Complex scalar

A finite system:

Variables : $\varphi = (\varphi^1, \dots, \varphi^n) \in \mathbb{C}^n$

notation $\bar{\varphi} = (\bar{\varphi}_1, \dots, \bar{\varphi}_n) := (\varphi^{1*}, \dots, \varphi^{n*})$

$$S_E(\varphi) = \sum_{i,j=1}^{n} \bar{\varphi}_i A^i_j \varphi^j =: \bar{\varphi} A \varphi$$

A hermitian, positive eigenvalues

$$d\bar{\varphi}\, d\varphi = d\bar{\varphi}_1 d\varphi^1 \cdots d\bar{\varphi}_n d\varphi^n = d\bar{\varphi}_n \cdots d\bar{\varphi}_1\, d\varphi^1 \cdots d\varphi^n$$

$$Z = \int d\bar{\varphi}\, d\varphi\, e^{-S_E(\varphi)} = \frac{(2\pi i)^n}{\det A}$$

$$\langle \varphi^{i_1} \cdots \varphi^{i_s} \bar{\varphi}_{j_1} \cdots \bar{\varphi}_{j_t} \rangle = ?$$

$$f(A, \bar{J}, J) := \int d\bar{\varphi}\, d\varphi\, e^{-S_E(\varphi) + \sum_i (\bar{J}_i \varphi^i + \bar{\varphi}_i J^i)}$$

$$\frac{\partial}{\partial \bar{J}_{i_1}} \cdots \frac{\partial}{\partial \bar{J}_{i_s}} \frac{\partial}{\partial J^{j_1}} \cdots \frac{\partial}{\partial J^{j_t}} f(A, J)$$

$$= \int d\bar{\varphi}\, d\varphi\, e^{-S_E(\varphi) + \bar{J}\varphi + \bar{\varphi} J}\, \varphi^{i_1} \cdots \varphi^{i_s} \bar{\varphi}_{j_1} \cdots \bar{\varphi}_{j_t}$$

$$\xrightarrow{\bar{J}, J \to 0} Z \langle \varphi^{i_1} \cdots \varphi^{i_s} \bar{\varphi}_{j_1} \cdots \bar{\varphi}_{j_t} \rangle$$

$f(A, \bar{J}, J)$ can be computed:

$$= \int d\bar{\varphi}\, d\varphi\, e^{-(\bar{\varphi} - \bar{J}A^{-1})A(\varphi - A^{-1}J) + \bar{J}A^{-1}J} = Z\, e^{\bar{J}A^{-1}J}$$

$$\therefore \langle \varphi^{i_1} \cdots \varphi^{i_s} \bar{\varphi}_{j_1} \cdots \bar{\varphi}_{j_t} \rangle = \underbrace{\frac{\partial}{\partial \bar{J}_{i_1}} \cdots \frac{\partial}{\partial \bar{J}_{i_s}}}_{\text{do this first}} \frac{\partial}{\partial J^{j_1}} \cdots \frac{\partial}{\partial J^{j_t}} e^{\bar{J}A^{-1}J} \Bigg|_{J, \bar{J} \to 0}$$

$$= \frac{\partial}{\partial J^{j_1}} \cdots \frac{\partial}{\partial J^{j_t}} (A^{-1}J)^{i_1} \cdots (A^{-1}J)^{i_s}\, e^{\bar{J}A^{-1}J} \Bigg|_{J, \bar{J} \to 0}$$

$$= \frac{\partial}{\partial J^{j_1}} \cdots \frac{\partial}{\partial J^{j_t}} (A^{-1}J)^{i_1} \cdots (A^{-1}J)^{i_s} \Bigg|_{J \to 0}$$

- When $s \neq t$, this vanishes.
- When $s = t$, this is the sum of $i_a - j_b$ parings $1 \leq a, b \leq s$.

e.g.

$$\langle \varphi^i \bar{\varphi}_j \rangle = \varphi^i \bar{\varphi}_j = A^{-1\,i}{}_j$$

$$\langle \varphi^i \varphi^j \bar{\varphi}_k \bar{\varphi}_\ell \rangle = \varphi^i \varphi^j \bar{\varphi}_k \bar{\varphi}_\ell + \varphi^i \varphi^j \bar{\varphi}_k \bar{\varphi}_\ell$$

$$= A^{-1\,i}{}_k A^{-1\,j}{}_\ell + A^{-1\,i}{}_\ell A^{-1\,j}{}_k$$

$$\vdots$$

$$\langle \varphi^{i_1} \cdots \varphi^{i_s} \bar{\varphi}_{j_1} \cdots \bar{\varphi}_{j_s} \rangle = \varphi^{i_1} \cdots \varphi^{i_s} \bar{\varphi}_{j_1} \cdots \bar{\varphi}_{j_s} + \cdots$$

$$= \sum_{\sigma \in \mathfrak{S}_s} A^{-1\,i_1}{}_{j_{\sigma(1)}} \cdots A^{-1\,i_s}{}_{j_{\sigma(s)}}.$$

complex scalar field in d dimensions

variable: $\phi(x)$ a $\mathbb{C}$-valued function of $x \in \mathbb{R}^d$, $\bar{\phi}(x) := \phi(x)^*$.

$$S_E[\phi] = \int d^d x_E \left(\partial\bar{\phi}\cdot\partial\phi + m^2\,\bar{\phi}\cdot\phi \right)$$

$$\overset{BC}{=} \int d^d x_E\; \bar{\phi}(x_E)\left(-\partial^2 + m^2\right)\phi(x_E).$$

$$\langle \phi(x_E)\,\bar{\phi}(y_E)\rangle_{\mathbb{R}^d_E} = \left(-\partial^2+m^2\right)^{-1}_{x_E y_E} = \int \frac{d^d p_E}{(2\pi)^d}\,\frac{e^{-ip_E(x_E-y_E)}}{p_E^2+m^2}.$$

$$\left(\text{Also, } \langle \phi(x_E)\,\phi(y_E)\rangle_{\mathbb{R}^d_E} = \langle \bar{\phi}(x_E)\,\bar{\phi}(x_E)\rangle_{\mathbb{R}^d_E} = 0. \right)$$

By reverse Wick rotation,

$$\langle \phi(x)\,\bar{\phi}(y)\rangle = \int_{\mathbb{R}^d} \frac{d^d p}{(2\pi)^d}\,\frac{i\,e^{-ip(x-y)}}{p^2-m^2+i0}$$

$$= \int \frac{d^{d-1}\mathbb{p}}{(2\pi)^{d-1}2\omega_{\mathbb{p}}}\, e^{-i\omega_{\mathbb{p}}|x^0-y^0| - i\mathbb{p}\cdot(\mathbb{x}-\mathbb{y})}$$

where $\omega_{\mathbb{p}} = \sqrt{\mathbb{p}^2+m^2}$, just as in the case of real scalar.

$$\left(\text{Also, } \langle \phi(x)\,\phi(y)\rangle = \langle \bar{\phi}(x)\,\bar{\phi}(y)\rangle = 0 \right)$$

And canonical quantization produces

$$\langle 0|T\,\phi(x)\bar{\phi}(y)|0\rangle = \int \frac{d^{d-1}\mathbb{p}}{(2\pi)^{d-1}2\omega_{\mathbb{p}}}\, e^{-i\omega_{\mathbb{p}}|x^0-y^0| - i\mathbb{p}\cdot(\mathbb{x}-\mathbb{y})},$$

$$\langle 0|T\,\phi(x)\phi(y)|0\rangle = \langle 0|T\,\bar{\phi}(x)\bar{\phi}(y)|0\rangle = 0.$$