$\Sigma_2(p)$ with a general gauge parameter ξ

In the lecture, we took $\xi = 1$ to compute one-loop amplitudes. $\Pi_2^{\mu\nu}(q)$ is independent of ξ as $\overbrace{\psi(x)\bar{\psi}(y)}$ is. But $\Sigma_2(p)$ and $\Gamma_2^{\mu}(p,q)$ do depend on ξ. Let us compute the former, and find the 1-loop expression for γ_ψ & γ_m for a general ξ.

For a general ξ, the free propagator of photon is

$$\overbrace{A_\mu(x)\, A_\nu(y)} = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik(x-y)} \left(\frac{\delta_{\mu\nu}}{k^2} + (\xi - 1)\frac{k_\mu k_\nu}{(k^2)^2} \right).$$

Let us assume that the free propagator of the regulator B_μ is

$$\overbrace{B_\mu(x)\, B_\nu(y)} = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik(x-y)} \left(\frac{\delta_{\mu\nu}}{k^2+\Lambda^2} + (\xi - 1)\frac{k_\mu k_\nu}{(k^2+\Lambda^2)^2} \right).$$

This is realized by the following kinetic term

$$\mathcal{L}_B^{\text{kin}} = \frac{1}{2} B_\mu \left(\delta^{\mu\nu}(-\partial^2 + \Lambda^2) + (1-\xi)\, \partial^\mu \frac{-\partial^2 + \Lambda^2}{-\xi\partial^2 + \Lambda^2}\, \partial^\nu \right) B_\nu$$

$$= \frac{1}{4} F_B^{\mu\nu} F_{\mu\nu}^B + \frac{\Lambda^2}{2} B^2 + \frac{1}{2\xi}(\partial\cdot B)^2 - \frac{1}{2}\left(1 - \frac{1}{\xi}\right)^2 \Lambda^2\, B_\mu \frac{\partial^\mu \partial^\nu}{-\partial^2 + \frac{1}{\xi}\Lambda^2} B_\nu .$$

This looks strange or at best complicated. If you do not like it, you may introduce an additional scalar ϕ and consider

$$\mathcal{L}_{kin}^{B,\phi} = \frac{1}{4} F_B^{\mu\nu} F^B_{\mu\nu} + \frac{\Lambda^2}{2} B^2 + \frac{1}{2\xi}(\partial\cdot B)^2$$

$$+ \frac{1}{2}(\partial\phi)^2 + \frac{1}{2\xi}\Lambda^2\phi^2 - i\left(1-\frac{1}{\xi}\right)\Lambda\,\phi\,\partial\cdot B$$

Integrating out ϕ, we obtain the above $\mathcal{L}^B_{kin}$.

In any case, we shall use the above $\overbrace{B_\mu(x)\, B_\nu(y)}$.

In this Pauli-Villars regularization,

$$\Sigma_2^{PV}(p) = -e^2 \int \frac{d^4k}{(2\pi)^4}\, \gamma^\mu \frac{1}{-\not{k}+m}\gamma^\nu \sum_{i=0,1} c_i \left(\frac{\delta_{\mu\nu}}{(k-p)^2+\Lambda_i^2} + (\xi-1)\frac{(k-p)_\mu (k-p)_\nu}{((k-p)^2+\Lambda_i^2)^2} \right)$$

where $(c_0, \Lambda_0) = (1, 0)$ and $(c_1, \Lambda_1) = (-1, \Lambda)$

Note that $\dfrac{1}{-\not{k}+m} = \dfrac{\not{k}+m}{k^2+m^2}$

- $\gamma^\mu(\not{k}+m)\gamma^\nu \delta_{\mu\nu} = 2\not{k} - 4m$

- $(\not{k}-\not{p})(\not{k}+m)(\not{k}-\not{p}) = (\not{k}-\not{p})(\not{k}-\not{p}+\not{p}+m)(\not{k}-\not{p})$

$$= \underbrace{(\not{k}-\not{p})^3}_{-(k-p)^2(\not{k}-\not{p})} + \underbrace{(\not{k}-\not{p})\not{p}(\not{k}-\not{p})}_{-2(k-p)\cdot p(\not{k}-\not{p}) - \not{p}(\not{k}-\not{p})^2} + \underbrace{m(\not{k}-\not{p})^2}_{-m(k-p)^2}$$

$\not{p}(k-p)^2$

$$= (k-p)^2(-\not{k}+2\not{p}-m) - 2(k-p)\cdot p\,(\not{k}-\not{p})$$

$$= ((k-p)^2+\Lambda^2)(-\not{k}+2\not{p}-m) - \Lambda^2(-\not{k}+2\not{p}-m) - 2(k-p)\cdot p\,(\not{k}-\not{p})$$

$$\Sigma_2^{PV}(p) = e^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2+m^2} \sum_i c_i \left[\frac{-2\not{k}+4m}{(k-p)^2+\Lambda_i^2} + (1-\xi)\frac{-\not{k}+2\not{p}-m}{(k-p)^2+\Lambda_i^2} \right.$$

$$\left. -(1-\xi)\frac{2(k-p)\cdot p\,(\cancel{\not{k}-\not{p}}) + \Lambda_i^2(-\not{k}+2\not{p}-m)}{((k-p)^2+\Lambda_i^2)^2} \right]$$

- As in the lecture,

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2+m^2} \sum_i c_i \frac{a\not{k}+b}{(k-p)^2+\Lambda_i^2}$$

$$= \int \frac{d^4k}{(2\pi)^4} \sum_i c_i (a\not{k}+b) \int_0^1 \frac{dx}{(\underbrace{(1-x)(k^2+m^2)+x((k-p)^2+\Lambda_i^2)}_{k^2-2xpk+xp^2+(1-x)m^2+x\Lambda_i^2})^2}$$

$$= (\underbrace{k-xp}_{\ell})^2 + \underbrace{x(1-x)p^2+(1-x)m^2+x\Lambda_i^2}_{\Delta_i}$$

$$= \int_0^1 dx \int \frac{d^4\ell}{(2\pi)^4} \sum_i c_i \frac{a(\not{\ell}+x\not{p})+b}{(\ell^2+\Delta_i)^2}$$

$$= \int_0^1 dx \frac{1}{(4\pi)^2} \int_0^\infty t\,dt \sum_i c_i \frac{ax\not{p}+b}{(t+\Delta_i)^2} \quad ; \quad \frac{t\,dt}{(t+\Delta_i)^2} = d\left[\log(t+\Delta_i) + \frac{\Delta_i}{t+\Delta_i}\right]$$

$$= \int_0^1 dx \frac{1}{(4\pi)^2}(ax\not{p}+b)\left[\sum_i c_i \left(\log(t+\Delta_i) + \frac{\Delta_i}{t+\Delta_i}\right)\right]_0^\infty$$

$$= \int_0^1 dx \frac{1}{(4\pi)^2}(ax\not{p}+b)\log\left(\frac{\Delta_1}{\Delta_0}\right)$$

$$\bullet \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2+m^2} \sum_i c_i \frac{(k-p)\cdot p \, (\not{k}-\not{p})}{((k-p)^2+\Lambda_i)^2}$$

$$= \int \frac{d^4k}{(2\pi)^4} (k-p)\cdot p \, (\not{k}-\not{p}) \sum_i c_i \int_0^1 \frac{2x\,dx}{\big(\underbrace{(1-x)(k^2+m^2) + x((k-p)^2+\Lambda_i^2)}_{(\underbrace{k-xp}_{\ell})^2+\Delta_i}\big)^3}$$

$$= \int_0^1 2x\,dx \int \frac{d^4\ell}{(2\pi)^4} \sum_i c_i \frac{(\ell-(1-x)p)\cdot p \, (\not{\ell}-(1-x)\not{p})}{(\ell^2+\Delta_i)^3}$$

$$\cdot \quad \ell\cdot p \, \not{\ell} = \ell^\mu p_\mu \gamma^\nu \ell_\nu \simeq \frac{1}{4}\delta^\mu_\nu \ell^2 p_\mu \gamma^\nu = \frac{1}{4}\ell^2 \not{p}$$

$$= \int_0^1 2x\,dx \underbrace{\int \frac{d^4\ell}{(2\pi)^4} \sum_i c_i \frac{\frac{1}{4}\ell^2\not{p} + (1-x)^2 p^2 \not{p}}{(\ell^2+\Delta_i)^3}}_{\frac{1}{(4\pi)^2}\int_0^\infty t\,dt \sum_i c_i \frac{\frac{1}{4}t\not{p} + (1-x)^2p^2\not{p}}{(t+\Delta_i)^3}}$$

$$\bullet \quad \frac{t^2\,dt}{(t+\Delta_i)^3} = d\left(\log(t+\Delta_i) + \frac{2\Delta_i}{t+\Delta_i} - \frac{1}{2}\frac{\Delta_i^2}{(t+\Delta_i)^2} \right)$$

$$\bullet \quad \frac{t\,dt}{(t+\Delta_i)^3} = d\left(-\frac{1}{t+\Delta_i} + \frac{1}{2}\frac{\Delta_i}{(t+\Delta_i)^2} \right)$$

$$= \int_0^1 2x\,dx \frac{1}{(4\pi)^2}\not{p} \sum_i c_i \left[\frac{1}{4}\left(-\log\Delta_i - 2 + \frac{1}{2}\right) + \frac{(1-x)^2p^2}{2\Delta_i} \right]$$

$$= \int_0^1 x\,dx \frac{1}{(4\pi)^2}\not{p} \left[\frac{1}{2}\log\left(\frac{\Delta_1}{\Delta_0}\right) + \frac{(1-x)^2p^2}{\Delta_0} \right] + O\left(\frac{p^2}{\Lambda^2}, \frac{m^2}{\Lambda^2}\right)$$

$$\bullet \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2+m^2} \frac{\Lambda^2(-\cancel{k}+2\cancel{p}-m)}{((k-p)^2+\Lambda^2)^2} = \int \frac{d^4k}{(2\pi)^4} \int_0^1 2x\,dx \frac{\Lambda^2(-\cancel{k}+2\cancel{p}-m)}{((\underbrace{k-xp}_{\ell})^2+\Delta_1)^3}$$

$$= \int_0^1 2x\,dx \int \frac{d^4\ell}{(2\pi)^4} \frac{\Lambda^2(-\cancel{\ell}-x\cancel{p}+2\cancel{p}-m)}{(\ell^2+\Delta_1)^3}$$

$$= \int_0^1 2x\,dx \frac{1}{(4\pi)^2} \int_0^\infty t\,dt \frac{\Lambda^2((2-x)\cancel{p}-m)}{(t+\Delta_1)^3}$$

$$= \int_0^1 2x\,dx \frac{1}{(4\pi)^2} \frac{\Lambda^2((2-x)\cancel{p}-m)}{2\Delta_1}$$

$$= \int_0^1 dx \frac{1}{(4\pi)^2} ((2-x)\cancel{p}-m) + \mathcal{O}\left(\frac{p^2}{\Lambda^2}, \frac{m^2}{\Lambda^2}\right)$$

Combining all, we find

$$\Sigma_2^{PV}(p) = \frac{e^2}{(4\pi)^2} \int_0^1 dx \Bigg[\left\{ (-2x\cancel{p}+4m) + (1-\xi)(-x\cancel{p}+2\cancel{p}-m) \right\} \log\left(\frac{\Delta_1}{\Delta_0}\right)$$

$$-2(1-\xi)x\cancel{p}\left[\frac{1}{2}\log\left(\frac{\Delta_1}{\Delta_0}\right) + \frac{(1-x)^2p^2}{\Delta_0}\right] - (1-\xi)\{(2-x)\cancel{p}-m\} \Bigg]$$

$$\Delta_0 = (1-x)(m^2+xp^2)$$

$$= \frac{e^2}{(4\pi)^2} \int_0^1 dx \Bigg[\left\{ (-2x+2(1-\xi)(1-x))\cancel{p} + 4m - (1-\xi)m \right\} \log\left(\frac{\Delta_1}{\Delta_0}\right)$$

$$-(1-\xi)\underbrace{\left\{ 2\frac{x(1-x)p^2}{m^2+xp^2} + 2 - x \right\}}_{4-3x-2\frac{(1-x)m^2}{m^2+xp^2}} \cancel{p} + (1-\xi)m \Bigg]$$

$$\therefore \Sigma_2(p) = A_2(p^2)\not{p} + B_2(p^2)m \qquad \text{where}$$

$$A_2(p^2) = \frac{e^2}{(4\pi)^2}\int_0^1 dx \left[\left(-2x + 2(1-\xi)(1-x)\right)\log\left(\frac{\Delta_1}{\Delta_0}\right) + (1-\xi)\left(-4+3x+\frac{2(1-x)m^2}{m^2+xp^2}\right)\right]$$

$$B_2(p^2) = \frac{e^2}{4\pi}\int_0^1 dx \left[(3+\xi)\log\left(\frac{\Delta_1}{\Delta_0}\right) + 1-\xi \right]$$

<u>On shell R.C.</u>

$$\delta_3^{(1)} = \Pi_2(0), \quad \delta_2^{(1)} = A_2(-m^2), \quad \delta_m^{(1)} = -B_2(-m^2)$$

$$\Pi^{(1)}(q^2) = \frac{8e^2}{(4\pi)^2}\int_0^1 dx\, x(1-x)\log\left(\frac{m^2+x(1-x)q^2}{m^2}\right)$$

(Same as $\xi = 1$ case)

$$A^{(1)}(p^2) = A_2(p^2) - A_2(-m^2)$$

$$= \frac{e^2}{(4\pi)^2}\int_0^1 dx \left[\left(-2x + 2(1-\xi)(1-x)\right)\log\left(\frac{(1-x)m^2}{m^2+xp^2}\right) - 2(1-\xi)\frac{x(m^2+p^2)}{m^2+xp^2}\right]$$

$$B^{(1)}(p^2) = B_2(p^2) - B_2(-m^2)$$

$$= \frac{e^2}{(4\pi)^2}\int_0^1 dx\,(3+\xi)\log\left(\frac{(1-x)m^2}{m^2+xp^2}\right)$$

Another R.C.

$$\delta_3^{(1)} = \Pi_2(\mu^2), \quad \delta_2^{(1)} = A_2(\mu^2), \quad \delta_m^{(1)} = -B_2(\mu^2)$$

$$\beta^{(1)} = \frac{1}{2} e \mu \frac{d}{d\mu} \delta_3^{(1)} = \frac{4e^3}{(4\pi)^2} \int_0^1 dx \; x(1-x) \frac{x(1-x)\, 2\mu^2}{m^2 + x(1-x)\mu^2},$$

(Same as $\xi = 1$ case)

$$\gamma_\psi^{(1)} = \frac{1}{2} \mu \frac{d}{d\mu} \delta_2^{(1)}$$

$$= \frac{2e^2}{(4\pi)^2} \int_0^1 dx \left[(x - (1-\xi)(1-x)) \frac{x\mu^2}{m^2 + x\mu^2} - (1-\xi) \frac{x(1-x) m^2 \mu^2}{m^2 + x\mu^2} \right],$$

$$\gamma_m^{(1)} = \mu \frac{d}{d\mu} \left(\delta_m^{(1)} - \delta_2^{(1)} \right)$$

$$= \frac{2e^2}{(4\pi)^2} \int_0^1 dx \left[(-2x + 2(1-\xi)(1-x) + 3 + \xi) \frac{x\mu^2}{m^2 + x\mu^2} + 2(1-\xi) \frac{x(1-x) m^2 \mu^2}{m^2 + x\mu^2} \right].$$

In the limit $m^2 \to 0$ (or $\mu^2 \gg m^2$),

$$\beta^{(1)} \to \frac{4e^3}{3(4\pi)^2}$$

$$\gamma_\psi^{(1)} \to \frac{e^2}{(4\pi)^2} \xi$$

$$\gamma_m^{(1)} \to \frac{6e^2}{(4\pi)^2}$$