Evaluation of the ratio of determinants

(taken (with a minor modification) from a japanese textbook
"QM of many degrees of freedom via path-integral" by Sakita-Kikkawa
(『経路積分による多自由度の量子力学』 崎田・吉川))

Let $\bar{x}(\tau)$ be the instanton with $\tau_1 = 0$ (so that $\bar{x}(-\tau) = -\bar{x}(\tau)$)

Consider $\left\{ \begin{array}{l} H = -\frac{d^2}{d\tau^2} + U''(\bar{x}(\tau)) \\ H_0 = -\frac{d^2}{d\tau^2} + \omega^2 \end{array} \right\}$ as Schrödinger operators for a particle moving in $\mathbb{R} = \{\tau\}$.

Put

$$\Delta(E) := \frac{\det(H - E)}{\det(H_0 - E)}$$

It is called "Fredholm determinant" and is used in Scattering theory.
There, $E > 0$ is considered ($E \to E + i0$ prescription needed).
Here, our interest is in $E \sim 0$, since

$$\frac{\det'(H)}{\det(H_0)} = \lim_{E \to 0} \frac{\Delta(E)}{-E} = -\Delta'(0).$$

Let $f_\pm(\tau, E)$ be solutions to the problem $(H - E)\psi(\tau) = 0$
which decays as

$$f_\pm(\tau, E) \longrightarrow e^{\mp \kappa \tau} \quad \text{as } \tau \to \pm\infty$$

where $\kappa^2 = \omega^2 - E$, $\kappa > 0$ (Note: $U''(\bar{x}(\tau)) \to \omega^2$ as $\tau \to \pm\infty$).

We write $f_\pm(\tau,E) = F_\pm(E)\, e^{\mp\kappa\tau} + A_\pm(E)\, e^{\pm\kappa\tau}$ as $\tau\to\mp\infty$

(growing) (decaying)

Define $W[f_+(\tau,E), f_-(\tau,E')] := f_+(\tau,E)\partial_\tau f_-(\tau,E') - \partial_\tau f_+(\tau,E)\cdot f_-(\tau,E')$.

We have $\partial_\tau W[f_+(\tau,E), f_-(\tau,E')] = (E-E')f_+(\tau,E)f_-(\tau,E')$. —— ⊛

$W[f_+(\tau,E), f_-(\tau,E)]$ is τ-independent, and $\longrightarrow 2\kappa F_\mp(E)$ as $\tau\to\pm\infty$.

Thus $F_+(E) = F_-(E) =: F(E)$.

Define $G(\tau,\tau';E) = \theta(\tau-\tau')\dfrac{f_+(\tau,E)f_-(\tau',E)}{2\kappa F(E)} + \theta(\tau'-\tau)\dfrac{f_+(\tau',E)f_-(\tau,E)}{2\kappa F(E)}$

It satisfies $\left(-\frac{d^2}{d\tau^2} + U''(\bar{x}(\tau)) - E\right)G(\tau,\tau';E) = \delta(\tau-\tau')$

and $G(\tau,\tau';E)\to 0$ as $\tau\to\pm\infty$, τ' fixed
and τ fixed, $\tau'\to\pm\infty$.

Thus we may interpret $G(\tau,\tau';E) = \langle\tau|\frac{1}{H-E}|\tau'\rangle$.

Let us compute $\frac{\partial}{\partial E}\log\Delta(E) = \frac{\partial}{\partial E}\log\frac{\det(H-E)}{\det(H_0-E)} = \frac{\partial}{\partial E}\left(\mathrm{tr}\log(H-E) - \mathrm{tr}\log(H_0-E)\right)$

$$= -\,\mathrm{tr}\left(\frac{1}{H-E}\right) + \mathrm{tr}\left(\frac{1}{H_0-E}\right) = -\int_{-\infty}^{\infty} d\tau\left(G(\tau,\tau;E) - G_0(\tau,\tau;E)\right)$$

$$= -\int_{-\infty}^{\infty} d\tau\left(\frac{f_+(\tau,E)f_-(\tau,E)}{2\kappa F(E)} - \frac{f_+^0(\tau,E)f_-^0(\tau,E)}{2\kappa F_0(E)}\right)$$

($F_0(E)$: 1)

(*)

$$= -\int_{-\infty}^{0} d\tau \lim_{E'\to E} \underbrace{\left[\frac{\partial_\tau W[f_+(\tau,E), f_-(\tau,E')]}{2K F(E)(E-E')} - \frac{\partial_\tau W[f_+^\circ(\tau,E), f_-^\circ(\tau,E')]}{2K(E-E')} \right]}_{\partial_\tau h(\tau)}$$

$$h(\tau) \overset{\tau\to\infty}{=} \frac{e^{-K\tau}\left(\overset{F(E')}{F_-(E')}K' e^{K'\tau} + A_-(E')(-K')e^{-K'\tau}\right) - (-K)e^{-K\tau}\left(F_-(E')e^{K'\tau} + A_-(E')e^{-K'\tau}\right)}{2K F(E)(E-E')}$$

$$- \frac{e^{-K\tau}K' e^{K'\tau} - (-K)e^{-K\tau}e^{K'\tau}}{2K(E-E')}$$

$$= \frac{e^{-(K-K')\tau}(K+K')(F(E')-F(E))}{2K F(E)(E-E')} + \frac{e^{-(K+K')\tau}A_-(E')(K-K')}{2K F(E)(E-E')}$$

$$\xrightarrow{E'\to E} -\frac{\partial_E F(E)}{F(E)} + \frac{e^{-2K\tau}A_-(E)\partial_E K}{2K F(E)} \quad \text{decaying}$$

$$h(\tau) \overset{\tau\to-\infty}{=} \frac{(F_+(E)e^{-K\tau} + A_+(E)e^{K\tau})K' e^{K'\tau} - (F_+(E)(-K)e^{-K\tau} + A_+(E)Ke^{K\tau})e^{K'\tau}}{2K F(E)(E-E')}$$

$$- \frac{e^{-K\tau}K' e^{K'\tau} - (-K)e^{-K\tau}e^{K'\tau}}{2K F(E)(E-E')}$$

$$\xrightarrow{E'\to E} -\frac{e^{2K\tau}A_+(E)}{2K F(E)}\partial_E K \quad \text{decaying}$$

$$\therefore \frac{\partial}{\partial E}\log\Delta(E) = \frac{\partial_E F(E)}{F(E)} = \frac{\partial}{\partial E}\log F(E) \quad \therefore \underline{\log\Delta(E) = \log F(E) + \text{Const.}}$$

Both $\Delta(E)$ & $F(E) \to 1$ as $E \to -\infty$. Thus, they are equal:

$$\boxed{\Delta(E) = F(E)}$$

Note that $x_1(\tau) = S_0^{-\frac{1}{2}} \frac{d\bar{x}(\tau)}{d\tau}$ solves $H x_1 = 0$ and

$$x_1(\tau) \longrightarrow A e^{-\omega|\tau|} \quad \text{as} \quad \tau \to \pm\infty.$$

[By $\tau_1 = 0$, $\bar{x}(-\tau) = -\bar{x}(\tau)$, we have $x_1(-\tau) = x_1(\tau)$. That is why the constant A is the same for $\tau \to +\infty$ and $\tau \to -\infty$.
Exercise : This A is the same A as in the class.]

Thus, for $E = 0$ ($\kappa = \omega$), we find

$$f_+(\tau, 0) = f_-(\tau, 0) = A^{-1} x_1(\tau).$$

($\Rightarrow F(0) = 0$ indeed) Using ⊗ again,

$$\frac{\partial}{\partial E}\frac{\partial}{\partial \tau} W[f_+(\tau,E), f_-(\tau,0)]\Big|_{E=0} = f_+(\tau,0) f_-(\tau,0) = A^{-2}(x_1(\tau))^2$$

$$\left[\frac{\partial}{\partial E} W[f_+(\tau,E), f_-(\tau,0)]\Big|_{E=0}\right]_{T''}^{T'} = \int_{T''}^{T'} d\tau\, A^{-2}(x_1(\tau))^2 \xrightarrow[T'' \to -\infty]{T' \to \infty} A^{-2}$$

$\| \; T' \to +\infty, \; T'' \to -\infty$

$$\Big[\partial_E(e^{-\kappa T'})\partial_\tau(e^{-\omega T'}) - \partial_E \partial_{T'}(e^{-\kappa T'}) e^{-\omega T'} \quad (\text{decaying})$$
$$- \partial_E(F(E) e^{-\kappa T''} + \text{decaying})\partial_{T''}(e^{\omega T''}) + \partial_E \partial_{T''}(F(E) e^{-\kappa T''} + \text{decaying}) e^{\omega T''}\Big]_{E=0}$$

$$\longrightarrow -2\omega F'(0) \qquad \therefore F'(0) = -\frac{1}{2\omega A^2}$$

$$\therefore \frac{\det' H}{\det H_0} = -\Delta'(0) = -F'(0) = \frac{1}{2\omega A^2}$$