d+1} commutes with $\not{D}_A^2$.

Of course, $\not{D}_A$ commutes with $\not{D}_A^2$.

$\therefore$ $\gamma_{d+1}\varphi$ and $\not{D}_A\varphi$ are also eigenvectors of $\not{D}_A^2$ with the same eigenvalue λ.

Suppose $\lambda > 0$. If φ is right handed ($\gamma_{d+1} = 1$), then $\not{D}_A\varphi$ is left handed ($\gamma_{d+1} = -1$). Furthermore,

$\not{D}_A(\not{D}_A\varphi) = \not{D}_A^2\varphi = \lambda\varphi \propto \varphi$.

I.e. there is a one-to-one correspondence between righ handed & left handed eigenvectors.

<u>Summary</u>: Let $S(E) = \bigoplus_\lambda S_\lambda(E)$ be the decomposition of spinors with values in E into the $\not{D}_A$ eigenspaces

Let $S_\lambda(E) = \underset{\gamma_{d+1}=1}{S_\lambda^R(E)} \oplus \underset{\gamma_{d+1}=-1}{S_\lambda^L(E)}$ be the R-L decomposition

For $\lambda > 0$, $\not{D}_A : S_\lambda^R(E) \xrightarrow{\cong} S_\lambda^L(E)$ is a linear isomorphism.

Then $\mathrm{Tr}_{S(E)}\left(\gamma_{d+1}\, e^{-\not{D}_A^2/\Lambda^2}\right) = \sum_\lambda \mathrm{Tr}_{S_\lambda(E)}(\gamma_{d+1})\, e^{-\lambda/\Lambda^2}$

and $\mathrm{Tr}_{S_\lambda(E)}(\gamma_{d+1}) = \mathrm{Tr}_{S_\lambda^R(E)}(+1) + \mathrm{Tr}_{S_\lambda^L(E)}(-1)$

$$= \dim S_\lambda^R(E) - \dim S_\lambda^L(E)$$

$$= 0 \text{ if } \lambda > 0$$

$\therefore\ \mathrm{Tr}_{S(E)}\left(\gamma_{d+1}\, e^{-\not{D}_A^2/\Lambda^2}\right) = \mathrm{Tr}_{S_0(E)}(\gamma_{d+1})$

$$= \dim S_0^R(E) - \dim S_0^L(E).$$

Note: $S_0^R(E) = \mathrm{Ker}\left(\not{D}_A : S^R(E) \to S^L(E)\right)$

$$S_0^L(E) = \mathrm{Ker}\big(\underbrace{\not{D}_A : S^L(E) \to S^R(E)}_{\text{adjoint of } \not{D}_A : S^R(E) \to S^L(E)}\big)$$

$$= \mathrm{Im}\left(\not{D}_A : S^R(E) \to S^L(E)\right)^\perp$$

$$= S^L(E) / \mathrm{Im}\left(\not{D}_A : S^R(E) \to S^L(E)\right)$$

$$=: \mathrm{Coker}\left(\not{D}_A : S^R(E) \to S^L(E)\right)$$

$$\therefore \operatorname{Tr}_{S(E)}\left(\gamma_{d+1}\, e^{-\cancel{D}_A^2/\Lambda^2}\right)$$

$$= \dim \operatorname{Ker}\left(\cancel{D}_A : S^R(E) \to S^L(E)\right)$$

$$- \dim \operatorname{Coker}\left(\cancel{D}_A : S^R(E) \to S^L(E)\right)$$

$$=: \operatorname{index}\left(\cancel{D}_A : S^R(E) \to S^L(E)\right)$$

Thus for a constant ϵ,

$$a_\epsilon^{d+1}[A] = 2\epsilon \cdot \operatorname{index}\left(\cancel{D}_A : S^R(E) \to S^L(E)\right)$$

We can see this also by mode expansion of $\psi, \bar{\psi}$

$$\mathcal{D}_A\left(\bar{\psi}\, e^{-i\epsilon\gamma_{d+1}}\right) \mathcal{D}_A\left(e^{-i\epsilon\gamma_{d+1}}\psi\right)$$

$$= e^{2i\epsilon \dim S_0^R(E)} \cdot e^{-2i\epsilon \dim S_0^L(E)} \cdot \mathcal{D}_A\bar{\psi}\, \mathcal{D}_A\psi$$

Atiyah-Singer index formula

$$\text{index}\left(\not{D}_A : S_R(E) \to S_L(E)\right) = \int_X ch(E)\,\hat{A}_X$$

where

$$ch(E) = tr_E\left(e^{\frac{i}{2\pi}F_A}\right) \quad \text{Chern character of } E$$

$$\hat{A}_X = \underbrace{1 - \frac{1}{24}p_1(X) + \cdots}_{\substack{\text{some power series} \\ \text{of Pontrjagin classes of } TX}} \quad \text{A-roof genus of } X$$

In the flat space, $\hat{A}_X = 1$, and it reads

$$\int_X \frac{1}{n!}\, tr_E\left(\left(\frac{i}{2\pi}F_A\right)^n\right)$$

... the same expression as found by Fujikawa's method.