BRST cohomology of Maxwell theory

For a spatial momentum $\mathbb{p} \neq 0$, a polarization ϵ is said to be <u>transversal to $\mathbb{p}$</u> if $\epsilon^0 = 0$ and $\boldsymbol{\epsilon} \cdot \mathbb{p} = 0$. A creation operator $\epsilon \cdot a(\mathbb{p})^\dagger$ is <u>transversal</u> if ϵ is transversal to $\mathbb{p}$.

We denote

$$\mathcal{H}_{\text{transv}} := \text{Span}_{\mathbb{C}} \{ \text{product of transversal } a^\dagger\text{'s on } |0\rangle \}$$

We would like to prove

<u>Theorem</u>

$$H^i(\mathcal{H}, Q_B) \cong \begin{cases} \mathcal{H}_{\text{transv}} & i = 0 \\ 0 & i \neq 0. \end{cases}$$

<u>preparation</u>

(a) Properties of $\mathcal{H}_{\text{transv}}$

(i) The inner product is positive definite on $\mathcal{H}_{\text{transv}}$.

(ii) $\mathcal{H}_{\text{transv}} \subset \operatorname{Ker} Q_B$.

(iii) $\mathcal{H}_{\text{transv}} \cap \operatorname{Im} Q_B = 0$.

(i) follows from $[a_i(\mathbb{p}), a_j(\mathbb{p}')^\dagger] = \delta_{ij}\,\delta^{d-1}(\mathbb{p}-\mathbb{p}')$.

(ii) holds since $[Q_B, \epsilon\cdot a(\mathbb{p})^\dagger] = 0$ if ϵ is transversal to $\mathbb{p}$.

To see (iii) take $v \in \mathcal{H}_{\text{transv}} \cap \operatorname{Im} Q_B$. Then $v = Q_B w$

for some $w \in \mathcal{H}$, and

$$(v, v) = (Q_B w, v) \overset{Q_B^\dagger = Q_B}{=} (w, \underbrace{Q_B v}_{\overset{(ii)}{=}0}) = 0 \quad \overset{(i)}{\Rightarrow} \quad v = 0. \quad //$$

(b) a_L & a_S

For $\mathbb{p} \neq 0$, write $a_L(\mathbb{p}) := -\sum_i p_i a_i(\mathbb{p}) - |\mathbb{p}|\, a_0(\mathbb{p})$,

$$a_S(\mathbb{p}) := \left(-\sum_i p_i a_i(\mathbb{p}) + |\mathbb{p}|\, a_0(\mathbb{p})\right)/2\mathbb{p}^2.$$

They obey

$$\left.\begin{aligned} &\{Q_B, \bar{b}(\mathbb{p})^\dagger\} = a_L(\mathbb{p})^\dagger, \quad [Q_B, a_L(\mathbb{p})^\dagger] = 0, \\ &[Q_B, a_S(\mathbb{p})^\dagger] = b(\mathbb{p})^\dagger, \quad \{Q_B, b(\mathbb{p})^\dagger\} = 0 \end{aligned}\right\} (*)$$

Note that $\{a_\mu(\mathbb{p})^\dagger\}_{\mu=0}^{d-1}$ is spanned by $a_L(\mathbb{p})^\dagger$, $a_S(\mathbb{p})^\dagger$ and transversal $\epsilon\cdot a(\mathbb{p})^\dagger$'s. Thus

$$\mathcal{H} = \operatorname{Span}_{\mathbb{C}}\left\{\begin{array}{l}\text{product of} \\ \text{transversal } a^\dagger\text{'s},\ a_L^\dagger,\ a_S^\dagger,\ b^\dagger,\ \bar{b}^\dagger \text{ on } |0\rangle\end{array}\right\}.$$

Proof of Theorem

Take any $V \in \operatorname{Ker} Q_B$.

By (b), V is a sum of products of transversal $a(p)^\dagger$'s, $a_L(p)^\dagger$'s, $a_S(p)^\dagger$'s, $b(p)^\dagger$'s, $\bar{b}(p)^\dagger$'s for various p's operated on $|0\rangle$. We shall show that the sum of terms which are not purely transversal is BRST exact.

Choose $p \neq 0$ and write $a_L^\dagger, a_S^\dagger, b^\dagger, \bar{b}^\dagger$ for $a_L(p)^\dagger, a_S(p)^\dagger, b(p)^\dagger, \bar{b}(p)^\dagger$. Recall from (*)

$$\bar{b}^\dagger \overset{Q_B}{\longmapsto} a_L^\dagger \overset{Q_B}{\longmapsto} 0, \qquad a_S^\dagger \overset{Q_B}{\longmapsto} b^\dagger \overset{Q_B}{\longmapsto} 0.$$

First, expand V in powers of $a_L^\dagger$ & $\bar{b}^\dagger$:

$$\begin{aligned} V = {} & V_0 + a_L^\dagger V_1 + a_L^{\dagger 2} V_2 + \cdots \\ & + \bar{b}^\dagger W_0 + \bar{b}^\dagger a_L^\dagger W_1 + \cdots \end{aligned}$$

where V_i, W_i are free of $a_L^\dagger$ and $\bar{b}^\dagger$.

$$\begin{aligned} 0 = Q_B V = {} & Q_B V_0 + a_L^\dagger Q_B V_1 + a_L^{\dagger 2} Q_B V_2 + \cdots \\ & + a_L^\dagger W_0 + a_L^{\dagger 2} W_1 + \cdots \\ & - \bar{b}^\dagger Q_B W_0 - \bar{b}^\dagger a_L^\dagger Q_B W_1 - \cdots \end{aligned}$$

Note that $Q_B v_i$, w_i, $Q_B w_i$ are free of $a_L^\dagger$ & $\bar{b}^\dagger$. Thus,

$$Q_B v_0 = 0, \quad Q_B v_i + w_{i-1} = 0 \ (i=1,2,\cdots), \quad Q_B w_i = 0 \ (i=0,1,\cdots).$$

Then, for $i=1,2,\cdots$,

$$Q_B(\bar{b}^\dagger a_L^{\dagger\, i-1} v_i) = a_L^{\dagger\, i} v_i - \bar{b}^\dagger a_L^{\dagger\, i-1} \underbrace{Q_B v_i}_{= -w_{i-1}}$$

$$= a_L^{\dagger\, i} v_i + \bar{b}^\dagger a_L^{\dagger\, i-1} w_{i-1}.$$

$$\therefore \ v = v_0 + \sum_{i \geq 1} (a_L^{\dagger\, i} v_i + \bar{b}^\dagger a_L^{\dagger\, i-1} w_{i-1})$$

$$= v_0 + Q_B \Big(\sum_{i \geq 1} \bar{b}^\dagger a_L^{\dagger\, i-1} v_i \Big).$$

Next, expand v_0 in powers of $a_S^\dagger$ & $b^\dagger$:

$$v_0 = v_{00} + a_S^\dagger v_{01} + a_S^{\dagger 2} v_{02} + \cdots$$

$$+ b^\dagger w_{00} + b^\dagger a_S^\dagger w_{01} + \cdots$$

where v_{0i}, w_{0i} are free of $a_S^\dagger$ and $b^\dagger$ (and of $a_L^\dagger$ and $\bar{b}^\dagger$).

$$0 = Q_B v_0 = Q_B v_{00} + a_S^\dagger Q_B v_{01} + a_S^{\dagger 2} Q_B v_{02} + \cdots$$

$$+ b^\dagger v_{01} + 2 b^\dagger a_S^\dagger v_{02} + \cdots$$

$$- b^\dagger Q_B w_{00} - b^\dagger a_S^\dagger Q_B w_{01} - \cdots$$

Note that $Q_B v_{0i}$, v_{0i}, $Q_B w_{0i}$ are free of $b^\dagger$ & $a_S^\dagger$. Thus,

$$Q_B v_{0i} = 0 \ (i=0,1,\cdots), \quad i v_{0i} - Q_B w_{0\, i-1} = 0 \ (i=1,2,\cdots).$$

Then, for $i = 1, 2, \cdots$

$$Q_B(a_S^{+i} w_{0\,i-1}) = i\, b^+ a_S^{+\,i-1} w_{0\,i-1} + a_S^{+i} \underbrace{Q_B w_{0\,i-1}}_{= i v_{0i}}$$

$$= i\left(a_S^{+i} v_{0i} + b^+ a_S^{+\,i-1} w_{0\,i-1}\right)$$

$$\therefore\ v_0 = v_{00} + \sum_{i\geq 1}\left(a_S^{+i} v_{0i} + b^+ a_S^{+\,i-1} w_{0\,i-1}\right)$$

$$= v_{00} + Q_B\left(\sum_{i\geq 1} \frac{1}{i} a_S^{+i} w_{0\,i-1}\right).$$

To summarize,

$$v = v_{00} + Q_B\left(\sum_{i\geq 1} \bar{b}^+ a_L^{+\,i-1} v_i + \sum_{i\geq 1} \frac{1}{i} a_S^{+i} w_{0\,i-1}\right)$$

where v_{00} is Q_B-closed and free of $a_L^+, a_S^+, b^+, \bar{b}^+$.

Repeating this for all p's that appear in the expansion of v by creation operators, we find

$$v = v_{\text{transv}} + Q_B(-), \quad v_{\text{transv}} \in \mathcal{H}_{\text{transv}}.$$

By (a-iii), v_{transv} is uniquely determined by the Q_B-cohomology class of v. Also, if $v \in \mathcal{H}_{\text{transv}}$, then $v_{\text{transv}} = v$.

Thus, we obtained a linear isomorphism

$$\Phi : \operatorname{Ker} Q_B / \operatorname{Im} Q_B \xrightarrow{\cong} \mathcal{H}_{\text{transv.}}$$

ghost #

In the above construction, all the steps respect the degree:

if $v \in \mathcal{H}^i$, $v_i \in \mathcal{H}^i$, $w_i \in \mathcal{H}^{i+1}$, $v_{0i} \in \mathcal{H}^i$, $w_{0i} \in \mathcal{H}^{i-1}$.

Hence, the isomorphism Φ respects the degree.

Thus it splits to degree-wise isomorphism

$$\Phi^i : H^i(\mathcal{H}, Q_B) \xrightarrow{\cong} \mathcal{H}^i_{\text{transv}}$$

Since $\mathcal{H}_{\text{transv}}$ consists only of ghost number 0,

this is what we wanted to prove.