A bound on Yang-Mills action (G simple)

$$S_E[A] = \int_{\mathbb{R}^4} d^4x \, \frac{1}{4e^2} \underbrace{\sum_{\mu,\nu=1}^{4} F_{\mu\nu} \cdot F_{\mu\nu}}_{-2\,\mathrm{Tr}(F_{\mu\nu}F_{\mu\nu})}$$

"standard trace"

$$= -\frac{1}{g^2} \int_{\mathbb{R}^4} \sum_{\mu<\nu} \mathrm{Tr}\, F_{\mu\nu} F_{\mu\nu} \qquad \geq 0$$

$A = A_\mu dx^\mu$ $\quad$ $\mathfrak{g}$-valued 1-form

$F_A = \frac{1}{2} F_{\mu\nu} dx^\mu dx^\nu = dA + \frac{1}{2}[A, A]$ $\quad$ $\mathfrak{g}$-valued 2-form

Hodge $*$ operation : p-form $\mapsto (4-p)$-form

η, ω : p-forms

$$\eta \wedge * \omega = \omega \wedge * \eta = \underbrace{(\eta, \omega)}_{\sum_{\mu_1 < \cdots < \mu_p} \eta_{\mu_1 \cdots \mu_p} \omega_{\mu_1 \cdots \mu_p}} d^4x$$

$= dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4$

$** = (-1)^p$ on p-forms in 4d

$*\omega \wedge * \eta = \omega \wedge \eta$ for ω p-form & η $(4-p)$-form

$$*(dx^\mu \wedge dx^\nu) = \frac{1}{2} \sum_{\rho,\lambda} \epsilon^{\mu\nu\rho\lambda} dx^\rho \wedge dx^\lambda$$

e.g. $*(dx^1 \wedge dx^2) = dx^3 \wedge dx^4$, $\; * dx^1 \wedge dx^3 = -dx^2 \wedge dx^4$, ...

$*$ can be defined on any oriented Riemannian mfd of any dimension.

$$S_E[A] = -\frac{1}{e^2}\int_{\mathbb{R}^4} \mathrm{Tr}(F_A \wedge * F_A)$$

$$= -\frac{1}{e^2}\int_{\mathbb{R}^4}\left\{\frac{1}{2}\mathrm{Tr}\big((F_A \pm * F_A)\wedge *(F_A \pm * F_A)\big) \mp \mathrm{Tr}(\underbrace{F_A \wedge F_A}_{F_A^2})\right\}$$

$$= \frac{1}{4e^2}\int_{\mathbb{R}^4} \| F_A \pm * F_A \|^2 d^4x \pm \frac{1}{e^2}\int_{\mathbb{R}^4} \mathrm{Tr}(F_A^2)$$

$$\geq \pm\frac{1}{e^2}\int_{\mathbb{R}^4}\mathrm{Tr}(F_A^2)$$

$$\geq \left|\frac{1}{e^2}\int_{\mathbb{R}^4}\mathrm{Tr}(F_A^2)\right| .$$

If $\int_{\mathbb{R}^4}\mathrm{Tr}(F_A^2) \gtrless 0$, the bound is saturated by A obeying

$$F_A \pm * F_A = 0 .$$

Note : $\underset{\substack{\uparrow \\ \text{1st order differential eqn}}}{F_A \pm * F_A = 0} \overset{(\cancel{*})}{\Rightarrow}$ EL eqn $\underset{\substack{\uparrow \\ \text{2nd order differential eqn}}}{\sum_\mu D_\mu F_{\mu\nu} = 0}$ (YM eqn)

Def <u>Pontrjagin index</u> of A

$$\nu[A] := -\frac{1}{8\pi^2}\int_{\mathbb{R}^4}\mathrm{Tr}(F_A^2)$$

Recall from Lecture 12:

- $S_E[A] < \infty \Rightarrow A \to g^{-1}dg$ at $|x| \to \infty$
- $\mathrm{Tr}(F_A^2) = d\,\mathrm{Tr}\left(A\,dA + \frac{2}{3}A^3\right)$

$$\nu[A] = -\frac{1}{8\pi^2}\int_{\mathbb{R}^4} d\,\mathrm{Tr}\left(A\,dA + \frac{2}{3}A^3\right)$$

$$= -\frac{1}{8\pi^2}\int_{S^3_\infty} \mathrm{Tr}\left(A\,dA + \frac{2}{3}A^3\right)\Big|_{A = g^{-1}dg}$$

$$= \frac{1}{24\pi^2}\int_{S^3_\infty} \mathrm{Tr}\left((g^{-1}dg)^3\right)$$

$=: n[g]$ the winding number of $g : S^3_\infty \to G$.

<u>Fact</u> G simple & simply connected (e.g. $G = SU(n), USp(n), Spin(n)$)

$$\pi_3(G) \xrightarrow{\cong} \mathbb{Z} \;;\; [g] \longleftrightarrow n[g]$$

<u>Exercise</u> $G = SU(2)$. For $g : S^3 \to G$ defined by

$$g(x) = \begin{pmatrix} x_4 + ix_3 & ix_1 + x_2 \\ ix_1 - x_2 & x_4 - ix_3 \end{pmatrix}; \; |x| = 1$$

Show that $n[g] = 1$.

θ vacua

Consider the Yang-Mills theory with simple & simply connected G.

Formulate it on a box of finite size $V \times T$ with a boundary condition s.t. A has a definite Pontryagin index $k \in \mathbb{Z}$.

We eventually take $V \to \infty$, $T \to \infty$.

$$Z(V, T, k) = \int_{\mathcal{A}/\mathcal{G}} \frac{\mathcal{D}A}{\mathrm{Vol}\,\mathcal{G}}\, e^{-S_E[A]} \underbrace{\delta_{\nu[A], k}}_{= \int_0^{2\pi} \frac{d\theta}{2\pi} e^{i\theta(\nu[A]-k)}}$$

$$= \int_0^{2\pi} \frac{d\theta}{2\pi} e^{-ik\theta} \underbrace{\int_{\mathcal{A}/\mathcal{G}} \frac{\mathcal{D}A}{\mathrm{Vol}\,\mathcal{G}}\, e^{-S_E[A] + i\theta\nu[A]}}_{Z(V, T, \theta)}$$

We assume that there are (anti-) instantons, i.e., localized solutions to

$$F_A \pm * F_A = 0, \quad \nu[A] = \pm 1$$

$$\Rightarrow \quad S_E[A] = \frac{8\pi^2}{e^2} =: S_0 .$$

By the dilute gas approximation, we have

$$Z(V,T,\theta)\Big|_{d.g.} = \sum_{n,\bar{n}} e^{-(n+\bar{n})S_0} K^{n+\bar{n}} \frac{(VT)^{n+\bar{n}}}{n!\,\bar{n}!} e^{i(n-\bar{n})\theta}$$

$$= \exp\left[KVT e^{-S_0+i\theta} + KVT e^{-S_0-i\theta} \right]$$

$$= \exp\left[-VT\,\mathcal{E}(\theta) \right]$$

$$\boxed{\mathcal{E}(\theta) = -2K e^{-8\pi^2/e^2} \cos(\theta)}$$

As in the case of periodic potential, we find a continuum of vacuum states, labelled by $\theta \in \mathbb{R}/2\pi\mathbb{Z}$

These are called the θ-vacua.

We shall discuss the meaning of this observation in the canonical/Hamitonian formulation of the theory.

Hamiltonian formulation of YM theory

Minkowski (real time) action

$$S[A] = \int -\frac{1}{4e^2} F^{\mu\nu} \cdot F_{\mu\nu} \, d^4x$$

$$= \int d^4x \left(\frac{1}{2e^2} \sum_i F_{0i}^2 - \frac{1}{2e^2} \sum_{i<j} F_{ij}^2 \right)$$

$\uparrow$ integrate out $\mathbb{E}$

$$S[\mathbb{A}, \mathbb{E}; A_0] = \int d^4x \left(\sum_i E_i F_{0i} - \frac{e^2}{2} \sum_i E_i^2 - \frac{1}{2e^2} \sum_{i<j} F_{ij}^2 \right)$$

$$= \int d^4x \left(\sum_i E_i \dot{A}_i - \frac{e^2}{2} \sum_i E_i^2 - \frac{1}{2e^2} \sum_{i<j} F_{ij}^2 + A_0 \sum_i D_i E_i \right)$$

- $A_0(\mathbf{x})$ is a Lagrange multiplier imposing a constraint

$$\mathbb{D} \cdot \mathbb{E} = 0 \qquad \text{Gauss law.}$$

- $A_i(\mathbf{x})$ & $E_i(\mathbf{x})$ are canonically conjugate variables

$$\{ A_{ia}(\mathbf{x}), E_{jb}(\mathbf{y}) \} = \delta_{ij} \delta_{ab} \delta(\mathbf{x}-\mathbf{y}).$$

- Hamiltonian

$$H(\mathbb{E}, \mathbb{A}) = \int d^3\mathbf{x} \left(\frac{e^2}{2} \sum_i E_i(\mathbf{x})^2 + \frac{1}{2e^2} \sum_{i<j} F_{ij}(\mathbf{x})^2 \right).$$

More on the constraint

$$\Phi(x) := \mathbb{D}\cdot\mathbb{E} = \sum_i \left(\partial_i E_i + [A_i, E_i]\right) = 0 :$$

For a $\mathfrak{g}$-valued function $\epsilon(x)$ of x, put

$$\Phi_\epsilon := \int d^{d-1}x\ \epsilon(x)\cdot\Phi(x) = -\int d^{d-1}x\ \mathbb{D}\epsilon(x)\cdot\mathbb{E}.$$

It obeys

$$\{\Phi_\epsilon, \mathbb{A}(x)\} = \mathbb{D}\epsilon(x)$$

$$\{\Phi_\epsilon, \mathbb{E}(x)\} = [\mathbb{E}, \epsilon](x)$$

$\therefore$ Φ_ϵ generates the gauge transformation by $\epsilon(x)$.

As H is gauge invariant, $\{\Phi_\epsilon, H\} = 0$.

Also, $\{\Phi_{\epsilon_1}, \Phi_{\epsilon_2}\} = \Phi_{[\epsilon_1,\epsilon_2]}$.

The Hamiltonian system of this type is called the system with a <u>first class constraint</u>.

Methods to quantize such a system have been developed.

See the additional note.

<u>One proposal</u>: physical states are wave functionals

$\Psi[\mathbb{A}]$ which satisfy the Gauss law

$$\widehat{\Phi}_\epsilon \Psi[\mathbb{A}] = 0 \qquad \forall \epsilon.$$

Since $\widehat{\mathbb{E}}(\mathbb{x}) = -i \frac{\delta}{\delta \mathbb{A}(\mathbb{x})}$, this means

$$\delta_\epsilon \Psi[\mathbb{A}] = 0, \qquad \forall \epsilon$$

i.e. invariance under infinitesimal gauge transformations.

To be precise, we need to impose a boundary condition at ∞.

As one natural choice, we take

$$\mathbb{A}(\mathbb{x}) \to 0 \qquad \text{as } |\mathbb{x}| \to \infty$$

$$\epsilon(\mathbb{x}) \to 0 \qquad \text{as } |\mathbb{x}| \to \infty$$

Let $\mathcal{A}$ be the space of such $\mathbb{A}(\mathbb{x})$'s and

$\mathfrak{g}$ be the Lie algebra of such $\epsilon(\mathbb{x})$'s.

$\mathfrak{g}$ generates the identity component $\mathcal{G}_0$ of the group

$$\mathcal{G} = \{ g : \mathbb{R}^3 \to G \mid g(\mathbb{x}) \to 1 \ \text{as } |\mathbb{x}| \to \infty \}.$$

An element $g \in \mathcal{G}$ defines a map $g : S^3 = \mathbb{R}^3 \cup \{\infty\} \longrightarrow G$.

It belongs to $\mathcal{G}_0$ if and only if it has no winding #:

$$g \in \mathcal{G}_0 \iff n[g] = 0.$$

Furtheremore,

$$\mathcal{G}/\mathcal{G}_0 \cong \pi_3(G) \cong \mathbb{Z}$$

$\mathbb{A} \mapsto \mathbb{A}^g$ by $g \in \mathcal{G} \setminus \mathcal{G}_0$ is called a

large gauge transformations.

The above proposal: physical states are functionals

Ψ on $\mathcal{A}$ which are invariant under $\mathcal{G}_0$:

$$\Psi[A^g] = \Psi[A] \quad \forall g \in \mathcal{G}_0$$

In other words, they are functionals on $\mathcal{A}/\mathcal{G}_0$.

It does not require invariance under large gauge transformations. Then, how should they transform?

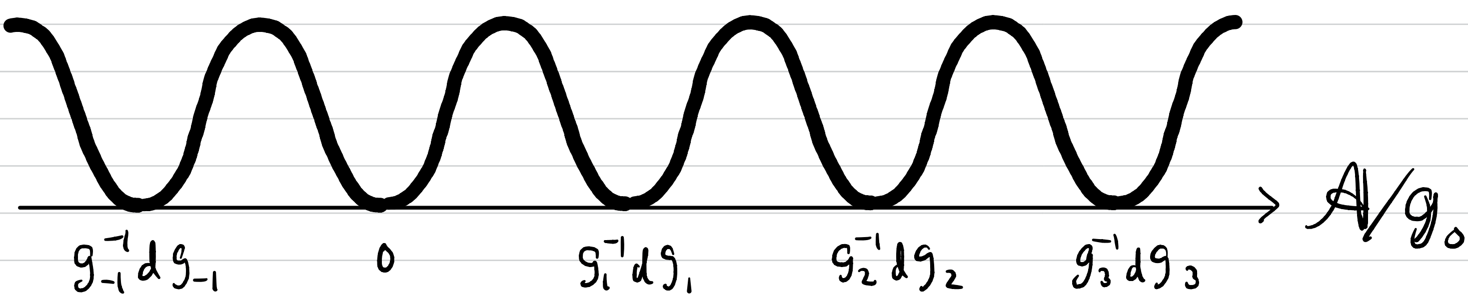
$$H = \int_{\mathbb{R}^3} d^3x \left(\frac{e^2}{2} \mathbb{E}(x)^2 + \frac{1}{2e^2} \mathbb{F}_{\mathbb{A}}(x)^2 \right) \Rightarrow$$

The potential is $U[\mathbb{A}] = \frac{1}{2e^2} \int_{\mathbb{R}^3} d^3x \, \mathbb{F}_{\mathbb{A}}(x)^2$

- It is invariant under $\mathbb{A} \mapsto \mathbb{A}^g$ for $\forall g \in \mathcal{G}$, including large gauge transformations.

- $U[\mathbb{A}] \geq 0$
  $= 0 \iff \mathbb{A} = g^{-1} dg$ for some $g \in \mathcal{G}$

$\therefore$ As a function on $\mathcal{A}/\mathcal{G}_0$, the potential looks like

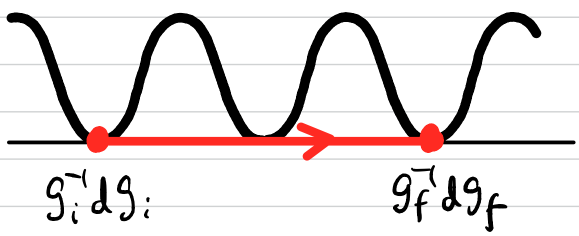


where $g_j \in \mathcal{G}$, $n[g_j] = j \in \mathbb{Z}$.

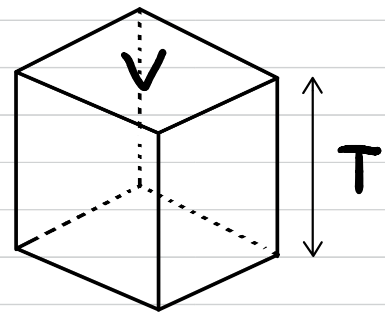
The system is similar to QM with periodic potential.

As in that case, we may consider tunnelling between different minima.

Tunnelling



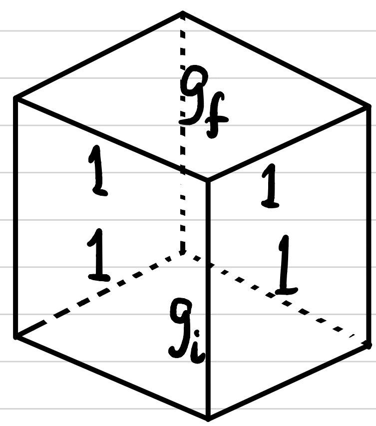
Put the system in a box



Boundary condition:

$$\left.\begin{array}{lll} A \to 0 & & x \to \partial V \\ A \to g_i^{-1} d g_i & & \tau \to -\frac{T}{2} \\ A \to g_f^{-1} d g_f & & \tau \to \frac{T}{2} \end{array}\right\} \quad A\big|_{\partial(V\times T)} = g^{-1} d g$$

where $g : \partial(V\times T) \to G$



$$\partial(V\times T) = \underset{\substack{\downarrow \\ 1}}{\partial V \times T} + \underset{\substack{\downarrow \\ g_i}}{V\times\{-\tfrac{T}{2}\}} - \underset{\substack{\downarrow \\ g_f}}{V\times\{\tfrac{T}{2}\}}$$

$$\therefore \quad \underset{\substack{\shortparallel \\ \nu[A]}}{n[g]} = n[g_i] - n[g_f]$$

$$Z_{V,T}(g_f^{-1}dg_f,\, g_i^{-1}dg_i)$$

$$= \sum_{n,\bar{n}} e^{-(n+\bar{n})S_0} K^{n+\bar{n}} \frac{(VT)^{n+\bar{n}}}{n!\,\bar{n}!} \underbrace{\delta_{n-\bar{n},\, n[g_i]-n[g_f]}}_{\int \frac{d\theta}{2\pi} e^{i\theta(n-\bar{n}-n[g_i]+n[g_f])}}$$

$$= \int \frac{d\theta}{2\pi} e^{i(n[g_f]-n[g_i])\theta} \exp\left(KVT e^{-8\pi^2/e^2+i\theta} + KVT e^{-8\pi^2/e^2-i\theta} \right)$$

$$= \int \frac{d\theta}{2\pi} \underbrace{e^{in[g_f]\theta - in[g_i]\theta} \exp\left(-VT\,\mathcal{E}(\theta)\right)}_{\text{should be identified with } \Psi_\theta[g_f^{-1}dg_f]\, e^{-VT\mathcal{E}(\theta)}\, \Psi_\theta[g_i^{-1}dg_i]^*}$$

$$\rightsquigarrow \Psi_\theta[g^{-1}dg] \propto e^{in[g]\theta}$$

The eigenstate $\Psi_\theta[A]$ is <u>not</u> invariant under large gauge transformations (unless $\theta \in 2\pi\mathbb{Z}$) since

$$\Psi_\theta[A^g] = e^{in[g]\theta}\, \Psi_\theta[A].$$

$\mathcal{G}$ (or $\mathcal{G}/\mathcal{G}_0 \cong \mathbb{Z}$) keeps the Hamiltonian invariant and hence is a symmetry of the system.

Therefore, the energy eigenstate Ψ_θ ($\theta \in \mathbb{R}/2\pi\mathbb{Z}$) is expected to be an eigenstate of this symmetry:

$$\Psi_\theta[A^g] = \underbrace{c_\theta(n[g])}_{\uparrow}\,\Psi_\theta[A].$$

depends only on the winding number $n[g]$ of g by the Gauss law.

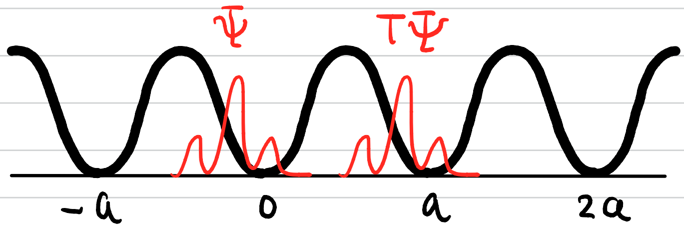
What we've seen, $\Psi_\theta[0^g] = e^{in[g]\theta}\,\Psi_\theta[0]$, is enough to determine the eigenvalue:

$$c_\theta(n) = e^{in\theta}.$$

Thus, we conclude

$$\boxed{\Psi_\theta[A^g] = e^{in[g]\theta}\cdot\Psi_\theta[A] \quad \forall g \in \mathcal{G}} \quad (\star)$$

Similar to the shift symmetry T of periodic potential



$$(T\Psi)(x) = \Psi(x-a)$$

localized state $|j\rangle$ at $x = ja$ with

$$H|j\rangle = E_0|j\rangle - \Delta|j-1\rangle - \Delta|j+1\rangle \quad \text{(tight binding)}$$

$$T|j\rangle = |j+1\rangle$$

$$\Rightarrow \quad \Psi_\theta = \sum_{j\in\mathbb{Z}} e^{ij\theta}|j\rangle \quad \text{eigenstate of } H \,\&\, T$$

$$H\Psi_\theta = (E_0 - 2\Delta\cos\theta)\Psi_\theta$$

$$T\Psi_\theta = e^{-i\theta}\Psi_\theta$$

$$\Psi_\theta(x-na) = e^{-in\theta}\Psi_\theta(x), \quad \cdots \text{ similar to } (\star).$$

$\rightsquigarrow$ Bloch wave function $\Psi_\theta(x) = e^{i\theta\frac{x}{a}}\underbrace{u_\theta(x)}_{\text{periodic.}}$

$\boxed{Q}$ Similar expression for $\Psi_\theta[A]$ in Yang-Mills theory?

What is the analog of $e^{i\theta\frac{x}{a}}$?

$\longrightarrow$ Chern-Simons functional

Chern-Simons functional

Y an oriented 3-dimensional manifold ($\mathbb{R}^3$, V, S^3 etc),

A a $\mathfrak{g}$-valued 1-form on Y.

The Chern-Simons functional of A on Y is

$$\boxed{CS_Y[A] := -\frac{1}{8\pi^2}\int_Y \mathrm{Tr}\left(A\,dA + \frac{2}{3}A^3\right)}$$

Some properties:

- For a variation of A, $\delta CS_Y[A] = -\frac{1}{4\pi^2}\int_Y \mathrm{Tr}(\delta A \wedge F_A)$

  $\therefore$ EL eqn is $F_A = 0$ (flatness)

- For $g : Y \to G$, $\mathrm{Tr}\left(A^g dA^g + \frac{2}{3}{A^g}^3\right)$

$$= \mathrm{Tr}\left(A\,dA + \frac{2}{3}A^3\right) - \frac{1}{3}\mathrm{Tr}(g^{-1}dg)^3 - d\,\mathrm{Tr}(dg\,g^{-1} \wedge A)$$

Thus, if $\partial Y = \phi$ or $\mathrm{Tr}(dg\,g^{-1}A)\big|_{\partial Y} = 0$,

$$CS_Y[A^g] = CS_Y[A] + \underbrace{\frac{1}{24\pi^2}\int_Y \mathrm{Tr}(g^{-1}dg)^3}_{=:\, n_Y[g]}$$

- Recall $\mathrm{Tr}\, F_A^2 = d\,\mathrm{Tr}(A dA + \frac{2}{3} A^3)$.

Thus for an oriented 4-dimensional manifold X possibly with boundary ∂X and a $\mathfrak{g}$-valued 1-form A on X

$$\nu_X[A] := -\frac{1}{8\pi^2}\int_X \mathrm{Tr}\, F_A^2 = CS_{\partial X}[A|_{\partial X}].$$

e.g. $X = \mathbb{R}^3 \times [\tau_i, \tau_f]$ (Euclidean)

$$\partial X = \underbrace{\partial \mathbb{R}^3 \times [\tau_i, \tau_f]}_{A=0 \text{ here}} + \mathbb{R}^3 \times \{\tau_i\} - \mathbb{R}^3 \times \{\tau_f\}$$

$$\nu_{\mathbb{R}^3 \times [\tau_i,\tau_f]}[A] = CS_{\mathbb{R}^3}[A|_{\tau_i}] - CS_{\mathbb{R}^3}[A|_{\tau_f}].$$

In Minkowski space

$$S_\theta[A] := \frac{\theta}{8\pi^2}\int_{\mathbb{R}^3 \times [t_i, t_f]} d^4x\, \epsilon^{ijk}\, \mathrm{Tr}(F_{0i} F_{jk})$$

$$= \theta\, CS_{\mathbb{R}^3}[A|_{t_i}] - \theta\, CS_{\mathbb{R}^3}[A|_{t_f}]$$

$CS[A] := CS_{\mathbb{R}^3}[A]$ obeys

$$CS[A^g] = CS[A] + n[g].$$

So, if we write

$$\Psi_\theta[A] = e^{i\theta CS[A]}\, \mathcal{U}_\theta[A]$$

Then, $\mathcal{U}_\theta[A]$ is invariant under all $g \in \mathcal{G}$.

"Bloch-type wave functional".

We may consider more general states $\Psi[A]$ with the same transformation property as $\Psi_\theta[A]$, i.e.

$$\Psi[A^g] = e^{in[g]\theta}\, \Psi[A] \quad \forall g \in \mathcal{G}.$$

If we write $\Psi[A] = e^{i\theta CS[A]}\, \mathcal{U}[A]$, then

$\mathcal{U}[A]$ is invariant under all $g \in \mathcal{G}$.

Let Ψ_i & Ψ_f be such states with a common θ.

Then, the transition amplitude between them is

$$(\Psi_f, e^{-i(t_f - t_i)H} \Psi_i)$$

$$= \text{"}\int\text{"}\ e^{iS[A]}\ \underbrace{\Psi_f[A(t_f)]^* \Psi_i[A(t_i)]}\ $$

$$e^{\overbrace{-i\theta CS[A(t_f)] + i\theta CS[A(t_i)]}^{}}\ \mathcal{U}_f[A(t_f)]^* \mathcal{U}_i[A(t_i)]$$

$$\underbrace{\qquad\qquad}_{iS_\theta[A]}$$

$$= \text{"}\int\text{"}\ e^{iS[A] + iS_\theta[A]}\ \mathcal{U}_f[A(t_f)]^* \mathcal{U}_i[A(t_i)]$$

We may consider the θ-sector in which

① All states transform as $\Psi[A^g] = e^{in[g]\theta}\Psi[A]$ for $g \in \mathcal{G}$ and the action is $S[A]$,

Or equivalently

② All states are invariant under $\mathcal{G}$ but the action is $S[A] + S_\theta[A]$

Note that the state Ψ_θ (or $\mathcal{U}_\theta$) belongs to this sector as the ground state. That's why call it the θ-vacuum.

All states obtained from Ψ_θ (or $\mathcal{U}_\theta$) by operating gauge invariant local operators are in the same sector.

This implies that sectors of different values of θ, e.g. θ_1-sector and θ_2-sector with $\theta_1 \not\equiv \theta_2 \pmod{2\pi\mathbb{Z}}$ do not mix with each other.

We shall consider different sectors to be different QFTs. In other words, to specify a theory, we need to specify the value of $\theta \in \mathbb{R}/2\pi\mathbb{Z}$.

If the states are as in (1) or (2) with a fixed $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, to avoid infinity, the path-integral must be over $\mathcal{A}/\mathcal{G}$ where $\mathcal{G}$ consists of g which does <u>not</u> have to satisfy $n[g|_t] = 0$:

$$\left(\Psi_f, e^{-i(t_f-t_i)H}\Psi_i\right)$$

$$= \int_{\mathcal{A}/\mathcal{G}} \frac{DA}{\mathrm{Vol}\,\mathcal{G}} e^{iS[A]} \Psi_f[A(t_f)]^* \Psi_i[A(t_i)] \quad \text{in } 1$$

$$= \int_{\mathcal{A}/\mathcal{G}} \frac{DA}{\mathrm{Vol}\,\mathcal{G}} e^{iS[A]+iS_\theta[A]} \mathcal{U}_f[A(t_f)]^* \mathcal{U}_i[A(t_i)] \quad \text{in } 2$$