Ghost system

We set $\hbar = 1$ in this note.

Consider the mechanics of a pair of anticommuting variables $c, \bar{c}$ with Lagrangian

$$L = i\dot{\bar{c}}\dot{c} - i\omega^2 \bar{c}c \qquad (\omega \in \mathbb{R})$$

- The system has two conserved charges:

$$E = i\dot{\bar{c}}\dot{c} + i\omega^2 \bar{c}c \qquad \text{energy},$$

$$N_{gh} = \bar{c}\dot{c} - \dot{\bar{c}}c \qquad \text{ghost number}.$$

Show that E and $-iN_{gh}$ are the Noether charges for time translation symmetry and symmetry $c \to e^{\alpha}c$, $\bar{c} \to e^{-\alpha}\bar{c}$.

- The canonical commutation relation among $\hat{c}, \hat{\bar{c}}, \hat{\dot{c}}, \hat{\dot{\bar{c}}}$ is

$$\{\hat{c}, \hat{\dot{\bar{c}}}\} = 1, \quad \{\hat{\bar{c}}, \hat{\dot{c}}\} = -1, \quad \text{all other anticommutators} = 0.$$

Show this using Ward identity.

- Lagrangian is real $L^* = L$ under $c^* = c$, $\bar{c}^* = \bar{c}$.

Accordingly, we'd like to find a representation with an inner product s.t. $\hat{c}^\dagger = \hat{c}$, $\hat{\dot{c}}^\dagger = \hat{\dot{c}}$, $\hat{\bar{c}}^\dagger = \hat{\bar{c}}$, $\hat{\dot{\bar{c}}}^\dagger = \hat{\dot{\bar{c}}}$.

- Define $\hat{b}$ and $\hat{\bar{b}}$ by

$$\hat{b} = \frac{1}{\sqrt{2}}\left(\sqrt{\omega}\,\hat{c} + \frac{i}{\sqrt{\omega}}\hat{\dot{c}}\right), \quad \hat{b}^{\dagger} = \frac{1}{\sqrt{2}}\left(\sqrt{\omega}\,\hat{c} - \frac{i}{\sqrt{\omega}}\hat{\dot{c}}\right),$$

$$\hat{\bar{b}} = \frac{-i}{\sqrt{2}}\left(\sqrt{\omega}\,\hat{\bar{c}} + \frac{i}{\sqrt{\omega}}\hat{\dot{\bar{c}}}\right), \quad \hat{\bar{b}}^{\dagger} = \frac{i}{\sqrt{2}}\left(\sqrt{\omega}\,\hat{\bar{c}} - \frac{i}{\sqrt{\omega}}\hat{\dot{\bar{c}}}\right).$$

We have also written $\hat{b}^{\dagger}$ and $\hat{\bar{b}}^{\dagger}$ anticipating $\hat{c}^{\dagger} = \hat{c}$ etc.

These four obey:

$$\{\hat{b}, \hat{\bar{b}}^{\dagger}\} = \{\hat{\bar{b}}, \hat{b}^{\dagger}\} = 1. \quad \text{all other anticommutators} = 0.$$

- Following again harmonic oscillator, let us prepare a state $|0\rangle$ annihilated by $\hat{b}$ and $\hat{\bar{b}}$, and build other states by operating $\hat{b}^{\dagger}$ and $\hat{\bar{b}}^{\dagger}$ on it. This gives an irreducible representation, spanned by four states

$$|0\rangle, \quad \hat{b}^{\dagger}|0\rangle, \quad \hat{\bar{b}}^{\dagger}|0\rangle, \quad \hat{\bar{b}}^{\dagger}\hat{b}^{\dagger}|0\rangle.$$

Compute the inner products among these states.

Observe that the inner product is **not positive definite !**

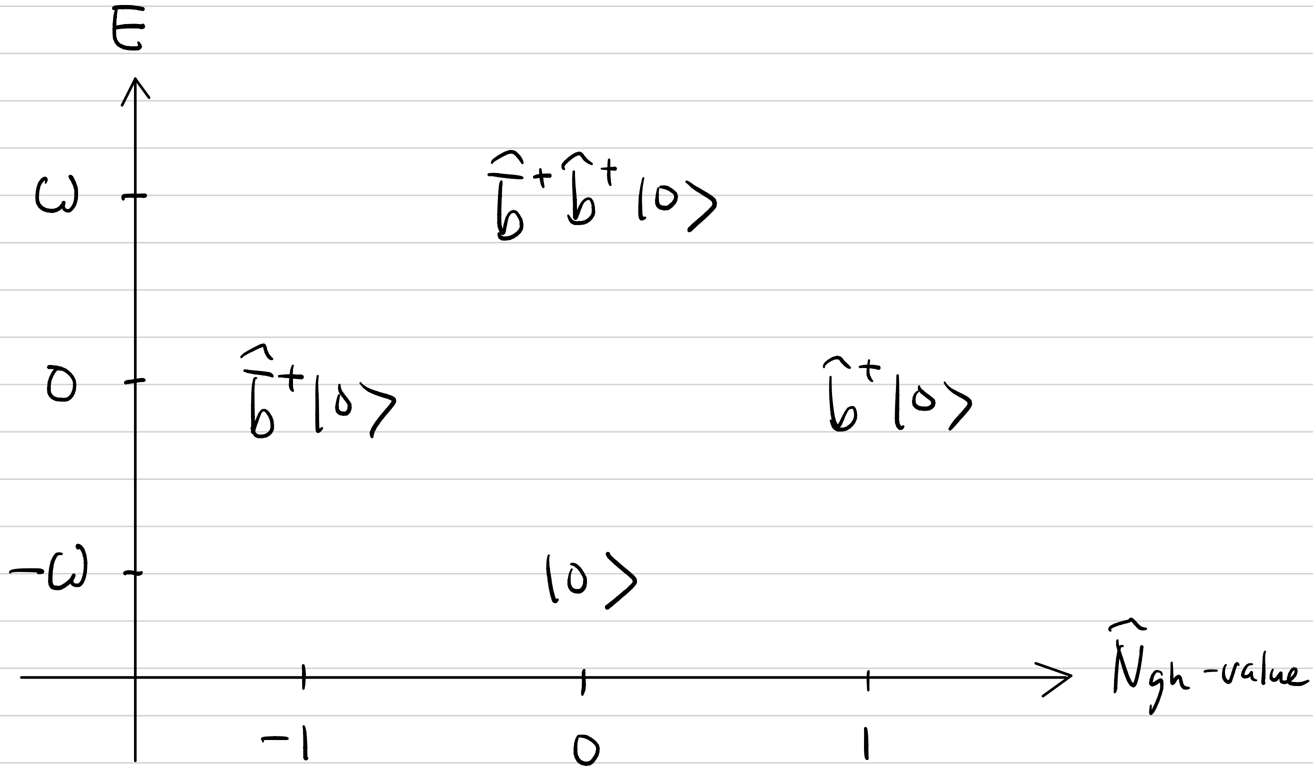
- As the operators corresponding to E and N_{gh}, we may take

$$\hat{H} = i\hat{\dot{\bar{c}}}\hat{\dot{c}} + i\omega^2\hat{\bar{c}}\hat{c} = \omega(\hat{\bar{b}}^\dagger\hat{b} + \hat{b}^\dagger\hat{\bar{b}} - 1)$$

$$\hat{N}_{gh} = \hat{\bar{c}}\hat{\dot{c}} - \hat{\dot{\bar{c}}}\hat{c} = \hat{b}^\dagger\hat{\bar{b}} - \hat{\bar{b}}^\dagger\hat{b}$$

(There is an ordering ambiguity for $\hat{N}_{gh}$ but not for $\hat{H}$.)

The four states above are eigenstates of these:



Note that $\hat{N}_{gh}$ is antihermitian, $\hat{N}_{gh}^\dagger = -\hat{N}_{gh}$, but has real eigenvalues $0, \pm 1$. This is possible because the inner product is not positive definite.

We call this a ghost system as it will appear in modes of "Faddeev-Popov ghost" in gauge theory.

What we have done here will be useful.