Ward identity in general dimensions

Consider a classical field theory in d space-time dimensions, with a set of variables $\phi(x)$ (scalar, spinor, gauge potential, ...) and a Lagrangian $\mathcal{L}(\phi, \partial\phi)$. The action on a region $R \subset \mathbb{R}^d$ is $S_R[\phi] = \int_R d^dx \, \mathcal{L}(\phi, \partial\phi)$.

Equation of motion is the E-L eqn $\delta S_R[\phi] = 0$ for variations vanishing at the boundary $\delta\phi|_{\partial R} = 0$.

Suppose there is a continuous symmetry, with infinitesimal variation $\phi \to \phi + \delta\phi$, $\delta\phi = \epsilon\, \mathcal{U}(\phi, \partial_\mu \phi)$, giving

$$\delta\mathcal{L} = \epsilon\, \partial_\mu (\cdots)^\mu \quad \text{total derivative.}$$

Allow the variational parameter to depend on position $\epsilon(x)$, s.t. it vanishes at the boundary $\epsilon|_{\partial R} = 0$.

Then, the variation of the action can be written as

$$\delta S_R[\phi] = \int_R d^dx \, \partial_\mu \epsilon\, J^\mu(\phi, \partial\phi)$$

This $J^\mu = J^\mu(\phi, \partial\phi)$ is called the Noether current.

For a solution of EOM, it obeys the conservation equation:

$$\partial_\mu J^\mu = 0.$$

☺ (Just as in the case $d=1$): A soln is s.t. $\delta S_R = 0$ for $\forall \delta\phi$ s.t. $\delta\phi|_{\partial R} = 0$. For $\forall \epsilon(x)$ s.t. $\epsilon|_{\partial R} = 0$, under $\phi \to \phi + \epsilon(x)\delta U(\phi, \partial\phi)$,

$$0 = \delta S_R = \int_R d^dx\, \partial_\mu \epsilon\, J^\mu = -\int_R d^dx\, \epsilon\, \partial_\mu J^\mu \quad \therefore \partial_\mu J^\mu = 0 \quad //$$

The charge $Q(t)$ at time t is the integral

$$Q(t) = \int_{\mathbb{R}^{d-1}} d^{d-1}\mathbf{x}\, J^0(t, \mathbf{x})$$

of the time component J^0 of the current on the time slice.

Then it is a constant of motion:

$$\frac{dQ(t)}{dt} = \int_{\mathbb{R}^{d-1}} d^{d-1}\mathbf{x}\, \underbrace{\partial_0 J^0(t,\mathbf{x})}_{= -\sum_{i=1}^{d-1} \partial_i J^i(t,\mathbf{x})} = 0$$

as long as $J^i(t,\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$.

This Q is the Noether charge.

Let us quantize the system with a path-integral measure $\mathcal{D}\phi$.

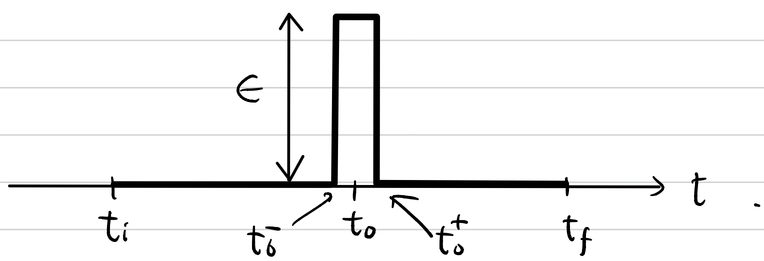
Suppose $\mathcal{D}\phi$ is also invariant under the above symmetry.

Let $\mathcal{O}$ be a local observable. Apply $\delta\phi = \epsilon(t)\, u(\phi, \partial\phi)$

in the integrand of $\int \mathcal{D}\phi\, e^{\frac{i}{\hbar} S_R[\phi]}\, \mathcal{O}(t_0, \mathbf{x}_0)$

for $R = [t_i, t_f] \times \mathbb{R}^{d-1}$

and $\epsilon(t)$ given by



Ward identity :

$$0 = \int \delta\left(\mathcal{D}\phi\, e^{\frac{i}{\hbar} S_R[\phi]}\, \mathcal{O}(t_0, \mathbf{x}_0)\right)$$

$$= \int \mathcal{D}\phi\, e^{\frac{i}{\hbar} S_R[\phi]} \left(\frac{i}{\hbar} \delta S_R[\phi]\, \mathcal{O}(t_0, \mathbf{x}_0) + \delta\mathcal{O}(t_0, \mathbf{x}_0)\right)$$

$$\delta S_R[\phi] = \int_R d^d x\; \partial_\mu \epsilon\, J^\mu$$

$$= \int_{t_i}^{t_f} dt \int_{\mathbb{R}^{d-1}} d^{d-1}\mathbf{x}\, \left(\epsilon\, \delta(t - t_0^-) - \epsilon\, \delta(t - t_0^+)\right) J^0(t, \mathbf{x})$$

$$= \epsilon\, Q(t_0^-) - \epsilon\, Q(t_0^+).$$

Then, just as in quantum mechanics, we find the operator form of Ward identity:

$$\boxed{\widehat{\delta\mathcal{O}(x_0)} = \frac{i}{\hbar}\left[\hat{Q}, \widehat{\mathcal{O}(x_0)}\right]}$$

"A continuous symmetry yields a conserved current in classical field theory, and the conserved charge generates the symmetry transformation after quantization." still holds.

Some drawbacks

- A choice of "time" or "time slice" has to be made to define Q or $\hat{Q}$.

- Convergence of integral $\int_{\mathbb{R}^{d-1}} d^{d-1}x \, J^0(t, x)$ not obvious.

  In fact, it is a serious problem when "Spontaneous symmetry breaking" occurs.

- $J^i(t, x) \to 0$ as $|x| \to \infty$ (used in $\frac{dQ}{dt} = 0$ from $\partial_\mu J^\mu = 0$) also not obvious.

These can be overcome by a slightly different formulation.
To describe it, it is easiest to use <u>the language of differential forms</u>.

Current (J^μ) is a "vector" but is more naturally a <u>$(d-1)$-form</u>

$$J = J^0 dx^1 \wedge \cdots \wedge dx^{d-1} - J^1 dx^0 \wedge dx^1 \wedge \cdots \wedge dx^{d-1} + \cdots + (-1)^{d-1} J^{d-1} dx^0 \wedge \cdots \wedge dx^{d-2}$$

$$= \sum_{\mu=0}^{d-1} (-1)^\mu J^\mu dx^0 \wedge \cdots \wedge \widehat{dx^\mu} \wedge \cdots \wedge dx^{d-1}$$

- $\delta S_R = \int_R d^d x \, \partial_\mu \epsilon \, J^\mu = \int_R d\epsilon \wedge J$
- Conservation equation $\partial_\mu J^\mu = 0 \iff dJ = 0$ (J is closed).
- $Q(t) = \int_{\{t\} \times \mathbb{R}^{d-1}} J$

We shall try to find identities satisfied by correlation functions on $\mathbb{R}^d$:

$$\langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \cdots \rangle$$

$= S_{\mathbb{R}^d}[\varphi]$

$$= \frac{1}{Z} \int \mathcal{D}\phi \, e^{\frac{i}{\hbar} S[\varphi]} \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \cdots$$

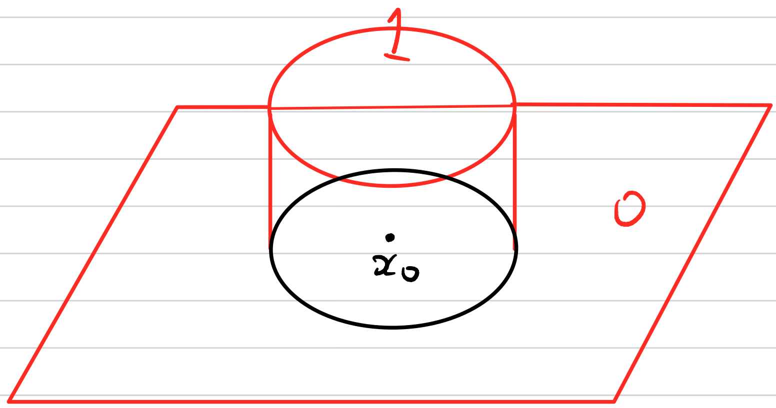
Consider a local observable $\mathcal{O}$ inserted at a point $x_0 \in \mathbb{R}^d$

Choose a d-dimensional ball $B \subset \mathbb{R}^d$ including x_0.

Apply the variation $\phi \to \phi + \epsilon(x)\, u(\phi, \partial\phi)$ in the integrand of

$\int \mathcal{D}\phi \, e^{\frac{i}{\hbar} S[\phi]} \mathcal{O}(x_0)$ where

$$\epsilon(x) = \begin{cases} 1 & x \in B \\ 0 & x \notin B \end{cases}$$



Ward identity:

$$0 = \int \delta\left(\mathcal{D}\phi \, e^{\frac{i}{\hbar} S[\phi]} \mathcal{O}(x_0) \right)$$

$$= \int \mathcal{D}\phi \, e^{\frac{i}{\hbar} S[\phi]} \left(\frac{i}{\hbar} \delta S[\phi] \, \mathcal{O}(x_0) + \delta \mathcal{O}(x_0) \right)$$

$$\delta S[\phi] = \int_{\mathbb{R}^d} d\epsilon \wedge J = \int_{\mathbb{R}^d} \left(d(\epsilon J) - \epsilon \, dJ \right)$$

$$= -\int_{\mathbb{R}^d} \epsilon \, dJ = -\int_B dJ = -\int_{\partial B} J$$

$$\therefore \left\langle \delta\mathcal{O}(x_0) \right\rangle = \left\langle \frac{i}{\hbar} \int_{\partial B} J \, \mathcal{O}(x_0) \right\rangle \quad \text{if } x_0 \in B.$$

The same holds also when other local observables are inserted outside B:

$$\left\langle \delta\mathcal{O}(x_0)\, \mathcal{O}_1(x_1)\, \mathcal{O}_2(x_2) \cdots \right\rangle$$

$$= \left\langle \frac{i}{\hbar} \int_{\partial B} J \, \mathcal{O}(x_0)\, \mathcal{O}_1(x_1)\, \mathcal{O}_2(x_2) \cdots \right\rangle$$

if $x_0 \in B, \quad x_1, x_2, \cdots \notin B.$

$(\star)$

Considering $\mathcal{O} = \mathrm{id}$ for which $\delta\mathcal{O} = 0$, we find

$$\left\langle \int_{\partial B} J \; \mathcal{O}_1(x_1)\, \mathcal{O}_2(x_2) \cdots \right\rangle = 0 \quad \text{if } x_1, x_2, \cdots \notin B.$$

In other words, as an identity inside correlators,

$$\boxed{\int_{\partial B} J = 0 \quad \text{if nothing is inserted inside } B.}$$

This is the quantum version of conservation law $dJ = 0$.

In particular, (★) continus to hold even if the surface $S = \partial B$ is deformed as long as x_0 is inside S and $x_1, x_2, \cdots$ are outside S.

In other words,

$$\boxed{\begin{array}{c} \delta \mathcal{O}(x_0) = \dfrac{i}{\hbar} \displaystyle\int_S J \, \mathcal{O}(x_0) \\ \text{for any surface } S \text{ enclosing } x_0 \\ \text{if nothing else is inserted inside } S \end{array}}$$

holds as an identity inside correlators.

Sometimes this is written as

$$\delta \mathcal{O}(x_0) = \frac{i}{\hbar} \oint_{x_0} J \, \mathcal{O}(x_0)$$

where $\oint_{x_0}$ stands for integration along a (small) surface enclosing only x_0.

<u>This is the form of Ward identity we were looking for.</u>

Remarks

(1) This holds whether or not there is an issue in convergence of $\int_{\mathbb{R}^{d-1}} d^{d-1}x\, J^0(t,x)$ (whether or not the symmetry is spontaneously broken).

(2) When $Q(t) = \int_{\mathbb{R}^{d-1}} d^{d-1}x\, J^0(t,x)$ is well-defined,

- $\widehat{\delta O} = \frac{i}{\hbar}[\hat{Q}, \hat{O}]$ can also be derived by considering $S = \partial(B_R^{d-1} \times [t_0^-, t_0^+])$ and taking the limits $R \to \infty$, $t_0^- \nearrow t_0 \swarrow t_0^+$.

- One can show that $\hat{Q}$ is a Lorentz scalar:

is contractible.

(3) There is a generalization :

J closed $(d-1)$-form

$\mathcal{O}(x)$ supported at a point x

$\rightsquigarrow \quad \delta \mathcal{O}(x) = \frac{i}{\hbar} \oint_x J \, \mathcal{O}(x)$

$\mapsto$

J_{d-1-p} closed $(d-1-p)$-form

$\mathcal{O}(\Sigma^p)$ supported at p-dimensional submanifold Σ^p

$\rightsquigarrow \quad \delta \mathcal{O}(\Sigma_p) = \frac{i}{\hbar} \oint_{\Sigma^p} J_{d-1-p} \, \mathcal{O}(\Sigma_p)$

"generalized global symmetries"

- We may also consider the exponentiated version.

- $\int_{S^{d-1}} J$ or $\int_{S^{d-1-p}} J_{d-1-p}$ or exp-version

  are called "topological defects".

(4) (Very technical)

In the above, we have secretly assumed that the spacetime $\mathbb{R}^d$ is oriented s.t. $dx^0 \wedge dx^1 \wedge \cdots \wedge dx^{d-1}$ is positive.

However, it is sometimes or more often assumed that $dx^1 \wedge \cdots \wedge dx^{d-1} \wedge dx^0$ is positive.

(This is especially so after Wick rotation: $dx^1 \wedge \cdots \wedge dx^d > 0$.)

In that case
$$\delta S = \int J \wedge d\epsilon = (-1)^{d-1} \int d\epsilon \wedge J$$

and the Ward identity takes the form

$$\delta \mathcal{O}(x) = (-1)^{d-1} \frac{i}{\hbar} \oint_x J \, \mathcal{O}(x).$$

(Or modify $J \to (-1)^{d-1} J$ and keep
$$\delta \mathcal{O}(x) = \frac{i}{\hbar} \oint_x J \, \mathcal{O}(x).$$
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