

The result

$$\begin{array}{c} p_a \\ \swarrow \\ \text{1-loop} \\ \searrow \\ p_b \end{array} = - (\delta^{\mu\nu} p^2 - p^\mu p^\nu) \delta^{ab} \frac{1}{(4\pi)^2} \int_0^1 dx \Gamma(2 - \frac{d}{2}) x$$

☆_A

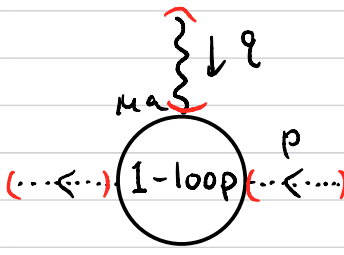
$$\left\{ \begin{array}{l} \tilde{\Delta}_{mf}^{\frac{d}{2}-2} T_{V_f} 8x(1-x) \\ + \tilde{\Delta}_{mb}^{\frac{d}{2}-2} T_{V_b} (2x-1)^2 \\ - \tilde{\Delta}_0^{\frac{d}{2}-2} h^\nu \left((1 - \frac{d}{2})(2x-1)^2 + 2 \right) \end{array} \right\}$$

where $\tilde{\Delta}_m := (x(1-x)p^2 + m^2) / 4\pi\mu_{DR}^2$

$$\begin{array}{c} \leftarrow \\ \text{1-loop} \\ \rightarrow \end{array} = C_2(V_f) \frac{e^2}{(4\pi)^2} \int_0^1 dx \Gamma(2 - \frac{d}{2}) \tilde{\Delta}_f^{\frac{d}{2}-2} \{ (d-2) x \not{p} - d m_f \}$$

where $\tilde{\Delta}_f := (x(1-x)p^2 + (1-x)m_f^2) / 4\pi\mu_{DR}^2$

$$\begin{array}{c} \dots \\ \text{1-loop} \\ \dots \end{array} = - h^\nu \frac{e^2}{(4\pi)^2} \int_0^1 dx \Gamma(2 - \frac{d}{2}) \tilde{\Delta}_0^{\frac{d}{2}-2} x p^2$$



$$(\dots \leftarrow \dots) \text{1-loop} (\dots \leftarrow \dots) = i \hbar^{\nu} a d e^{\alpha} \cdot \frac{e^2}{(4\pi)^2} \int_{\triangle} dy dz \Gamma\left(2 - \frac{d}{2}\right) \tilde{\Delta}_\nu^{\frac{d}{2}-2}$$

$$\left\{ (p+q)^\mu \frac{d}{4} + (4-d) \frac{p^\mu R + q^\mu S}{\Delta_\nu} \right\}$$

where $\triangle := \left\{ (y, z) \in \mathbb{R}^2 \mid \begin{array}{l} y, z \geq 0 \\ y+z \leq 1 \end{array} \right\}$

$$\Delta_\nu := y(1-y)q^2 + z(1-z)p^2 + 2yzqP$$

$$\tilde{\Delta}_\nu = \Delta_\nu / 4\pi M_{\text{Pl}}^2$$

R, S : linear in q^2, p^2 & qP ,

and at most linear in y & z .

∴ The functions that appear in the renormalization condition are, at 1-loop,

$$\Pi(p^2) = \frac{1}{e^2} + \frac{1}{(4\pi)^2} \int_0^1 dx \Gamma(2-\frac{d}{2}) \star_A + \left(\frac{Z_A}{e^2}\right)''$$

$$A(p^2) = -1 - C_2(V_f) \frac{e^2}{(4\pi)^2} \int_0^1 dx \Gamma(2-\frac{d}{2}) \tilde{\Delta}_f^{\frac{d}{2}-2} (d-2)x - Z_\psi''$$

$$B(p^2) = m_f + C_2(V_f) \frac{e^2}{(4\pi)^2} \int_0^1 dx \Gamma(2-\frac{d}{2}) \tilde{\Delta}_f^{\frac{d}{2}-2} dm_f + (Z_\psi m_0)''$$

$$\Pi_{gh}(p^2) = -1 + h^\nu \frac{e^2}{(4\pi)^2} \int_0^1 dx \Gamma(2-\frac{d}{2}) \tilde{\Delta}_0^{\frac{d}{2}-2} x - Z_c''$$

$$C_0(p, q) = -i - i h^\nu \frac{e^2}{(4\pi)^2} \int_{\triangle} dy dz \Gamma(2-\frac{d}{2}) \tilde{\Delta}_\nu^{\frac{d}{2}-2} \left(\frac{d}{4} + (4-d) \frac{R}{\Delta_\nu} \right) - i(Z_c'' + \sqrt{Z_A''})$$

The last terms are the contributions by the one-loop counter terms. The renormalization condition

$$\Pi(\mu^2) = \frac{1}{e^2}, \quad A(\mu^2) = -1, \quad B(\mu^2) = m_f, \quad \Pi_{gh}(\mu^2) = -1$$

$$C_0(p, q) \Big|_{p^2=q^2=(p+q)^2=\mu^2} = -i$$

determines them:

$$\left(\frac{Z_A}{e_s^2}\right)^{(1)} = -\frac{1}{(4\pi)^2} \int_0^1 dx \Gamma\left(2-\frac{d}{2}\right) \cancel{A} \Big|_{p^2=\mu^2}$$

$$Z_\psi^{(1)} = -C_2(V_f) \frac{e^2}{(4\pi)^2} \int_0^1 dx \Gamma\left(2-\frac{d}{2}\right) \tilde{\Delta}_f^{\frac{d}{2}-2} (d-2)x \Big|_{p^2=\mu^2}$$

$$(Z_{\psi m_0})^{(1)} = -C_2(V_f) \frac{e^2}{(4\pi)^2} \int_0^1 dx \Gamma\left(2-\frac{d}{2}\right) \tilde{\Delta}_f^{\frac{d}{2}-2} dm_f \Big|_{p^2=\mu^2}$$

$$Z_c^{(1)} = h^\nu \frac{e^2}{(4\pi)^2} \int_0^1 dx \Gamma\left(2-\frac{d}{2}\right) \tilde{\Delta}_0^{\frac{d}{2}-2} x \Big|_{p^2=\mu^2}$$

$$Z_c^{(1)} + \sqrt{Z_A}^{(1)}$$

$$= -h^\nu \frac{e^2}{(4\pi)^2} \int_{\triangle} dy dz \Gamma\left(2-\frac{d}{2}\right) \tilde{\Delta}_\nu^{\frac{d}{2}-2} \left(\frac{d}{4} + (4-d) \frac{R}{\Delta_\nu}\right) \Big|_{\substack{p^2=q^2 \\ =(p+r)^2=\mu^2}}$$

Note : $\Gamma\left(2-\frac{d}{2}\right) \tilde{\Delta}^{\frac{d}{2}-2} (\alpha + (d-4)\beta)$ for $d=4-\epsilon$

$$= \Gamma\left(\frac{\epsilon}{2}\right) \tilde{\Delta}^{-\frac{\epsilon}{2}} (\alpha + \epsilon\beta)$$

$$= \left(\frac{2}{\epsilon} - \gamma + O(\epsilon)\right) \left(1 - \frac{\epsilon}{2} \log \tilde{\Delta} + O(\epsilon)\right) (\alpha + \epsilon\beta)$$

$$= \left(\frac{2}{\epsilon} - \gamma - \log \tilde{\Delta}\right) \alpha + 2\beta + O(\epsilon)$$

Let us first consider the case $m_f, m_b \ll \mu$ and ignore $O(m_f^2/\mu^2, m_b^2/\mu^2)$ terms. Then, all of $\log \tilde{\Delta}$'s at the renormalization point = $\log \mu^2 + \mu$ -independent.

Thus, with $X := \frac{2}{\epsilon} - \log \mu^2$,

$$\left(\frac{\tilde{Z}_A}{e_0^2}\right)^{(\prime\prime)} = -\frac{1}{(4\pi)^2} \int_0^1 dx X \left\{ T_{V_f} 8x(1-x) + T_{V_b} (2x-1)^2 - h^v (-(2x-1)^2 + 2) \right\} + \dots$$

↖ μ independent terms + $O(\epsilon)$

$$= -\frac{1}{(4\pi)^2} X \left\{ \frac{4}{3} T_{V_f} + \frac{1}{3} T_{V_b} - \frac{5}{3} h^v \right\} + \dots$$

$$\tilde{Z}_\psi^{(\prime\prime)} = -C_2(V_f) \frac{e^2}{(4\pi)^2} \int_0^1 dx X \underbrace{2x}_{=1} + \dots$$

$$\left(\tilde{Z}_\psi m_0\right)^{(\prime\prime)} = -C_2(V_f) \frac{e^2}{(4\pi)^2} \int_0^1 dx X \underbrace{4m_f}_{=1} + \dots$$

$$\tilde{Z}_c^{(\prime\prime)} = h^v \frac{e^2}{(4\pi)^2} \int_0^1 dx X \underbrace{x}_{\frac{1}{2}} + \dots$$

$$\tilde{Z}_c^{(\prime\prime)} + \sqrt{\tilde{Z}_A}^{(\prime\prime)} = -h^v \frac{e^2}{(4\pi)^2} \int_{\triangle} dy dz X \underbrace{\quad}_{\frac{1}{2}} + \dots$$

$$\Rightarrow \sqrt{Z_A^{(1)}} = -h^\nu \frac{e^2}{(4\pi)^2} X + \dots, \quad Z_A^{(1)} = -2 h^\nu \frac{e^2}{(4\pi)^2} X + \dots$$

$$\left(\frac{Z_A}{e^2}\right)^{(1)} = \left(\frac{1}{e^2}\right)^{(1)} + \frac{Z_A^{(1)}}{e^2} \Rightarrow$$

$$\left(\frac{1}{e^2}\right)^{(1)} = -\frac{1}{(4\pi)^2} X \left(\frac{4}{3} T_{V_f} + \frac{1}{3} T_{V_b} - \frac{5}{3} h^\nu - 2 h^\nu \right) + \dots$$

$\underbrace{-\frac{11}{3} h^\nu}$

$$\therefore \frac{1}{e_0^2} = \frac{1}{e^2} - \frac{1}{(4\pi)^2} \left(\frac{2}{\epsilon} - \log \mu^2 \right) \left(\frac{4}{3} T_{V_f} + \frac{1}{3} T_{V_b} - \frac{11}{3} h^\nu \right) + \dots$$

↑
μ-indep
& higher loop

$$(Z_\psi m_0)^{(1)} = Z_\psi^{(1)} m_f + m_0^{(1)} \Rightarrow$$

$$m_0^{(1)} = -3 C_2(V_f) \frac{e^2}{(4\pi)^2} X m_f + \dots$$

$$\therefore m_0 = m_f - 3 C_2(V_f) \frac{e^2}{(4\pi)^2} \left(\frac{2}{\epsilon} - \log \mu^2 \right) m_f + \dots$$

Using these, we can determine the β & γ functions of the renormalization group, at 1-loop:

$$0 = \mu \frac{d}{d\mu} \left(\frac{1}{e^2} \right) = \underbrace{\mu \frac{d}{d\mu} \left(\frac{1}{e^2} \right)}_{-2e^{-3} \mu \frac{d}{d\mu} e} + \frac{2}{(4\pi)^2} \left(\frac{4}{3} T_{V_f} + \frac{1}{3} T_{V_b} - \frac{11}{3} h^\nu \right) + \dots \text{ (higher loop)}$$

$$0 = \mu \frac{d}{d\mu} m_0 = \underbrace{\mu \frac{d}{d\mu} m_f}_{-\gamma_{m_f} m_f} + 6 C_2(V_f) \frac{e^2}{(4\pi)^2} + \dots$$

$$0 = \mu \frac{d}{d\mu} \phi_0 = \mu \frac{d}{d\mu} Z_0^{\frac{1}{2}} \cdot \phi + Z_0^{\frac{1}{2}} \underbrace{\mu \frac{d\phi}{d\mu}}_{-\gamma_\phi \phi} + \dots$$

$$\beta = \mu \frac{d}{d\mu} e = \frac{e^3}{(4\pi)^2} \left(\frac{4}{3} T_{V_f} + \frac{1}{3} T_{V_b} - \frac{11}{3} h^\nu \right) + \dots$$

$$\gamma_{m_f} = -\mu \frac{d}{d\mu} \log m_f = 6 C_2(V_f) \frac{e^2}{(4\pi)^2} + \dots$$

$$\gamma_A = \mu \frac{d}{d\mu} \log \sqrt{Z_A} = 2 h^\nu \frac{e^2}{(4\pi)^2} + \dots$$

$$\gamma_\psi = \mu \frac{d}{d\mu} \log \sqrt{Z_\psi} = C_2(V_f) \frac{e^2}{(4\pi)^2} + \dots$$

$$\gamma_c = \mu \frac{d}{d\mu} \log \sqrt{Z_c} = -\frac{1}{2} h^\nu \frac{e^2}{(4\pi)^2} + \dots$$

For general m_f & m_b ,

$$\beta = \frac{e^3}{(4\pi)^2} \left(T_{V_f} F_f(m_f, \mu) + T_{V_b} F_b(m_b, \mu) - \frac{11}{3} h^v \right) + \dots$$

$$F_f(m_f, \mu) = \int_0^1 dx \frac{x(1-x)\mu^2}{x(1-x)\mu^2 + m_f^2} 8x(1-x) \sim \begin{cases} 4/3 & \mu \gg m_f \\ 0 & \mu \ll m_f \end{cases}$$

$$F_b(m_b, \mu) = \int_0^1 dx \frac{x(1-x)\mu^2}{x(1-x)\mu^2 + m_b^2} (2x-1)^2 \sim \begin{cases} 1/3 & \mu \gg m_b \\ 0 & \mu \ll m_b \end{cases}$$

$$\gamma_{m_f} = C_2(V_f) \frac{e^2}{(4\pi)^2} \int_0^1 dx \frac{2x\mu^2}{x\mu^2 + m_f^2} (4-2x) \sim \begin{cases} 6C_2(V_f) \frac{e^2}{(4\pi)^2} & \mu \gg m_f \\ 0 & \mu \ll m_f \end{cases}$$

$$\gamma_\psi = C_2(V_f) \frac{e^2}{(4\pi)^2} \int_0^1 dx \frac{2x\mu^2}{x\mu^2 + m_f^2} x \sim \begin{cases} C_2(V_f) \frac{e^2}{4\pi^2} & \mu \gg m_f \\ 0 & \mu \ll m_f \end{cases}$$

γ_A, γ_c remain the same.

Heavy fields ($m \gg \mu$) do not contribute to the running of the gauge coupling constant at scale μ .

(They must be integrated out and are absent in the effective theory at scale μ .)

Only the light fields ($m \ll \mu$) contribute.

Closer look at

$$\beta = - \frac{e^3}{(4\pi)^2} \underbrace{\left(\frac{11}{3} h^v - \frac{4}{3} T_{V_f} - \frac{1}{3} T_{V_b} \right)}_{b_1} + \dots$$

$$\mu \frac{d}{d\mu} \left(\frac{1}{e^2} \right) = \frac{2b_1}{(4\pi)^2} + \mathcal{O}(e^2)$$

$$\frac{1}{e^2(\mu)} - \frac{1}{e^2(\mu_0)} = \frac{b_1}{8\pi^2} \log(\mu/\mu_0) + \dots$$

$$e^2(\mu) = \frac{e^2(\mu_0)}{1 + \frac{e^2(\mu_0)}{8\pi^2} b_1 \log(\mu/\mu_0) + \dots}$$

Suppose $b_1 > 0$

$$\mu \rightarrow \infty : e(\mu) \rightarrow 0$$

UV free “asymptotic freedom”

Or equivalently,

μ smaller : $e(\mu)$ larger

gauge coupling is stronger at lower energies.

At some $\mu = \Lambda$, it diverges:

$$1 + \frac{e^2(\mu_0)}{8\pi^2} b_1 \log(\mu/\mu_0) = 0 \quad \text{i.e.} \quad \Lambda = \mu_0 \exp\left(-\frac{8\pi^2}{b_1 e^2(\mu_0)}\right)$$

... characteristic energy of the system, the "dynamical scale".

It determines how the gauge coupling runs:

$$\exp\left(-\frac{8\pi^2}{e^2(\mu)}\right) = \left(\frac{\Lambda}{\mu}\right)^{b_1}$$

Λ , not e , is the appropriate parameter of the theory.

... "dimensional transmutation"

At $\mu \ll \Lambda$, $e^2(\mu) \gg 1$: "gauge theory" is

NOT a good description!

If $b_1 > 0$, finding the low energy behaviour of the system is a difficult problem. "Interesting theory"

When is $b_1 > 0$?

More general systems

- general gauge group G :

$$\{e^a\} \subset \mathfrak{g} \text{ an arbitrary basis, } F_{\mu\nu} = \sum_a e^a F_{a\mu\nu},$$

$$\mathcal{L}_E = \frac{1}{4} \sum_{a,b} \left(\frac{1}{e^2}\right)^{ab} F_a^{\mu\nu} F_{b\mu\nu} + \dots$$

$$\mu \frac{d}{d\mu} \left(\frac{1}{e^2}\right)^{ab} = \frac{1}{8\pi^2} \left(\frac{11}{3} T_{\mathfrak{g}}^{ab} - \frac{4}{3} T_{V_f}^{ab} - \frac{1}{3} T_{V_b}^{ab} \right)$$

$$T_V^{ab} = -\text{tr}_V(e^a e^b)$$

For $G = \prod_{\alpha} U(1)_{\alpha} \times \prod_I G_I$ with \leftarrow simple factors

$$\mathcal{L}_E = \sum_{\alpha,\beta} \frac{1}{4e_{\alpha,\beta}^2} F_{\alpha}^{\mu\nu} F_{\beta\mu\nu} + \sum_I \frac{1}{4e_I^2} F_I^{\mu\nu} F_{I\mu\nu} + \dots$$

$$\bullet \quad b_I^{\alpha\beta} = \frac{11}{3} h_I^{\alpha\beta} - \frac{4}{3} T_{V_f}^{I\alpha\beta} - \frac{1}{3} T_{V_b}^{I\alpha\beta}$$

$$V^I = V \text{ as a representation of } G_I$$

$$\bullet \quad \mu \frac{d}{d\mu} \frac{8\pi^2}{e_{\alpha,\beta}^2} = -\frac{4}{3} T_{V_f}^{\alpha\beta} - \frac{1}{3} T_{V_b}^{\alpha\beta}$$

$$T_V^{\alpha\beta} = \sum_i Q_i^{\alpha} Q_i^{\beta} \quad (\{Q_i^{\alpha}\} U(1)_{\alpha} \text{ charges of } V)$$

$$\left(\begin{array}{l} \text{If there is just one } U(1) \text{ factor,} \\ b_1^{U(1)} = -\frac{4}{3} T_{V_f}^{U(1)} - \frac{1}{3} T_{V_b}^{U(1)}, \\ T_V^{U(1)} = \sum_i Q_i^2. \end{array} \right)$$

- Dirac fermion Ψ

→ Majorana fermion ($\Psi^c = \Psi$):

V_f must be a representation/R

or → chiral fermion ($\Psi = \Psi_L$ or Ψ_R):

* anomaly must be absent
in the full system

Contribution to T_{V_f} is halved.

- Complex scalar Φ

→ real scalar ($\bar{\Phi} = \Phi$):

V_b must be a representation/R.

Contribution to T_{V_b} is halved.

Examples

- QED

$$G = U(1), \quad V_f = \oplus_i \mathbb{C}(Q_i), \quad V_b = 0$$

$$b_1 = -\frac{4}{3} \sum_i Q_i^2 < 0$$

∴ Infra-red free.

- Pure Yang-Mills theory

$$G = \text{simple}, \quad V_f = V_b = 0.$$

$$b_1 = \frac{11}{3} h^v > 0$$

∴ Asymptotically free.

- $SU(N_c)$ QCD with N_f flavors

$$G = SU(N_c), \quad V_f = (\mathbb{C}^{N_c})^{\oplus N_f}, \quad V_b = 0$$

$$h^v = N_c, \quad T_{\mathbb{C}^v} = \frac{1}{2} \quad \therefore V_{V_f} = \frac{1}{2} \times N_f.$$

$$b_1 = \frac{11}{3} N_c - \frac{4}{3} \cdot \frac{N_f}{2} = \frac{1}{3} (11 N_c - 2 N_f)$$

∴ Asymptotically free if $N_f < \frac{11}{2} N_c$.

• The standard model $G = SU(3) \times SU(2) \times U(1)$

$$V_{f,L} = \left[\underbrace{(1, 2, -\frac{1}{2})}_{\text{Leptons}} \oplus \underbrace{(3, 2, \frac{1}{6})}_{\text{Quarks}} \right]^{\oplus 3} \text{ generations}$$

$$V_{f,R} = \left[(1, 1, -1) \oplus (3, 1, \frac{2}{3}) \oplus (3, 1, -\frac{1}{3}) \right]^{\oplus 3}$$

$$V_b = (1, 2, \frac{1}{2}) \text{ Higgs}$$

$$SU(3): T_{V_f} = \frac{1}{2} \left[\frac{1}{2} \times 2 \right] \times 3 + \frac{1}{2} \left[\frac{1}{2} \times (1+1) \right] \times 3 = 3, T_{V_b} = 0$$

$$b_1^{SU(3)} = \frac{11}{3} \cdot 3 - \frac{4}{3} \cdot 3 = 11 - 4 = 7 > 0 \quad \underline{\text{A.F.}}$$

$$SU(2): T_{V_f} = \frac{1}{2} \left[\frac{1}{2} \times (1+3) \right] \times 3 = 3, T_{V_b} = \frac{1}{2} \times 1 = \frac{1}{2}$$

$$b_1^{SU(2)} = \frac{11}{2} \cdot 2 - \frac{4}{3} \cdot 3 - \frac{1}{3} \cdot \frac{1}{2} = \frac{19}{6} = 3.166... > 0 \quad \underline{\text{A.F.}}$$

$$U(1): T_{V_f} = \frac{1}{2} \left[\left(-\frac{1}{2}\right)^2 \times 2 + \left(\frac{1}{6}\right)^2 \times 3 \cdot 2 \right] \times 3$$

$$+ \frac{1}{2} \left[\left(-1\right)^2 \times 1 + \left(\frac{2}{3}\right)^2 \times 3 + \left(-\frac{1}{3}\right)^2 \times 3 \right] \times 3 = 5$$

$$T_{V_b} = \left(\frac{1}{2}\right)^2 \times 2 = \frac{1}{2}$$

$$b_1^{U(1)} = -\frac{4}{3} \cdot 5 - \frac{1}{3} \cdot \frac{1}{2} = -\frac{41}{6} = -6.833... < 0 \quad \underline{\text{IR free}}$$

- $SO(N_c)$ QCD with N_f flavors

$$G = SO(N_c), \quad V_f = (\mathbb{R}^{N_c})^{\oplus N_f} \text{ (Majorana)}, \quad V_b = 0$$

$$h^V = N_c - 2, \quad T_{\mathbb{C}^N} = 1 \quad \therefore T_{V_f} = 1 \times N_f \times \frac{1}{2} = \frac{N_f}{2}$$

$$b_1 = \frac{11}{3}(N_c - 2) - \frac{4}{3} \cdot \frac{N_f}{2} = \frac{1}{3}(11N_c - 22 - 2N_f)$$

\therefore Asymptotically free if $N_f < \frac{11}{2}N_c - 11$

- $USp(N_c)$ QCD with N_f flavors ($N_c = 2U, N_f$ both even)

$$G = USp(N_c) = \left\{ g \in U(N_c) \mid g^T \begin{pmatrix} 0 & -\mathbb{1}_U \\ \mathbb{1}_U & 0 \end{pmatrix} g = \begin{pmatrix} 0 & -\mathbb{1}_U \\ \mathbb{1}_U & 0 \end{pmatrix} \right\}$$

$$h^V = U + 1 = \frac{1}{2}(N_c + 2), \quad T_{\mathbb{C}^{2U}} = \frac{1}{2}$$

$$V_f = (\mathbb{C}^{N_c})^{\oplus N_f/2}, \quad V_b = 0$$

$$T_{V_f} = \frac{1}{2} \times \frac{N_f}{2} = \frac{N_f}{4}$$

$$b_1 = \frac{11}{3} \cdot \frac{N_c + 2}{2} - \frac{4}{3} \cdot \frac{N_f}{4} = \frac{1}{6}(11N_c + 22 - 2N_f)$$

\therefore Asymptotically free if $N_f < \frac{11}{2}N_c + 11$

- $\mathcal{N}=1$ supersymmetric Yang-Mills theory (SYM)

$$G = \text{simple}, \quad V_f = \mathfrak{g} (\text{Majorana}), \quad V_b = 0$$

$$T_{V_f} = h^\nu \times \frac{1}{2}$$

$$b_1 = \frac{11}{3} h^\nu - \frac{4}{3} \cdot \frac{h^\nu}{2} = 3h^\nu > 0$$

\therefore Asymptotically free.

- $\mathcal{N}=1$ $SU(N_c)$ supersymmetric QCD (SQCD) with N_f flavors

$$G = SU(N_c), \quad V_f = \mathfrak{g} (\text{Majorana}) \oplus (\mathbb{C}^{N_c})^{\oplus N_f}$$

$$V_b = (\mathbb{C}^{N_c} \oplus \mathbb{C}^{N_c^*})^{\oplus N_f}$$

$$T_{V_f} = N_c \times \frac{1}{2} + \frac{1}{2} \times N_f, \quad T_{V_b} = \left(\frac{1}{2} + \frac{1}{2}\right) \times N_f$$

$$b_1 = \frac{11}{3} N_c - \frac{4}{3} \left(\frac{N_c}{2} + \frac{N_f}{2}\right) - \frac{1}{3} N_f = 3N_c - N_f$$

\therefore Asymptotically free if $N_f < 3N_c$.

• $\mathcal{N}=1$ $SO(N_c)$ SQCD with N_f flavors

$$G = SO(N_c), \quad V_f = \mathfrak{g} \oplus (\mathbb{R}^{N_c})^{\oplus N_f} \quad (\text{all } \underline{\text{Majorana}})$$

$$V_b = (\mathbb{C}^{N_c})^{\oplus N_f}$$

$$T_{V_f} = (N_c - 2 + 1 \times N_f) \times \frac{1}{2}, \quad T_{V_b} = 1 \times N_f$$

$$b_1 = \frac{11}{3}(N_c - 2) - \frac{4}{3} \frac{N_c - 2 + N_f}{2} - \frac{1}{3} N_f = 3(N_c - 2) - N_f$$

\therefore Asymptotically free if $N_f < 3(N_c - 2)$.

• $\mathcal{N}=1$ $USp(N_c)$ SQCD with N_f flavors (N_c, N_f both even)

$$G = USp(N_c), \quad V_f = \mathfrak{g} (\underline{\text{Majorana}}) \oplus (\mathbb{C}^{N_c})^{\oplus \frac{N_f}{2}}$$

$$V_b = (\mathbb{C}^{N_c})^{\oplus N_f}$$

$$T_{V_f} = \frac{N_c + 2}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{N_f}{2}, \quad T_{V_b} = \frac{1}{2} \times N_f$$

$$b_1 = \frac{11}{3} \frac{N_c + 2}{2} - \frac{4}{3} \frac{N_c + 2 + N_f}{4} - \frac{1}{3} \frac{N_f}{2} = \frac{3(N_c + 2) - N_f}{2}$$

\therefore Asymptotically free if $N_f < 3(N_c + 2)$.

- $\mathcal{N}=2$ SYM

$$G = \text{simple}, \quad V_f = \mathfrak{g}_G, \quad V_b = \mathfrak{g}_G$$

$$T_{V_f} = T_{V_b} = h^\vee$$

$$b_1 = \frac{11}{3} h^\vee - \frac{4}{3} h^\vee - \frac{1}{3} h^\vee = 2 h^\vee > 0$$

\therefore Asymptotically free

- $\mathcal{N}=2$ $SU(N_c)$ SQCD with N_f flavours

$$G = SU(N_c), \quad V_f = \mathfrak{g}_G \oplus (\mathbb{C}^{N_c})^{\oplus N_f}$$

$$V_b = \mathfrak{g}_G \oplus (\mathbb{C}^{N_c} \oplus (\mathbb{C}^{N_c})^*)^{\oplus N_f}$$

$$T_{V_f} = N_c + \frac{1}{2} \times N_f, \quad T_{V_b} = N_c + \left(\frac{1}{2} + \frac{1}{2}\right) \times N_f$$

$$b_1 = \frac{11}{3} N_c - \frac{4}{3} \left(N_c + \frac{N_f}{2}\right) - \frac{1}{3} (N_c + N_f) = 2N_c - N_f$$

\therefore Asymptotically free if $N_f < 2N_c$

- $\mathcal{N}=2$ $SO(N_c)$ SQCD with N_f flavors

$$G = SO(N_c), \quad V_f = \mathfrak{g}_G \oplus (\mathbb{R}^{N_c})^{\oplus N_f} \text{ (Majorana)}$$

$$V_b = \mathfrak{g}_G \oplus (\mathbb{C}^{N_c})^{\oplus N_f}$$

$$T_{V_f} = N_c - 2 + 1 \times N_f \times \frac{1}{2}, \quad T_{V_b} = N_c - 2 + 1 \times N_f$$

$$\begin{aligned} b_1 &= \frac{11}{3}(N_c - 2) - \frac{4}{3}\left(N_c - 2 + \frac{N_f}{2}\right) - \frac{1}{3}(N_c - 2 + N_f) \\ &= 2(N_c - 2) - N_f \end{aligned}$$

\therefore Asymptotically free if $N_f < 2(N_c - 2)$.

- $\mathcal{N}=2$ $USp(N_c)$ SQCD with N_f flavors (N_c, N_f both even)

$$G = USp(N_c), \quad V_f = \mathfrak{g}_G \oplus (\mathbb{C}^{N_c})^{\oplus \frac{N_f}{2}}$$

$$V_b = \mathfrak{g}_G \oplus (\mathbb{C}^{N_c})^{\oplus N_f}$$

$$T_{V_f} = \frac{N_c + 2}{2} + \frac{1}{2} \cdot \frac{N_f}{2}, \quad T_{V_b} = \frac{N_c + 2}{2} + \frac{1}{2} N_f$$

$$b_1 = \frac{11}{3} \cdot \frac{N_c + 2}{2} - \frac{4}{3} \left(\frac{N_c + 2}{2} + \frac{N_f}{4} \right) - \frac{1}{3} \frac{N_c + 2 + N_f}{2} = N_c + 2 - \frac{N_f}{2}$$

\therefore Asymptotically free if $N_f < 2(N_c + 2)$.

• $N=4$ SYM

$$G = \text{simple}, \quad V_f = \mathfrak{g}^{\oplus 4} (\text{Majorana}) = \mathfrak{g}_{\mathbb{C}}^{\oplus 2}$$

$$V_b = \mathfrak{g}^{\oplus 6} (\text{real}) = \mathfrak{g}_{\mathbb{R}}^{\oplus 3}$$

$$T_{V_f} = h^{\vee} \times 2, \quad T_{V_b} = h^{\vee} \times 3$$

$$b_1 = \frac{11}{3} h^{\vee} - \frac{4}{3} \cdot 2h^{\vee} - \frac{1}{3} \cdot 3h^{\vee} = 0.$$

Note : $b_1 = 0$ for

$$\left. \begin{array}{l} \mathcal{N}=1 \text{ SQCD, } SU(N_c), N_f = 3N_c \\ SO(N_c), N_f = 3(N_c - 2) \\ USp(N_c), N_f = 3(N_c + 2) \end{array} \right\} \beta > 0 \text{ from higher loop} \\ \text{(IR free)}$$

$$\left. \begin{array}{l} \mathcal{N}=2 \text{ SQCD, } SU(N_c), N_f = 2N_c \\ SO(N_c), N_f = 2(N_c - 2) \\ USp(N_c), N_f = 2(N_c + 2) \end{array} \right\} \beta \equiv 0 \text{ exactly} \\ \text{all loop \& non-perturbatively}$$

$\mathcal{N}=4$ SYM

finite, scale invariant.

They are believed to be invariant
also under (super) conformal transformations.

“(super) conformal field theories”