Renormalization group

Choices of renormalization conditions :

"On shell", "intermediate", "Another (μ)", ...

All these originate from the same classical Lagrangian

$\longrightarrow$ Same physics.

But we need a dictionary :

renormalization condition I $\rightarrow$ $\phi_I, \lambda_I, \cdots$

renormalization condition II $\rightarrow$ $\phi_{II}, \lambda_{II}, \cdots$

relation ?

e.g. in 4d ϕ^4 theory

"another R.C." parametrized by a mass scale μ

renormalization point

$$\begin{cases} \Gamma(-p,p)\big|_{p^2=\mu^2} = \mu^2 + m^2 \\ \dfrac{d}{dp^2}\Gamma(-p,p)\Big|_{p^2=\mu^2} = 1 \\ \Gamma(p_1,p_2,p_3,p_4)\Big|_{p_i\cdot p_j = \begin{cases} \mu^2 & i=j \\ -\mu^2/3 & i\neq j \end{cases}} = \lambda \end{cases}$$

at μ : ϕ, m, λ

at μ' : ϕ', m', λ'

To describe the same physics,

how are they related?

Answer: so that the bare fields/couplings are the same.

$$Z_0^{\frac{1}{2}}(m,\lambda;\mu,\Lambda)\,\phi = \phi_0 = Z_0^{\frac{1}{2}}(m',\lambda';\mu',\Lambda)\,\phi'$$

$$m_0(m,\lambda;\mu,\Lambda) = m_0 = m_0(m',\lambda';\mu',\Lambda)$$

$$\lambda_0(m,\lambda;\mu,\Lambda) = \lambda_0 = \lambda_0(m',\lambda';\mu',\Lambda)$$

$$\Gamma_0(\phi_0, m_0, \lambda_0; \Lambda)$$

$$\Gamma(\phi,m,\lambda;\mu,\Lambda) = \Gamma_0(\phi_0, m_0, \lambda_0; \Lambda) = \Gamma(\phi',m',\lambda';\mu',\Lambda) \qquad \bigstar$$

The change $(\phi, m, \lambda) \to (\phi', m', \lambda')$ for $\mu \to \mu'$ is called the <u>renormalization group (RG) transformation</u>, and the equality $\bigstar$ is called the <u>RG equation</u>.

The relation between the renormalized fields/couplings has a limit as $\Lambda \to \infty$, and

$$\Gamma(\phi,m,\lambda;\mu) := \lim_{\Lambda\to\infty} \Gamma(\phi,m,\lambda;\mu,\Lambda) \text{ satisfies}$$

$$\Gamma(\phi,m,\lambda;\mu) = \Gamma(\phi',m',\lambda';\mu').$$

The RG transformation may be written as

$$\phi' = Z^{-\frac{1}{2}}(m,\lambda;\mu',\mu)\,\phi,$$

$$m' = R^{m}(m,\lambda;\mu',\mu),$$

$$\lambda' = R^{\lambda}(m,\lambda;\mu',\mu).$$

Put

$$\mu'\frac{\partial}{\partial\mu'} Z^{\frac{1}{2}}(m,\lambda;\mu',\mu)\Big|_{\mu'=\mu} =: \gamma(m,\lambda;\mu),$$

$$\mu'\frac{\partial}{\partial\mu'} R^{m}(m,\lambda;\mu',\mu)\Big|_{\mu'=\mu} =: -\gamma_m(m,\lambda;\mu)\,m,$$

$$\mu'\frac{\partial}{\partial\mu'} R^{\lambda}(m,\lambda;\mu',\mu)\Big|_{\mu'=\mu} =: \beta(m,\lambda;\mu).$$

Then, the <u>infinitesimal RG transformation</u> (<u>RG flow</u>) is

$$\mu\frac{d}{d\mu}\phi = -\gamma(m,\lambda;\mu)\,\phi,$$

$$\mu\frac{d}{d\mu}m = -\gamma_m(m,\lambda;\mu)\,m,$$

$$\mu\frac{d}{d\mu}\lambda = \beta(m,\lambda;\mu),$$

and the <u>infinitesimal RG equation</u> is

$$\left[-\gamma\,\phi\cdot\frac{\delta}{\delta\phi} - \gamma_m\, m\frac{\partial}{\partial m} + \beta\frac{\partial}{\partial\lambda} + \mu\frac{\partial}{\partial\mu}\right]\Gamma(\phi, m,\lambda;\mu) = 0.$$

Computation in 4d φ^4 theory

Recall $\quad Z_0 = 1 + \hbar\lambda a_1 + \hbar^2\lambda^2 a_2 + \cdots$

$$Z_0 m_0^2 = m^2 + \hbar\lambda b_1 + \hbar^2\lambda^2 b_2 + \cdots$$

$$Z_0^2\lambda_0 = \lambda + \hbar\lambda^2 c_1 + \hbar^2\lambda^3 c_2 + \cdots$$

$$a_1 = 0$$

$$b_1 = -\frac{1}{2(4\pi)^2}\left[\Lambda^2 - m^2\left(\log\left(\frac{\Lambda^2}{m^2}\right) + 1 - \gamma\right) + m^2\, O\!\left(\frac{m^2}{\Lambda^2}\right)\right]$$

$$c_1 = \frac{3}{2(4\pi)^2}\left[\log\left(\frac{\Lambda^2}{2m^2}\right) - \gamma - 1 - \int_0^1 dx \log\left(1 + x(1-x)\frac{4\mu^2}{3m^2}\right) + O\!\left(\frac{m^2}{\Lambda^2}\right)\right]$$

Thus, to 1-loop, $Z_0 = 1 + O(\hbar^2)$, and

$$0 = \mu\frac{d}{d\mu}\varphi_0 = \mu\frac{d}{d\mu}\varphi + O(\hbar^2)$$

$$0 = \mu\frac{d}{d\mu}m_0^2 = \mu\frac{d}{d\mu}m^2 + \hbar\left(\mu\frac{d}{d\mu}\lambda\, b_1 + \lambda\,\mu\frac{d}{d\mu}b_1\right) + O(\hbar^2)$$

$$0 = \mu\frac{d}{d\mu}\lambda_0 = \mu\frac{d}{d\mu}\lambda + \hbar\left(\mu\frac{d}{d\mu}\lambda^2 c_1 + \lambda^2\mu\frac{d}{d\mu}c_1\right) + O(\hbar^2)$$

$$\Rightarrow \quad \mu\frac{d}{d\mu}\varphi = O(\hbar^2)$$

$$\mu\frac{d}{d\mu}\lambda = -\hbar\lambda^2\mu\frac{d}{d\mu}c_1 + O(\hbar^2)$$

$$\mu\frac{d}{d\mu}m = O(\hbar^2)$$

$$\therefore \quad \gamma(m,\lambda;\mu) = O(\hbar^2)$$

$$\gamma_m(m,\lambda;\mu) = O(\hbar^2)$$

$$\beta(m,\lambda;\mu) = -\hbar\lambda^2 \mu \frac{d}{d\mu} C_1$$

$$= \frac{3\hbar\lambda^2}{2(4\pi)^2} \int_0^1 dx \, \frac{x(1-x)\frac{8\mu^2}{3m^2}}{1+x(1-x)\frac{4\mu^2}{3m^2}} + O(\hbar^2)$$

$$= \begin{cases} \dfrac{3\hbar\lambda^2}{(4\pi)^2} + O(\hbar^2) & \mu \gg m \\[2ex] \dfrac{2\hbar\lambda^2}{3(4\pi)^2}\dfrac{\mu^2}{m^2} + O(\hbar^2) & \mu \ll m \end{cases}$$

Let us consider the limit $m \to 0$.

As λ & Z are dimensionless, β & γ are functions of λ only:

$$\beta = \beta(\lambda), \quad \gamma = \gamma(\lambda).$$

Indeed, in this limit,

$$\gamma = 0, \quad \beta = \frac{3\lambda^2}{(4\pi)^2} \quad \text{at 1-loop.}$$

The RG flow : $\mu \frac{d}{d\mu} \lambda = \beta(\lambda)$

$$\mu \frac{d}{d\mu} \phi = -\gamma(\lambda)\, \phi$$

The RG eqn :

$$\left[\mu \frac{\partial}{\partial \mu} + \beta(\lambda) \frac{\partial}{\partial \lambda} - \gamma(\lambda)\, \phi \cdot \frac{\delta}{\delta \phi} \right] \Gamma(\phi, \lambda; \mu) = 0.$$

Instead of μ, we $t = \log(\mu/\mu_0)$ or $\mu = e^t \mu_0$.

Then, the RG flow takes the form

$$\lambda = \bar{\lambda}(t) \quad \left(\leftarrow \text{a solution to } \frac{d}{dt} \lambda = \beta(\lambda) \right)$$

$$\phi = \bar{\phi}(t) = \bar{\phi}(0) \cdot e^{-\int_0^t dt' \, \gamma(\bar{\lambda}(t'))}$$

and the RGE :

$\Gamma(\bar{\phi}(t), \bar{\lambda}(t); e^t \mu_0)$ is t-independent.

Write

$$\Gamma = \sum_{n=0}^{\infty} \frac{1}{n!} \int \prod_{i=1}^{n} \frac{d^4 p_i}{(2\pi)^4} \cdot (2\pi)^4 \delta(p_1 + \cdots + p_n)$$

$$\Gamma(p_1, \cdots, p_n, \lambda; \mu)\, \phi(p_1) \cdots \phi(p_n)$$

RGE :

$$e^{-n\int_0^t dt' \gamma(\bar\lambda(t'))} \Gamma(p_1, \cdots, p_n, \bar\lambda(t); e^t\mu_0) \quad \text{is } t\text{-independent}$$

$$= \Gamma(p_1, \cdots, p_n, \bar\lambda(0); \mu_0)$$

On the other hand, the canonical dimensions are

$$[\mu] = 1, \ [\phi] = 1, \ [\lambda] = 0, \ [\Gamma] = 0$$

$$\therefore [\Gamma(p_1, \cdots, p_n, \lambda; \mu)] = 4-n \Rightarrow$$

$$\Gamma(e^t p_1, \cdots, e^t p_n, \lambda; e^t\mu) = e^{(4-n)t} \Gamma(p_1, \cdots, p_n, \lambda; \mu).$$

"dimensional analysis"

Combining,

$$\boxed{\begin{aligned} &\Gamma(e^t p_1, \cdots, e^t p_n, \bar\lambda(0); \mu_0) \\ &\overset{\text{RGE}}{=} e^{-n\int_0^t dt' \gamma(\bar\lambda(t'))} \Gamma(e^t p_1, \cdots, e^t p_n, \bar\lambda(t); e^t\mu_0) \\ &\overset{\text{dim. an.}}{=} e^{4t - n\int_0^t dt' (1+\gamma(\bar\lambda(t')))} \Gamma(p_1, \cdots, p_n, \bar\lambda(t); \mu_0) \end{aligned}}$$

This means

(1) If we uniformly rescale the momenta as

$$p_i \to e^t p_i ,$$

the coupling λ effectively changes as

$$\bar{\lambda}(0) \to \bar{\lambda}(t).$$

$\bar{\lambda}(t)$ is the "effective coupling constant."

(2) The dimension of ϕ has also changed as

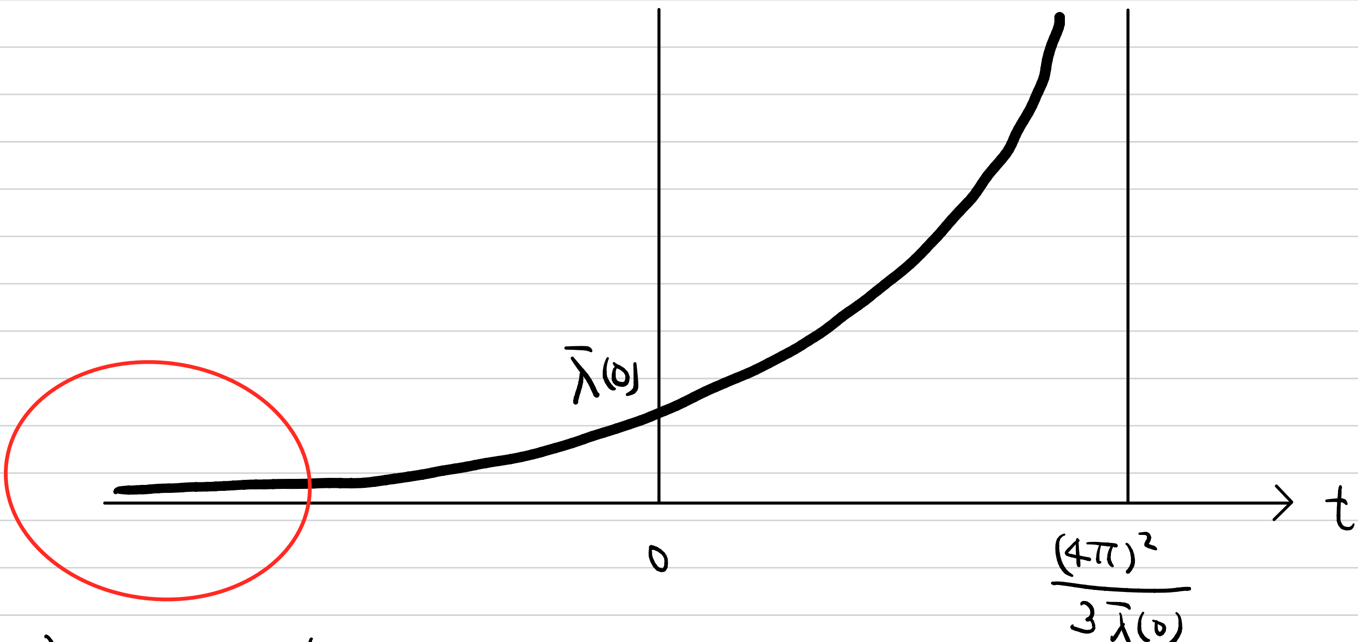
$$1 \to 1 + \gamma(\bar{\lambda}(t))$$

$\gamma(\lambda)$ is the "anomalous dimension" of ϕ.

At 1-loop, $\dfrac{d\lambda}{dt} = \dfrac{3\lambda^2}{(4\pi)^2}$.

$$\int \frac{d\lambda}{\lambda^2} = \int \frac{3dt}{(4\pi)^2} \sim -\frac{1}{\bar\lambda(t)} + \frac{1}{\bar\lambda(0)} = \frac{3t}{(4\pi)^2}$$

$$\boxed{\bar\lambda(t) = \frac{\bar\lambda(0)}{1 - \frac{3t}{(4\pi)^2}\bar\lambda(0)}}$$



$\lambda \to 0$ as $t \to -\infty$

$\mu = e^t \mu_0 \to 0$

The coupling is weaker at lower energies

or stronger at higher energies.

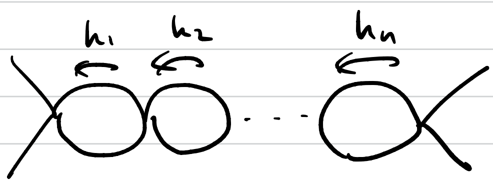
- $$\lambda(\mu) = \frac{\lambda(\mu_0)}{1 - \frac{3\lambda(\mu_0)}{(4\pi)^2} \log\left(\mu/\mu_0\right)}$$

  is valid for $\mu \ll \mu_0$ even if $\left|\log(\mu/\mu_0)\right|$ may be large.

- The series expansion

  $$\lambda(\mu) = \sum_{n=0}^{\infty} \lambda(\mu_0)\left(\frac{3\lambda(\mu_0)}{(4\pi)^2} \log\left(\mu/\mu_0\right)\right)^n$$

  has a Feynman diagram interpretation:

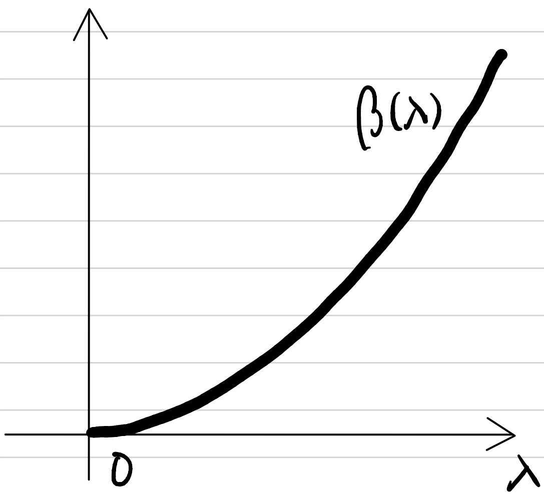


$$\sim \lambda^{n+1} \int_{\mu}^{\mu_0} \frac{d^4 k_1}{(k_1^2)^2} \cdots \int_{\mu}^{\mu_0} \frac{d^4 k_n}{(k_n^2)^2}$$

$$\sim \lambda^{n+1}\left(\log \mu_0/\mu\right)^n$$

"RG sums up a series of Feynman diagrams"

Various possibilities



$\lambda \to 0$ in the IR limit

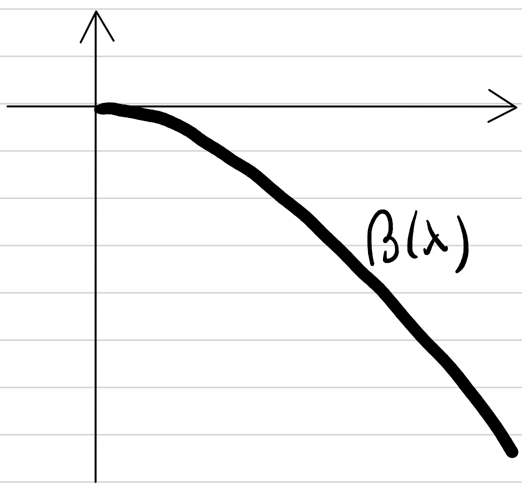
infra-red free theory

We've just seen — QFT II

e.g. 4d ϕ^4 theory, QED_4

QCD_4 with large # of flavors

next



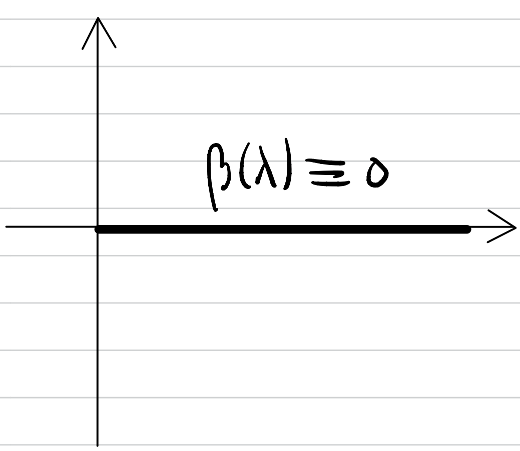
$\lambda \to 0$ in the UV limit

asymptotically free theory

e.g. 4d Yang-Mills theory

QCD_4 with small # of flavors

next



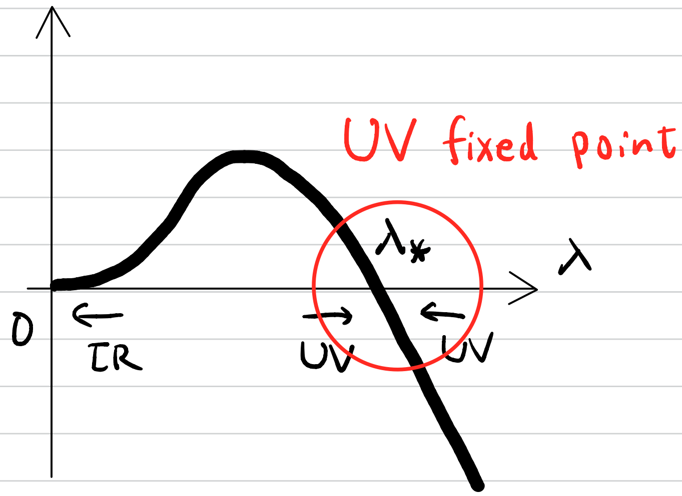
λ does not run!

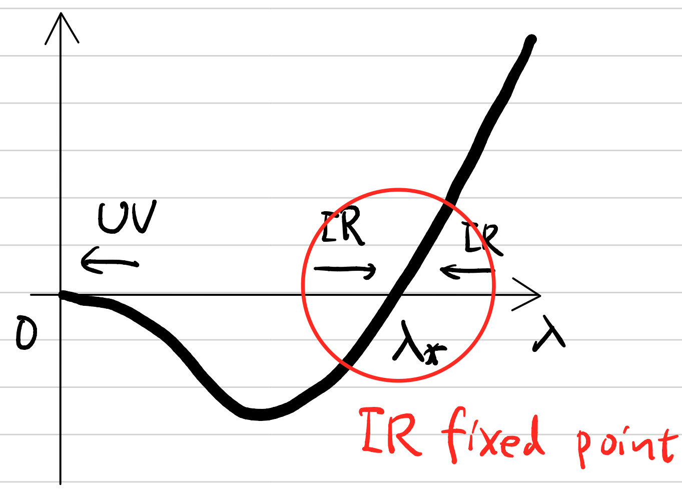
finite theory

e.g. 4d $\mathcal{N}=4$ supersymmetric Yang-Mills

next

Other possibilities :





$\therefore\ \exists$ <u>non-trivial fixed point of RG flow</u>.

At such a point λ_*, with $\gamma_* := \gamma(\lambda_*)$,

$$\Gamma(e^t p_1, \cdots, e^t p_n, \lambda_* ; \mu_0)$$

$$= e^{(4-n(1+\gamma_*))t}\, \Gamma(p_1, \cdots, p_n, \lambda_* ; \mu_0)$$

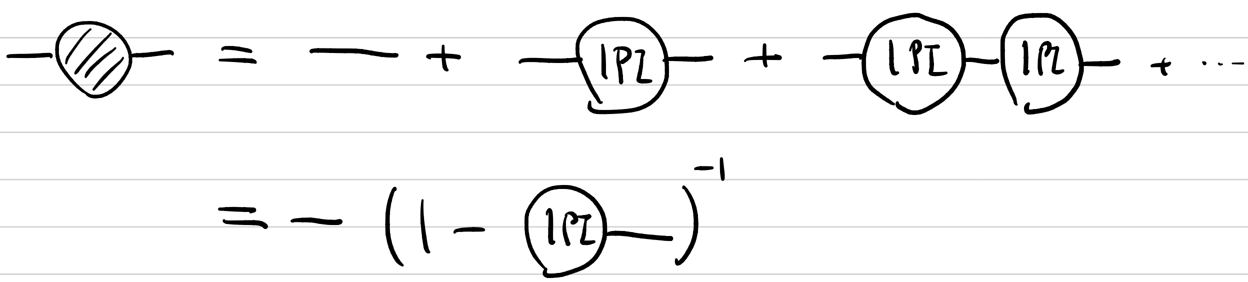
Correlation functions scales in a simple way.

"scale invariant theory".

e.g. $\Gamma(-p, p) = \text{const} \cdot (p^2)^{1-\gamma_*}$

$$\Gamma(\phi) = \frac{1}{2}\phi_i A_{ij}\phi_j - \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{i_1\cdots i_n}\lambda_{1PI}^{i_1\cdots i_n}\phi_{i_1}\cdots\phi_{i_n}$$

$$\Gamma(-p,p) = p^2 - \lambda^{(2)}_{1PI}(p^2)$$



$$= \frac{1}{p^2}\left(1-\lambda^{(2)}_{1PI}(p^2)\frac{1}{p^2}\right)^{-1} = \frac{1}{p^2-\lambda^{(2)}_{1PI}(p^2)} = \frac{1}{\Gamma(-p,p)}$$

$$\langle\phi(x)\phi(0)\rangle = \int\frac{d^4p}{(2\pi)^4}\frac{e^{-ipx}}{\Gamma(-p,p)} \quad \propto (p^2)^{1-\gamma_*}$$

$$\propto \frac{1}{|x|^{2+2\gamma_*}}$$

RG of 4d non-Abelian gauge theories

Consider a 4d gauge theory with gauge group G and a Dirac fermion ψ in a $\mathbb{C}$-representation V_f (and/or a scalar ϕ in a $\mathbb{C}$-representation V_b).

Aim: Compute the functions β, γ that determines the RG flow of couplings and fields.

In particular, the β-function for the gauge coupling constant e.

For this, we choose a renormalization condition and find out

$$Z_A,\ Z_\psi,\ Z_c,\ e_0,\ m_0$$

as a function of renormalized couplings e, m, renormalization point μ, and cut-off Λ.

What kind of renormalization condition can we impose?

We may impose a condition on the 1PI effective action $\Gamma[A, \psi, c, \bar{c}, B ; e, m, \xi ; \Lambda]$ so that $Z_A, Z_\psi, Z_c, e_0, m_0$ are fixed at each order in the loop expansion.

Recall

$$S[X_0, \overset{=0}{K_0} ; e_0, m_0, \xi_0] = O(1, \hbar, \cdots, \hbar^{N-1}, \hbar^N) \text{ from old}$$

$$+ \hbar^N \int d^4x \left\{ \frac{1}{4} \left(\frac{Z_A}{e_0^2}\right)^{(N)}_{new} F^{\mu\nu} \cdot F_{\mu\nu} + \frac{1}{2} \frac{\sqrt{Z_A}^{(N)}_{new}}{e^2} F^{\mu\nu} \cdot [A_\mu, A_\nu] \right.$$

$$- i Z_{\psi\,new}^{(N)} \bar{\psi} \not{\partial} \psi - i \sqrt{Z_A}^{(N)}_{new} \bar{\psi} \not{A} \psi + \bar{\psi} (Z_\psi m_0)^{(N)}_{new} \psi$$

$$\left. + \bar{c}\, \partial^\mu \left(Z_{c\,new}^{(N)} \partial_\mu c + \left(Z_{c\,new}^{(N)} + \sqrt{Z_A}^{(N)}_{new}\right) [A_\mu, c] \right) \right\}$$

$+ O(\hbar^{N+1})$.

$\rightsquigarrow$

A renormalization condition that constrains the terms

$\frac{1}{4e^2}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2$, $-i\bar{\psi}\not{\partial}\psi$, $\bar{\psi} m \psi$, $\bar{c}\partial^2 c$, $\bar{c}\partial^\mu [A_\mu, c]$

in Γ, for example.

Let us write

$$
\begin{aligned}
\Gamma = \int \frac{d^4p}{(2\pi)^4} \Big\{ & \frac{1}{2} A_{\mu a}(-p)\, \Gamma^{\mu\nu, ab}(-p, p)\, A_{\nu b}(p) \\
& + \bar{\Psi}(-p)\, \Gamma_\Psi(-p, p)\, \Psi(p) \\
& + \bar{C}(-p)\, \Gamma_{gh}(-p, p)\, C(p) \quad \Big\}
\end{aligned}
$$

$$
+ \int \frac{d^4p}{(2\pi)^4} \frac{d^4q}{(2\pi)^4}\, \bar{C}(-p-q)\, \Gamma_{gh}^{\mu a}(-p-q, q, p)\, A_{\mu a}(q)\, C(p)
$$

$$
+ \cdots .
$$

$$\Gamma_\Psi(-p, p) \in \mathrm{End}(V_f \otimes S)$$

$$\Gamma_{gh}(-p, p),\ \Gamma_{gh}^{\mu a}(-p-q, q, p) \in \mathrm{End}(\mathfrak{g})$$

By Ward identities associated with symmetries (Lorentz, global G, ghost #, BRST, ...) and additional arguments, these coefficient functions are constrained as:

$$\Gamma^{\mu\nu,ab}(-p,p) = \left(\delta^{\mu\nu}p^2 - p^\mu p^\nu\right)\delta^{ab}\,\Pi(p^2)$$

$$\Gamma_\psi(-p,p) = A(p^2)\,\not{p} + B(p^2)$$

$$\Gamma_{gh}(-p,p) = p^2\,\Pi_{gh}(p^2)$$

$$\Gamma_{gh}^{\mu a}(-p-q,q,p) = (p+q)^\mu e^a C(p,q) + q^\mu e^a D(p,q)$$

A, B, Π_{gh}, C, D must commute with G

$C(p,q)$ & $D(p,q)$ are functions of p^2, q^2, $p\cdot q$.

As a renormalization condition, we may take

$$\Pi(\mu^2) = \frac{1}{e^2}$$

$$A(\mu^2) = -1\,, \quad B(\mu^2) = m$$

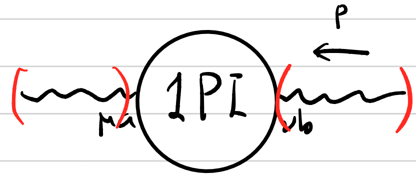
$$\Pi_{gh}(\mu^2) = -1$$

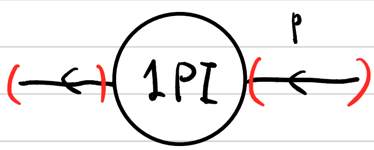
$$C(p,q)\Big|_{p^2=q^2=(p+q)^2=\mu^2} = -i$$

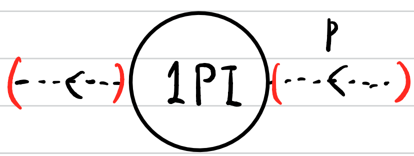
Recall

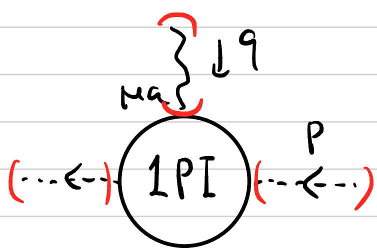
$$\Gamma = S_{free} - \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{i_1 \cdots i_n} \underbrace{\lambda_{1PI}^{i_1 \cdots i_n}}_{\text{1PI vertex}} \phi_{i_1} \cdots \phi_{i_n}$$

$\rightsquigarrow$

$$\Gamma^{\mu\nu,ab}(-p,p) = \frac{1}{e^2}\left(\delta^{\mu\nu}p^2 - p^\mu p^\nu\right)\delta^{ab} -$$ 

$$\Gamma_\psi(-p,p) = -\not{p} + m^2 -$$ 

$$\Gamma_{gh}(-p,p) = -p^2 -$$ 

$$\Gamma_{gh}^{\mu a}(-p-q,q,p) = -$$ 

Tree and one-loop contributions to 1PI vertices :