① : momentum cut-off

$$U_1 = \frac{1}{2}\int \frac{d^4k}{(2\pi)^2} \log\left(k^2+m^2+\frac{\lambda}{2}\phi^2\right)$$

$$= \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\left\{ \underline{\log(k^2+m^2)} + \log\left(1+\boxed{\frac{1}{k^2+m^2}}\,\frac{\lambda}{2}\phi^2\right)\right\}$$

field independent constant

drop.

$$\boxed{\frac{e^{-\frac{k^2+m^2}{\Lambda^2}}}{k^2+m^2}}$$

$$U_1^{①} = \frac{1}{2}\int \frac{d^dk}{(2\pi)^4} \log\left(1+\frac{e^{-\frac{k^2+m^2}{\Lambda^2}}}{k^2+m^2}\,\frac{\lambda}{2}\phi^2\right)$$

$$= \frac{1}{2}\cdot\frac{\mathrm{Vol}(S^3)}{2(2\pi)^4}\int_0^\infty k^2\,dk^2 \log\left(1+\frac{e^{-\frac{k^2+m^2}{\Lambda^2}}}{k^2+m^2}\,\frac{\lambda}{2}\phi^2\right)$$

$$k^2+m^2 = \Lambda^2 t$$

$$= \frac{1}{2}\cdot\frac{1}{(4\pi)^2}\int_{m^2/\Lambda^2}^\infty (\Lambda^2 t - m^2)\,\Lambda^2 dt \log\left(1+\frac{e^{-t}}{\Lambda^2 t}\,\frac{\lambda}{2}\phi^2\right)$$

$$\Delta := m^2/\Lambda^2 \quad , \quad \frac{\lambda\phi^2}{2m^2} := A$$

$$= \frac{m^4}{2(4\pi)^2}\int_\Delta^\infty \left(\frac{t}{\Delta^2}-\frac{1}{\Delta}\right) dt \log\left(1+\frac{e^{-t}}{t}\Delta\cdot A\right)$$

$$I = \int_{\Delta}^{\infty} \left(\frac{t}{\Delta^2} - \frac{1}{\Delta}\right) dt \, \underbrace{\log\left(1 + \frac{e^{-t}}{t}\Delta \cdot A\right)}_{\log(t + e^{-t}\Delta \cdot A) - \log t}$$

<u>Note</u>: $\frac{d}{dt}\left(\frac{t^2}{2}\left(\log t - \frac{1}{2}\right)\right) = t \log t$, $\frac{d}{dt}\left[t(\log t - 1)\right] = \log t$.

$$\frac{d}{dt}\left[\frac{(t+e^{-t}\Delta A)^2}{2\Delta^2}\left(\log(t+e^{-t}\Delta A) - \frac{1}{2}\right) - \frac{t+e^{-t}\Delta A}{\Delta}\left(\log(t+e^{-t}\Delta A) - 1\right) - \frac{t^2}{2\Delta^2}\left(\log t - \frac{1}{2}\right) + \frac{t}{\Delta}(\log t - 1) \right]$$

$$= \left(\frac{t+e^{-t}\Delta A}{\Delta^2} - \frac{1}{\Delta}\right)\log(t+e^{-t}\Delta A)\cdot(1 - e^{-t}\Delta A) - \left(\frac{t}{\Delta^2} - \frac{1}{\Delta}\right)\log t$$

$$= \left(\frac{t}{\Delta^2} - \frac{1}{\Delta}\right)\log(t+e^{-t}\Delta A)\,(1 \underbrace{- e^{-t}\Delta A}_{\text{unwanted } 2}) + \underbrace{\frac{e^{-t}A}{\Delta}\log(t+e^{-t}\Delta A)(1-e^{-t}\Delta A)}_{\text{unwanted } 1} - \left(\frac{t}{\Delta^2} - \frac{1}{\Delta}\right)\log t$$

$$\frac{d}{dt}\left[\frac{e^{-t}A}{\Delta}(t+e^{-t}\Delta A)\left(\log(t+e^{-t}\Delta A) - 1\right)\right]$$

$$= \underbrace{\frac{e^{-t}A}{\Delta}\log(t+e^{-t}\Delta A)(1-e^{-t}\Delta A)}_{1 \text{ perfectly}} - \frac{e^{-t}A}{\Delta}\underbrace{(t+e^{-t}\Delta A)\left(\log(t+e^{-t}\Delta A) - 1\right)}_{2 \text{ partially}}$$

$$\frac{d}{dt}\Bigg[\frac{(t+e^{-t}\Delta A)^2}{2\Delta^2}\left(\log(t+e^{-t}\Delta A)-\frac{1}{2}\right)-\frac{t+e^{-t}\Delta A}{\Delta}\left(\log(t+e^{-t}\Delta A)-1\right)$$

$$-\frac{t^2}{2\Delta^2}\left(\log t-\frac{1}{2}\right)+\frac{t}{\Delta}\left(\log t-1\right)$$

$$-\frac{e^{-t}A}{\Delta}(t+e^{-t}\Delta A)\left(\log(t+e^{-t}\Delta A)-1\right)\Bigg] \quad \overset{\because}{=} \frac{d}{dt}F(t)$$

$$=\left(\frac{t}{\Delta^2}-\frac{1}{\Delta}\right)\log(t+e^{-t}\Delta A)-\left(\frac{t}{\Delta^2}-\frac{1}{\Delta}\right)\log t$$

$$+e^{-t}A\log(t+e^{-t}\Delta A)+(e^{-t}A)^2\log(t+e^{-t}\Delta A)$$

$$-(e^{-t}A)^2-\frac{e^{-t}A}{\Delta}t$$

$$\left.\begin{aligned}=&\left(\frac{t}{\Delta^2}-\frac{1}{\Delta}\right)\log\left(1+\frac{e^{-t}}{t}\Delta A\right)\\&+\left(e^{-t}A+(e^{-t}A)^2\right)\left\{\log t+\log\left(1+\frac{e^{-t}}{t}\Delta A\right)\right\}\\&-(e^{-t}A)^2-\frac{e^{-t}A}{\Delta}t\end{aligned}\right\} \circledast$$

Note: $\displaystyle\int_\Delta^\infty dt\,\left(e^{-t}A+(e^{-t}A)^2\right)\log\left(1+\frac{e^{-t}}{t}\Delta A\right)\xrightarrow{\Delta\to 0} 0$

$$(\Delta\log\Delta\to 0 \text{ as } \Delta\to 0)$$

$\displaystyle\int_\Delta^\infty dt\,[\text{the rest of } \circledast]$ is computable.

$$F(t) = \frac{(t+e^{-t}\Delta A)^2}{2\Delta^2}\left(\log(t+e^{-t}\Delta A)-\frac{1}{2}\right) - \frac{t+e^{-t}\Delta A}{\Delta}\left(\log(t+e^{-t}\Delta A)-1\right)$$

$$-\frac{t^2}{2\Delta^2}\left(\log t-\frac{1}{2}\right)+\frac{t}{\Delta}\left(\log t-1\right)$$

$$-\frac{e^{-t}A}{\Delta}(t+e^{-t}\Delta A)\left(\log(t+e^{-t}\Delta A)-1\right)$$

$$=\left(\frac{t^2}{2\Delta^2}-\frac{t}{\Delta}\right)\log\left(1+\frac{e^{-t}}{t}\Delta A\right)+e^{-t}\left\{\text{Polynomial of } t \text{ \& } \log t\right\}$$

$$\longrightarrow 0 \quad \text{as} \quad t\to\infty$$

$$F(\Delta)=\frac{1}{2}(1+e^{-\Delta}A)^2\left(\log\Delta+\log(1+e^{-\Delta}A)-\frac{1}{2}\right)-\frac{1}{2}\left(\log\Delta-\frac{1}{2}\right)$$

$$-(1+e^{-\Delta}A)\left(\log\Delta+\log(1+e^{-\Delta}A)-1\right)+\left(\log\Delta-1\right)$$

$$-e^{-\Delta}A(1+e^{-\Delta}A)\left(\log\Delta+\log(1+e^{-\Delta}A)-1\right)$$

$$=-\frac{1}{2}(1+e^{-\Delta}A)^2\left(\log\Delta+\log(1+e^{-\Delta}A)\right)+\frac{1}{2}\log\Delta$$

$$+\frac{3}{4}\left\{(1+e^{-\Delta}A)^2-1\right\}$$

$$=-\frac{1}{2}(1+A)^2\log(1+A)+\left(-\frac{1}{2}\log\Delta+\frac{3}{4}\right)(2A+A^2)$$

$$+O(\Delta)$$

$$\cdot \int_\Delta^\infty e^{-nt}\log t\,dt = \int_\Delta^\infty d\left(-\frac{1}{n}e^{-nt}\right)\log t = \frac{e^{-n\Delta}}{n}\log\Delta + \int_0^\infty \frac{e^{-nt}dt}{nt}$$

$$= \frac{1}{n}\log\Delta + \frac{1}{n}\left(-\log(n\Delta)-\gamma\right) + O(\Delta) = -\frac{1}{n}\left(\log n+\gamma\right)+O(\Delta)$$

$$\cdot \int_\Delta^\infty e^{-t}t\,dt = 1, \quad \int_\Delta^\infty e^{-2t}dt = \frac{1}{2}.$$

$$\therefore F(\infty)-F(\Delta) = \underbrace{\int_\Delta^\infty dt\left(\frac{t}{\Delta^2}-\frac{1}{\Delta}\right)\log\left(1+\frac{e^{-t}}{t}\Delta A\right)}_{= I}$$

$$-\gamma A - \frac{1}{2}(\log 2+\gamma)A^2 - \frac{1}{2}A^2 - \frac{1}{\Delta}A + O(\Delta)$$

$$\therefore I = -F(\Delta) + \gamma A + \frac{1}{2}\left((\log 2+\gamma)\right)A^2 + \frac{1}{2}A^2 + \frac{1}{\Delta}A + O(\Delta)$$

$$= \frac{1}{2}(1+A)^2\log(1+A) + \frac{1}{2}A^2\left(\log(2\Delta)+\gamma-\frac{1}{2}\right)$$

$$+ A\left(\frac{1}{\Delta}+\log\Delta+\gamma-\frac{3}{2}\right)$$

$$\delta_1 U = \frac{\lambda b_1}{2}\phi^2 + \frac{\lambda^2 c_1}{4!}\phi^4 = m^2 b_1 A + \frac{m^4 c_1}{6}A^2$$

$$b_1(\Lambda) = \frac{-1}{2(4\pi)^2}\left[\Lambda^2 - m^2\left(\log\left(\frac{\Lambda^2}{m^2}\right)+1-\gamma\right)\right] = \frac{-m^2}{2(4\pi)^2}\left[\frac{1}{\Delta}+\log\Delta-1+\gamma\right]$$

$$c_1(\Lambda) = \frac{3}{2(4\pi)^2}\left[\log\left(\frac{\Lambda^2}{2m^2}\right)-\gamma-1-K\right] = \frac{3}{2(4\pi)^2}\left[-\log(2\Delta)-\gamma-1-K\right]$$

$$\therefore\ \delta_1 U = \frac{m^4}{2(4\pi)^2}\left\{ -\left(\frac{1}{\Delta} + \log\Delta - 1 + \gamma\right)A - \frac{1}{2}\left(\log(2\Delta) + \gamma + 1 + \kappa\right)A^2 \right\}$$

$$\therefore\ U_{\mathrm{eff}}^{L\leq 1} = U + \delta_1 U + \frac{m^4}{2(4\pi)^2}\underbrace{I}_{\frac{1}{2}(1+A)^2\log(1+A) + \frac{1}{2}A^2\left(\log(2\Delta)+\gamma-\frac{1}{2}\right) + A\left(\frac{1}{\Delta}+\log\Delta+\gamma-\frac{3}{2}\right)}$$

$$= U + \frac{m^4}{2(4\pi)^2}\left\{ \frac{1}{2}(1+A)^2\log(1+A) - \frac{1}{2}A - \left(\frac{3}{4}+\frac{\kappa}{2}\right)A^2 \right\}$$

$$= \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4$$

$$+ \frac{1}{2(4\pi)^2}\left\{ \frac{1}{2}\left(m^2 + \frac{\lambda\phi^2}{2}\right)^2\log\left(1+\frac{\lambda\phi^2}{2m^2}\right) - \frac{\lambda}{4}m^2\phi^2 - \frac{1}{4}\left(\frac{3}{4}+\frac{\kappa}{2}\right)\lambda^2\phi^4 \right\}$$

③ : dimensional regularization

$$U_1 = \frac{\mu_{DR}^{4-d}}{2}\int \frac{d^d k}{(2\pi)^d} \underbrace{\log(k^2+U'')}_{\sim \frac{\partial}{\partial n}(k^2+U'')^{-n}\Big|_{n=0}}$$

$$= -\frac{\partial}{\partial n}\frac{\mu_{DR}^{4-d}}{2}\int \frac{d^d k}{(2\pi)^d}(k^2+U'')^{-n}\Big|_{n=0}$$

$$\int \frac{d^d k}{(2\pi)^d}(k^2+U'')^{-n} = \int \frac{d^d k}{(2\pi)^d}\int_0^\infty \alpha^{n-1} d\alpha \, e^{-\alpha(k^2+U'')} \Big/ \Gamma(n)$$

$$= \int_0^\infty \alpha^{n-1} d\alpha \, e^{-\alpha U''} \underbrace{\int \frac{d^d k}{(2\pi)^d} e^{-\alpha k^2}}_{\frac{\mathrm{Vol}(S^{d-1})}{2(2\pi)^d}\int_0^\infty k^{d-2} dk^2 \, e^{-\alpha k^2}} \Big/ \Gamma(n)$$

$$= \frac{1}{(4\pi)^{d/2}\cancel{\Gamma(d/2)}} \cdot \alpha^{-\frac{d}{2}} \cancel{\Gamma(d/2)} = \frac{\alpha^{-\frac{d}{2}}}{(4\pi)^{d/2}}$$

$$= \frac{1}{(4\pi)^{d/2}}\int_0^\infty \alpha^{n-d/2-1} d\alpha \, e^{-\alpha U''} \Big/ \Gamma(n)$$

$$= \frac{1}{(4\pi)^{d/2}} (U'')^{d/2-n} \frac{\Gamma(n-d/2)}{\Gamma(n)}$$

$$\therefore\ U_1 = -\frac{\partial}{\partial n} \frac{\mu_{DR}^{4-d}}{2(4\pi)^{d/2}} \left(U''\right)^{d/2-n} \frac{\Gamma(n-d/2)}{\Gamma(n)} \Bigg|_{n=0}$$

$$\boxed{\frac{1}{\Gamma(n)} = \frac{1}{\frac{1}{n}-\gamma+O(n)} = \frac{n}{1-\gamma n+O(n^2)} = n + O(n^2)}$$

$$= -\frac{\mu_{DR}^{4-d}}{2(4\pi)^{d/2}} \left(U''\right)^{\frac{d}{2}} \underbrace{\Gamma\left(-\frac{d}{2}\right)}_{\overset{d=4-\epsilon}{=}\ \frac{1}{2}\left(\frac{2}{\epsilon}+\frac{3}{2}-\gamma+O(\epsilon)\right)}$$

$d=4-\epsilon$

$$= -\frac{(U'')^2}{2(4\pi)^2} \frac{\mu_{DR}^{\epsilon}(U'')^{-\frac{\epsilon}{2}}}{(4\pi)^{-\epsilon/2}} \cdot \frac{1}{2}\left(\frac{2}{\epsilon}+\frac{3}{2}-\gamma+O(\epsilon)\right)$$

$$= -\frac{(U'')^2}{4(4\pi)^2}\left(\frac{2}{\epsilon}-\log\left(\frac{U''}{4\pi\mu_{DR}^2}\right)+\frac{3}{2}-\gamma+O(\epsilon)\right)$$

$U''=m^2+\frac{\lambda\phi^2}{2}$

$$= -\frac{1}{4(4\pi)^2}\left(m^2+\frac{\lambda\phi^2}{2}\right)^2\left[\frac{2}{\epsilon}-\log\left(\frac{m^2}{4\pi\mu_{DR}^2}\right)-\log\left(1+\frac{\lambda\phi^2}{2m^2}\right)+\frac{3}{2}-\gamma+O(\epsilon)\right]$$

$$= \frac{1}{4(4\pi)^2}\left(m^2+\frac{\lambda\phi^2}{2}\right)^2 \log\left(1+\frac{\lambda\phi^2}{2m^2}\right) + \text{continued}$$

continuation

$$+\frac{1}{4(4\pi)^2}\left(m^4+\lambda m^2\phi^2+\frac{1}{4}\lambda^2\phi^4\right)\left(-\frac{2}{\epsilon}+\log\left(\frac{m^2}{4\pi\mu_{DR}^2}\right)-\frac{3}{2}+\gamma\right)$$

Recall $\delta_1 U = \frac{\lambda b_1(\epsilon)}{2}\phi^2 + \frac{\lambda^2 c_1(\epsilon)}{4!}\phi^4$

$$b_1(\epsilon) = \frac{m^2}{2(4\pi)^2}\left[\frac{2}{\epsilon}-\log\left(\frac{m^2}{4\pi\mu_{DR}^2}\right)-\gamma+1\right]$$

$$C_1(\epsilon) = \frac{3}{2(4\pi)^2}\left[\frac{2}{\epsilon}-\log\left(\frac{m^2}{4\pi\mu_{DR}^2}\right)-\gamma-K\right]$$

$$U_{eff}^{\epsilon 1} = U + \delta_1 U + U_1$$

$$= \frac{m^4}{4(4\pi)^2}\left(-\frac{2}{\epsilon}+\log\left(\frac{m^2}{4\pi\mu_{DR}^2}\right)-\frac{3}{2}+\gamma\right) \leftarrow \text{const } (\phi\text{-indep})$$

$$+\frac{m^2}{2}\phi^2+\frac{\lambda}{4!}\phi^4$$

$$+\frac{1}{4(4\pi)^2}\left[\left(m^2+\frac{\lambda\phi^2}{2}\right)^2\log\left(1+\frac{\lambda\phi^2}{2m^2}\right) - \frac{1}{2}\lambda m^2\phi^2 - \left(\frac{3}{2}+K\right)\frac{\lambda^2}{4}\phi^4\right]$$

For each R.C., 1 & 3 gave the same result.