LSZ for m $\longrightarrow$ n S-matrix

In Lecture 10, we obtained a formula for $2 \to n$ S-matrix.

We would like to extend it to $m \to n$ S-matrix for $m > 2$.

For this, we need to know the limit of ${}_{\text{out}}\langle g_1, \cdots, g_n | \mathcal{O}_f(T)$ as $T \to +\infty$.

<u>Claim</u> When $g_1, \cdots, g_n$ has no overlap as $t \to +\infty$,

$$\lim_{T \to +\infty} {}_{\text{out}}\langle g_1, \cdots, g_n | \mathcal{O}_f(T)$$

$$\overset{!}{=} \langle g_1 | f \rangle \cdot {}_{\text{out}}\langle g_2, \cdots, g_n | + \cdots + \langle g_n | f \rangle \cdot {}_{\text{out}}\langle g_1, \cdots, g_{n-1} |$$

In particular, if $\langle g_a | f \rangle = 0 \ \forall a$,

$$\lim_{T \to +\infty} {}_{\text{out}}\langle g_1, \cdots, g_n | \mathcal{O}_f(T) \overset{!}{=} 0.$$

This is like the second expressions for $|f_1, \cdots, f_n\rangle_{\text{in/out}}$:

This is physically reasonable but a mathematical proof is hard to find.

If we admit the Claim, the procedure used in Lecture 10 leads us to the formula for ${}_{\text{out}}\langle g_1, \cdots, g_n | f_1, \cdots, f_m \rangle_{\text{in}}$.

$$X_{T_1 \cdots T_n, T_1', \cdots, T_m'} :=$$

$$\prod_{i=1}^{n} \int_{-T_i}^{T_i} d^d y_i \, g_i(y_i)^* \frac{i}{\sqrt{Z}} (\partial_{y_i}^2 + m^2) \prod_{j=1}^{m} \int_{-T_j'}^{T_j'} d^d x_j \, f_j(x_j) \frac{i}{\sqrt{Z}} (\partial_{x_j}^2 + m^2)$$

$$\langle 0 | T \, \mathcal{O}(y_1) \cdots \mathcal{O}(y_n) \mathcal{O}(x_1) \cdots \mathcal{O}(x_m) | 0 \rangle .$$

$$\xrightarrow{T_1, \cdots, T_n \to \infty} \prod_{j=1}^{m} \int_{-T_j'}^{T_j'} d^d x_j \, f_j(x_j) \frac{i}{\sqrt{Z}} (\partial_{x_j}^2 + m^2)$$

$${}_{out}\langle g_1, \cdots, g_n | T \, \mathcal{O}(x_1) \cdots \mathcal{O}(x_n) | 0 \rangle$$

First, let us consider the case $\langle g_a | f_b \rangle = 0 \ \forall (a, b)$. By Claim, ${}_{out}\langle g_1, \cdots, g_n | \mathcal{O}_{f_a}(T_a') \to 0$ as $T_a' \to \infty$, and we obtain

$$X_{T_1 \cdots T_n, T_1' \cdots T_m'} \xrightarrow{T_1, \cdots, T_n, T_1', \cdots, T_m' \to \infty} {}_{out}\langle g_1, \cdots, g_n | f_1, \cdots, f_n \rangle_{in} .$$

That is,

$$\langle g_1, \cdots, g_n, \text{free} | S | f_1, \cdots, f_m, \text{free} \rangle$$

$$= \prod_{i=1}^{n} \int d^d y_i \, g_i(y_i)^* \frac{i}{\sqrt{Z}} (\partial_{y_i}^2 + m^2) \prod_{j=1}^{m} \int d^d x_j \, f_j(x_j) \frac{i}{\sqrt{Z}} (\partial_{x_j}^2 + m^2)$$

$$\langle 0 | T \, \mathcal{O}(y_1) \cdots \mathcal{O}(y_n) \mathcal{O}(x_1) \cdots \mathcal{O}(x_m) | 0 \rangle .$$

Next, consider the general case where $\langle g_a | f_b \rangle$ may not be zero. Using the Claim, we find

$$\lim_{T_1, \dots, T_n, T_1', \dots, T_m' \to \infty} X_{T_1, \dots, T_n; T_1', \dots, T_m'}$$

$$= {}_{out}\langle g_1, \dots, g_n | f_1, \dots, f_m \rangle_{in}$$

omit g_a omit f_b

$$- \sum_{\substack{1 \le a \le n \\ 1 \le b \le m}} \langle g_a | f_b \rangle \, {}_{out}\langle \cdots \hat{g}_a \cdots | \cdots \hat{f}_b \cdots \rangle_{in}$$

$$+ \sum_{\substack{1 \le a_1 < a_2 \le n \\ 1 \le b_1 < b_2 \le m}} {}_{out}\langle g_{a_1}, g_{a_2} | f_{b_1}, f_{b_2} \rangle_{out} \; {}_{out}\langle \cdots \hat{g}_{a_1} \cdots \hat{g}_{a_2} \cdots | \cdots \hat{f}_{b_1} \cdots \hat{f}_{b_2} \cdots \rangle_{in}$$

$$- \cdots$$

Inverting the relation, we find

$$\langle g_1, \dots, g_n, \text{free} | (S-1) | f_1, \dots, f_m, \text{free} \rangle$$

$$= LSZ(g_1, \dots, g_n | f_1, \dots, f_m)$$

$$+ \sum_{\substack{1 \le a \le n \\ 1 \le b \le m}} \langle g_a, \text{free} | f_b, \text{free} \rangle \, LSZ(\cdots \hat{g}_a \cdots | \cdots \hat{f}_b \cdots)$$

$$+ \cdots$$

$$= \sum_{\ell \geq 0} \sum_{\substack{1 \leq a_1 < \cdots < a_\ell \leq n \\ 1 \leq b_1 < \cdots < b_\ell \leq m}} \langle g_{a_1}, \cdots, g_{a_\ell}, \text{free} \,|\, f_{b_1}, \cdots, f_{b_\ell}, \text{free} \rangle \cdot LSZ(\cdots \widehat{g}_{a_1} \cdots \widehat{g}_{a_\ell} \cdots \,|\, \cdots \widehat{f}_{b_1} \cdots \widehat{f}_{b_\ell} \cdots)$$

where $LSZ(g_1, \cdots, g_N \,|\, f_1, \cdots, f_M)$

$$:= \prod_{i=1}^{N} \int d^d y_i \, g_i(y_i)^* \frac{i}{\sqrt{Z}} (\partial_{y_i}^2 + m^2) \prod_{j=1}^{M} \int d^d x_j \, f_j(x_j) \frac{i}{\sqrt{Z}} (\partial_{x_j}^2 + m^2)$$

$$\langle 0 | T \, \mathcal{O}(y_1) \cdots \mathcal{O}(y_N) \mathcal{O}(x_1) \cdots \mathcal{O}(x_M) | 0 \rangle.$$