Decomposition to connected parts

Here we explain

(i) $Z_{\text{pert}}/Z_{\text{free}} \overset{!}{=} \exp\left(\text{sum of connected vacuum diagrams}\right)$

(ii) $\langle \Phi_1 \cdots \Phi_s \rangle_{\text{pert}}$

$$\overset{!!}{=} \sum_{\{1,\cdots,s\} = I_1 \cup \cdots \cup I_\ell} \underbrace{\pm}_{\text{permutation of fermionic } \Phi_i\text{'s}} \left\langle \prod_{i \in I_1} \Phi_{i_1} \right\rangle_{\text{conn}} \cdots \left\langle \prod_{i_\ell \in I_\ell} \Phi_{i_s} \right\rangle_{\text{conn}}$$

Sum over decompositions of $\{1,\cdots,s\}$ to non-empty subsets $I_1,\cdots,I_\ell \subset \{1,\cdots,s\}$

* In particular, no term has a vacuum diagram factor.

(ii) follows from (i) Define

$$Z_{\text{pert}}(J) := \left[\int \mathcal{D}\Phi \; e^{-S_{\text{free}}(\Phi) - S_{\text{int}}(\Phi) + J\cdot\Phi}\right]_{\text{pert}}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \int \mathcal{D}\Phi \; e^{-S_{\text{free}}(\Phi)} \Big(-S_{\text{int}}(\Phi) + \underbrace{J\cdot\Phi}_{\text{a part of interaction}}\Big)^n.$$

Then,

$$\langle \Phi_1 \cdots \Phi_s \rangle_{pert} = \frac{1}{Z_{pert}(J)} \frac{\partial}{\partial J_1} \cdots \frac{\partial}{\partial J_s} Z_{pert}(J) \Big|_{J=0} .$$

On the other hand, (i) implies

$$Z_{pert}(J) = Z_{free} \cdot \exp\left(Z_{conn}(J) \right), \text{ where}$$

$$Z_{conn}(J) = \text{Sum of connected vacuum diagrams}$$

($J \cdot \Phi$ is a part of interaction and corresponds to a vertex of the form J —— .)

Thus,

$$\langle \Phi_1 \cdots \Phi_s \rangle_{pert} = e^{-Z_{conn}(J)} \frac{\partial}{\partial J_1} \cdots \frac{\partial}{\partial J_s} e^{Z_{conn}(J)} \Big|_{J=0}$$

$$= \sum_{\{1,\cdots,s\} = I_1 \sqcup \cdots \sqcup I_l} \pm \prod_{i_1 \in I_1} \frac{\partial}{\partial J_{i_1}} Z_{conn}(J) \cdots \prod_{i_l \in I_l} \frac{\partial}{\partial J_{i_l}} Z_{conn}(J) \Big|_{J=0}$$

$$= \sum_{\{1,\cdots,s\} = I_1 \sqcup \cdots \sqcup I_l} \pm \left\langle \prod_{i_1 \in I_1} \Phi_{i_1} \right\rangle_{conn} \cdots \left\langle \prod_{i_l \in I_l} \Phi_{i_l} \right\rangle_{conn}$$

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Thus, it remains to show (i)

Notation in this discussion: for a diagram D, we write $[D]$ for the contribution of D to Z_{pert}/Z_{free}.

Thus $$Z_{pert}/Z_{free} = \sum_D [D]$$

Case 1 $-S_{int} = V$, a single type of vertex

e.g. ϕ^4 theory without source term.

Then $$Z_{pert}/Z_{free} = \sum_{n=0}^{\infty} \frac{1}{n!} \langle V^n \rangle_{free}$$

Suppose a connected diagram C has v_C vertices.

Then $[C]$ is a term in $\frac{1}{v_C!} \langle V^{v_C} \rangle_{free}$.

Also, $[\underbrace{C \cdots C}_{m}]$ is a term in $\frac{1}{(m v_C)!} \langle V^{m v_C} \rangle_{free}$,

and is included in its part

$$\frac{1}{(m v_C)!} \underbrace{\langle V^{v_C} \rangle_{free} \cdots \langle V^{v_C} \rangle_{free}}_{m} \times \left(\begin{array}{l} \text{number of ways to} \\ \text{decompose } m v_C \text{ elements} \\ \text{to } m \text{ groups of } v_C \text{ elements} \end{array} \right)$$

$$\begin{pmatrix}\text{number of ways to} \\ \text{decompose } m\nu_c \text{ elements} \\ \text{to } m \text{ groups of } \nu_c \text{ elements}\end{pmatrix}$$

$$= \underbrace{\binom{m\nu_c}{\nu_c}\binom{m\nu_c-\nu_c}{\nu_c}\cdots\binom{2\nu_c}{\nu_c}\binom{\nu_c}{\nu_c}}_{\text{number of ways to put } m\nu_c \text{ elements to } m \text{ labeled boxes}} \times \underbrace{\frac{1}{m!}}_{\text{forget the labels of the boxes}}$$

$$= \frac{(m\nu_c)!}{(\nu_c!)^m\, m!}$$

$\therefore [\underbrace{C \cdots C}_{m}]$ is a term in

$$\frac{1}{\cancel{(m\nu_c)!}} \langle V^{\nu_c} \rangle_{\text{free}} \cdots \langle V^{\nu_c} \rangle_{\text{free}} \times \frac{\cancel{(m\nu_c)!}}{(\nu_c!)^m\, m!}$$

$$= \frac{1}{m!}\Big(\underbrace{\frac{1}{\nu_c!}\langle V^{\nu_c}\rangle_{\text{free}}}_{[C] + \text{others}}\Big)^m$$

$$\therefore [\underbrace{C \cdots C}_{m}] = \frac{1}{m!}[C]^m$$

If $C_1, \cdots, C_k$ are connected diagrams of $v_{C_1}, \cdots, v_{C_k}$ vertices,

$$[\underbrace{C_1 \cdots C_1}_{m_1} \ \underbrace{C_2 \cdots C_2}_{m_2} \cdots \underbrace{C_k \cdots C_k}_{m_k}]$$

is a term in $\dfrac{1}{(m_1 V_{C_1} + \cdots + mV_{C_k})!} \left\langle V^{m_1 V_{C_1} + \cdots + mV_{C_2}} \right\rangle_{\text{free}}$

and is included in

$$\frac{1}{(m_1 V_{C_1} + \cdots + mV_{C_k})!} \left\langle V^{v_{C_1}} \right\rangle_{\text{free}}^{m_1} \cdots \left\langle V^{v_{C_k}} \right\rangle_{\text{free}}^{m_k}$$

$$\times \underbrace{\begin{pmatrix} \text{number of ways to decompose } m_1 V_{C_1} + \cdots + mV_{C_k} \\ \text{elements to } m_1 \text{ groups of } V_{C_1} \text{ elements}, \cdots, \\ \cdots, m_k \text{ groups of } V_{C_k} \text{ elements} \end{pmatrix}}_{\dfrac{(m_1 V_{C_1} + \cdots + mV_{C_k})!}{(V_{C_1}!)^{m_1} \cdots (V_{C_k}!)^{m_k}} \ \dfrac{1}{m_1! \cdots m_k!}}$$

$$= \frac{1}{m_1!} \left(\frac{1}{V_{C_1}!} \left\langle V^{V_{C_1}} \right\rangle_{\text{free}} \right)^{m_1} \cdots \frac{1}{m_k!} \left(\frac{1}{V_{C_k}!} \left\langle V^{V_{C_k}} \right\rangle_{\text{free}} \right)^{m_k}$$

$$\therefore \ [\underbrace{C_1 \cdots C_1}_{m_1} \cdots \underbrace{C_k \cdots C_k}_{m_k}] = \frac{1}{m!} [C_1]^{m_1} \cdots \frac{1}{m_k!} [C_k]^{m_k}$$

Thus

$$Z_{pert}/Z_{free} = \sum_{D} [D]$$

$$= \sum_{\substack{C_1, \cdots, C_k \\ \text{connected}}} [C_1 \cdots C_1 \cdots C_k \cdots C_k]$$

$$[\underbrace{C_1 \cdots C_1}_{m_1} \cdots \underbrace{C_k \cdots C_k}_{m_k}] = \frac{1}{m_1!}[C_1]^{m_1} \cdots \frac{1}{m_k!}[C_k]^{m_k}$$

$$= \prod_{C \text{ connected}} \underbrace{\sum_{m_c=0}^{\infty} \frac{1}{m_c!}[C]^{m_c}}_{\exp([C])}$$

$$= \exp\left(\sum_{C \text{ connected}} [C]\right) \qquad //$$

<u>Case 2</u> $-S_{int} = V_1 + \cdots + V_N$: multiple types of vertices

e.g. ϕ^4 theory with a source $J \cdot \phi$

e.g. QCD with or without a source.

Then, $$Z_{pert}/Z_{free} = \sum_{n_1,\cdots,n_N} \frac{1}{n_1!\cdots n_N!}\langle V_1^{n_1}\cdots V_N^{n_N}\rangle_{free}$$

Suppose a connected diagram C has V_C^1 vertices of type V_1, V_C^2 vertices of type V_2, $\cdots$, V_C^N vertices of type V_N.

Then $[C]$ is a term in $$\frac{1}{V_C^1!\cdots V_C^N!}\langle V_1^{V_C^1}\cdots V_N^{V_C^N}\rangle_{free}.$$

Also $[\underbrace{C\cdots C}_{m}]$ is a term in $$\frac{1}{(mV_C^1)!\cdots(mV_C^N)!}\langle V_1^{mV_C^1}\cdots V_N^{mV_C^N}\rangle_{free}$$

and is included in its part

$$\frac{1}{(mV_C^1)!\cdots(mV_C^N)!}\left(\langle V_1^{V_C^1}\cdots V_N^{V_C^N}\rangle_{free}\right)^m$$

$$\times \underbrace{\left(\begin{array}{l}\text{number of ways to distribute } mV_C^1 \text{ elements of type } 1,\\ mV_C^2 \text{ elements to type } 2, \cdots, mV_C^N \text{ elements of type } N\\ \text{to } m \text{ unlabeled boxes, where each box admit}\\ V_C^1 \text{ elements of type } 1, \cdots, V_C^N \text{ elements of type } N\end{array}\right)}_{\frac{(mV_C^1)!}{(V_C^1!)^m}\cdots\frac{(mV_C^N)!}{(V_C^N!)^m}\cdot\frac{1}{m!}}$$

$$= \frac{1}{m!}\left(\frac{1}{V_c^1! \cdots V_c^N!} \left\langle V_1^{V_c^1} \cdots V_N^{V_c^N} \right\rangle_{\text{free}} \right)^m$$

$$\therefore \quad [\underbrace{C \cdots C}_{m}] = \frac{1}{m!} [C]^m$$

For $i = 1, \cdots, k$, let C_i be a connected diagram with $V_{C_i}^j$ vertices of type V_j $(j = 1, \cdots, N)$. Then

$[\underbrace{C_1 \cdots C_1}_{m_1} \cdots \underbrace{C_k \cdots C_k}_{m_k}]$ is a term in

$$\prod_{j=1}^{N} \frac{1}{\left(\sum_{i=1}^{k} m_i V_{C_i}^j\right)!} \left\langle V_1^{\sum_{i=1}^{k} m_i V_{C_i}^1} \cdots V_N^{\sum_{i=1}^{k} m_i V_{C_i}^N} \right\rangle_{\text{free}}$$

and is included in its part

$$\prod_{j=1}^{N} \frac{1}{\left(\sum_{i=1}^{k} m_i V_{C_i}^j\right)!} \prod_{i=1}^{k} \left\langle V_1^{V_{C_i}^1} \cdots V_N^{V_{C_i}^N} \right\rangle_{\text{free}}^{m_i}$$

$\times$ (number of ways to distribute $\sum_{i=1}^{k} m_i V_{C_i}^j$ elements of type j $(j = 1, \cdots, N)$ to m_i unlabeled boxes, where each box admit $V_{C_i}^1$ elements of type $1, \cdots, V_{C_i}^N$ elements of type N $(i = 1, \cdots, k)$.)

$$= \prod_{j=1}^{N} \frac{1}{\left(\sum_{i=1}^{k} m_i V_{c_i}^{j}\right)!} \prod_{i=1}^{k} \left\langle V_1^{V_{c_i}^{1}} \cdots V_N^{V_{c_i}^{N}} \right\rangle_{free}^{m_i}$$

$$\times \prod_{j=1}^{N} \left(\frac{\left(\sum_{i=1}^{k} m_i V_{c_i}^{j}\right)!}{\prod_{i=1}^{k} (m_i V_{c_i}^{j})!} \cdot \prod_{i=1}^{k} \frac{(m_i V_{c_i}^{j})!}{(V_{c_i}^{j}!)^{m_i}} \right) \frac{1}{m_1! \cdots m_k!}$$

$$= \prod_{i=1}^{k} \frac{1}{m_i!} \left(\frac{1}{V_{c_i}^{1}! \cdots V_{c_i}^{N}!} \left\langle V_1^{V_{c_i}^{1}} \cdots V_N^{V_{c_i}^{N}} \right\rangle_{free} \right)^{m_i}$$

Thus,

$$[\underbrace{C_1 \cdots C_1}_{m_1} \cdots \underbrace{C_k \cdots C_k}_{m_k}] = \frac{1}{m_1!}[C_1]^{m_1} \cdots \frac{1}{m_k!}[C_k]^{m_k} \longrightarrow *$$

$$Z_{pert}/Z_{free} = \sum_{D} [D] = \sum_{C_1, \cdots, C_k \text{ conn}} [C_1 \cdots C_1 \cdots C_k \cdots C_k]$$

$$\overset{*}{=} \prod_{C \text{ conn}} \sum_{m_C=0}^{\infty} \frac{1}{m_C!} [C]^{m_C}$$

$$= \prod_{C \text{ conn}} \exp[C] = \exp\left(\sum_{C \text{ conn}} [C]\right)$$

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