

Lagrangian $\xleftrightarrow{\text{Legendre}}$ Hamiltonian

$$S = \int_{t_i}^{t_f} L(q, \dot{q}) dt$$

$$\{A, B\} = \frac{\partial A}{\partial q} \frac{\partial B}{\partial p} - \frac{\partial A}{\partial p} \frac{\partial B}{\partial q}$$

\downarrow extremize fixing $q(t_i)$
 $q(t_f)$

$H(p, q)$ Hamiltonian

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) = \frac{\partial L}{\partial q}$$

$$\frac{d}{dt} A = \{A, H\}$$

\Downarrow

\Downarrow

Path-integral \longleftrightarrow Operator

Transition amplitude
 $(t_i, q_i) \rightarrow (t_f, q_f)$

$$= \int \mathcal{D}q e^{\frac{i}{\hbar} \int_{t_i}^{t_f} L(q, \dot{q}) dt}$$

$$q(t_f) = q_f, \quad q(t_i) = q_i$$

$$[\hat{q}, \hat{p}] = i\hbar \{q, p\} = i\hbar$$

States \leftrightarrow vectors in \mathcal{H}

Time evolution

$$U_{t_f, t_i} = e^{-i \frac{t_f - t_i}{\hbar} \hat{H}}$$

A reminder : Legendre transform

$$\boxed{L \rightarrow H} \quad L(q, \dot{q}) \text{ given}$$

$$\text{Solve } \frac{\partial L}{\partial \dot{q}^i}(q, \dot{q}) = p_i \text{ for } \dot{q} : \dot{q}^i = \dot{q}^i(p, q)$$

$$H(p, q) := \sum_i p_i \dot{q}^i(p, q) - L(q, \dot{q}(p, q))$$

$$\boxed{H \rightarrow L} \quad H(p, q) \text{ given}$$

$$\text{Solve } \frac{\partial H}{\partial p_i}(p, q) = \dot{q}^i \text{ for } p : p_i = p_i(q, \dot{q})$$

$$L(q, \dot{q}) := \sum_i p_i(q, \dot{q}) \dot{q}^i - H(q, p(q, \dot{q}))$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} = \frac{\partial L}{\partial q^i} \quad \begin{array}{c} \boxed{L \rightarrow H} \\ \rightleftarrows \\ \boxed{L \leftarrow H} \end{array} \quad \dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}$$

RHS can also be obtained by extremizing

$$S = \int_{t_i}^{t_f} dt (p \dot{q} - H(p, q)) \text{ fixing } q(t_i) \text{ \& } q(t_f).$$

Operator → Path-integral

transition amplitude in operator formalism:

$$Z(t_f, q_f; t_i, q_i) = \langle q_f | e^{-\frac{i}{\hbar}(t_f - t_i)\hat{H}} | q_i \rangle$$

time



$$t_f - t_i = N\epsilon$$

$$e^{-\frac{i}{\hbar}(t_f - t_i)\hat{H}} = e^{-\frac{i}{\hbar}N\epsilon\hat{H}} = \underbrace{e^{-\frac{i\epsilon}{\hbar}\hat{H}} \dots e^{-\frac{i\epsilon}{\hbar}\hat{H}}}_N$$

$$Z(t_f, q_f; t_i, q_i)$$

$$= \langle q_f | e^{-\frac{i\epsilon}{\hbar}\hat{H}} e^{-\frac{i\epsilon}{\hbar}\hat{H}} \dots e^{-\frac{i\epsilon}{\hbar}\hat{H}} e^{-\frac{i\epsilon}{\hbar}\hat{H}} | q_i \rangle$$
$$\int dq_{N-1} |q_{N-1}\rangle \langle q_{N-1}| \dots \int dq_1 |q_1\rangle \langle q_1|$$

$$= \int \prod_{j=1}^{N-1} dq_j \langle q_f | e^{-\frac{i\epsilon}{\hbar}\hat{H}} | q_{N-1} \rangle \langle q_{N-1} | e^{-\frac{i\epsilon}{\hbar}\hat{H}} | q_{N-2} \rangle \dots$$
$$\dots \langle q_2 | e^{-\frac{i\epsilon}{\hbar}\hat{H}} | q_1 \rangle \langle q_1 | e^{-\frac{i\epsilon}{\hbar}\hat{H}} | q_i \rangle$$

$$\langle q_{j+1} | e^{-\frac{i\epsilon}{\hbar} \hat{H}} | q_j \rangle = \int dp_j \langle q_{j+1} | p_j \rangle \langle p_j | e^{-\frac{i\epsilon}{\hbar} \hat{H}} | q_j \rangle$$

$$\bullet \langle q_{j+1} | p_j \rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{i p_j q_{j+1} / \hbar}$$

$$\bullet \langle p_j | e^{-\frac{i\epsilon}{\hbar} \hat{H}} | q_j \rangle = \langle p_j | (1 - \frac{i\epsilon}{\hbar} \hat{H} + O(\epsilon^2)) | q_j \rangle$$

Suppose $\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{q})$, then

$$\langle p_j | \hat{H} | q_j \rangle = \frac{p_j^2}{2m} + V(q_j) = H(p_j, q_j)$$

$$= \langle p_j | q_j \rangle (1 - \frac{i\epsilon}{\hbar} H(p_j, q_j) + O(\epsilon^2))$$

$$= \frac{1}{\sqrt{2\pi\hbar}} e^{-\frac{i p_j q_j}{\hbar} - \frac{i\epsilon}{\hbar} H(p_j, q_j) + O(\epsilon^2)}$$

$$= \int \frac{dp_j}{2\pi\hbar} e^{\frac{i}{\hbar} p_j (q_{j+1} - q_j) - \frac{i\epsilon}{\hbar} H(p_j, q_j) + O(\epsilon^2)}$$

$$Z(t_f, q_f; t_i, q_i)$$

$$= \int \prod_{j=1}^{N-1} dq_j \prod_{j=0}^{N-1} \frac{dp_j}{2\pi\hbar} e^{\frac{i}{\hbar} \sum_{j=0}^{N-1} \left\{ p_j (q_{j+1} - q_j) - \epsilon H(p_j, q_j) \right\}} + O(N\epsilon^2)$$

$$q_N = q_f, q_0 = q_i$$

$N \rightarrow \infty$ holding

$N\epsilon = t_f - t_i$ fixed

$$\sum_{j=0}^{N-1} \epsilon \left\{ p_j \frac{q_{j+1} - q_j}{\epsilon} - H(p_j, q_j) \right\}$$

$$= \int \mathcal{D}q \mathcal{D}p e^{\frac{i}{\hbar} \int_{t_i}^{t_f} dt (p \dot{q} - H(p, q))}$$

$$q(t_f) = q_f, q(t_0) = q_i$$

Integrate out p : solve $\dot{q} - \frac{\partial H}{\partial p}(p, q) = 0$ for p
and insert the answer.
— Legendre transform

$$= \int \mathcal{D}q e^{\frac{i}{\hbar} \int_{t_i}^{t_f} dt \mathcal{L}(q, \dot{q})}$$

$$q(t_f) = q_f, q(t_0) = q_i$$

More concretely, for $H = \frac{p^2}{2m} + V(q)$,

$$\int \frac{dp_j}{2\pi\hbar} e^{\frac{i}{\hbar} \epsilon \left\{ p_j \frac{q_{j+1} - q_j}{\epsilon} - H(p_j, q_j) \right\}}$$

$$= -\frac{p_j^2}{2m} + p_j \frac{q_{j+1} - q_j}{\epsilon} - V(q_j)$$

$$= -\frac{1}{2m} \left(p_j - m \frac{q_{j+1} - q_j}{\epsilon} \right)^2 + \frac{m}{2} \left(\frac{q_{j+1} - q_j}{\epsilon} \right)^2 - V(q_j)$$

$$= C_\epsilon e^{\frac{i\epsilon}{\hbar} \left(\frac{m}{2} \left(\frac{q_{j+1} - q_j}{\epsilon} \right)^2 - V(q_j) \right)}$$

$$C_\epsilon := \int \frac{dp_j}{2\pi\hbar} e^{-\frac{i\epsilon}{2m\hbar} p_j^2} = \sqrt{\frac{m}{2\pi\hbar i \epsilon}}$$

$$Z(t_f, q_f; t_i, q_i)$$

$$= \int C_\epsilon^N \prod_{j=1}^{N-1} dq_j e^{\frac{i}{\hbar} \sum_{j=0}^{N-1} \epsilon \left(\frac{m}{2} \left(\frac{q_{j+1} - q_j}{\epsilon} \right)^2 - V(q_j) \right)}$$

$$q_N = q_f, q_0 = q_i$$

$$N \rightarrow \infty$$

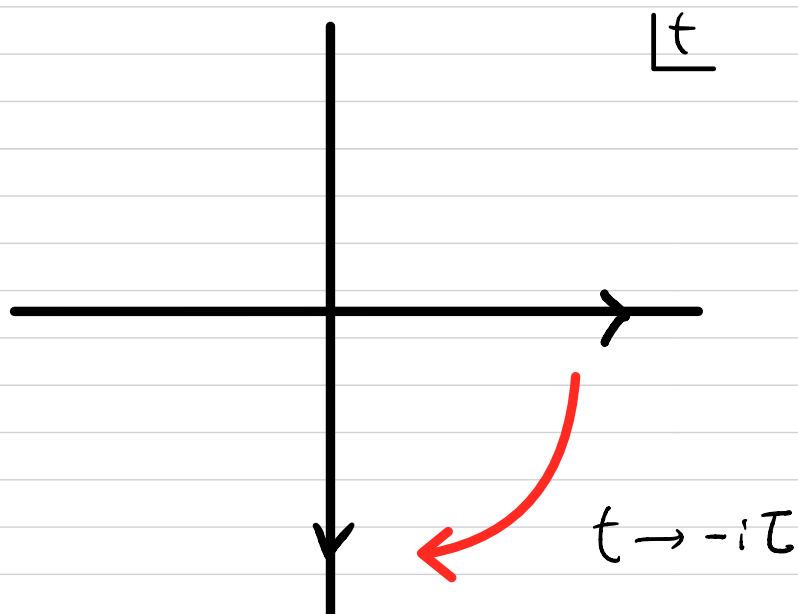
$$N\epsilon = t_f - t_i$$

$$= \int \mathcal{D}q e^{\frac{i}{\hbar} \int_{t_i}^{t_f} dt \left(\frac{m}{2} \dot{q}^2 - V(q) \right)}$$

$$\mathcal{L}(q, \dot{q})$$

$$q(t_f) = q_f, q(t_i) = q_i$$

Wick rotation



In the path-integral, $\epsilon \rightarrow -i\epsilon$ ($\epsilon > 0$).

Oscillatory integral

Absolutely convergent
integral

$$\int \frac{dp_j}{2\pi\hbar} e^{\frac{i}{\hbar} \epsilon \left(-\frac{1}{2m} (p_j + \dots)^2\right)} \rightarrow \int \frac{dp_j}{2\pi\hbar} e^{-\frac{\epsilon}{\hbar} \frac{1}{2m} (p_j + \dots)^2},$$

$$e^{\frac{i\epsilon}{\hbar} \left(\frac{m}{2} \left(\frac{q_{j+1} - q_j}{\epsilon}\right)^2 - V(q_j)\right)}$$

$$\rightarrow e^{\frac{i(-i\epsilon)}{\hbar} \left(\frac{m}{2} \left(\frac{q_{j+1} - q_j}{-i\epsilon}\right)^2 - V(q_j)\right)}$$

$$= e^{-\frac{\epsilon}{\hbar} \left(\frac{m}{2} \left(\frac{q_{j+1} - q_j}{\epsilon}\right)^2 + V(q_j)\right)}.$$

$$e^{\frac{i}{\hbar} S[q]} = e^{\frac{i}{\hbar} \int_{t_i}^{t_f} dt \left(\frac{m}{2} \dot{q}^2 - V(q) \right)}$$

$$\rightarrow e^{-\frac{1}{\hbar} \int_{\tau_i}^{\tau_f} d\tau \underbrace{\left(\frac{m}{2} \left(\frac{dq}{d\tau} \right)^2 + V(q) \right)}_{L_E(q, \frac{dq}{d\tau})} = e^{-\frac{1}{\hbar} S_E[q]}$$

Euclidean Lagrangian/action

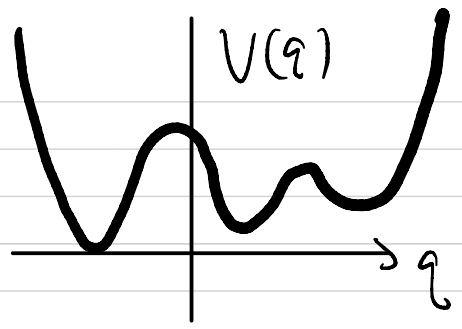
In general, $L_E(q, \frac{dq}{d\tau}) = -L(q, i \frac{dq}{d\tau})$.

$$\langle q_f | e^{-i \frac{t_f - t_i}{\hbar} \hat{H}} | q_i \rangle = \int_{q(t_f)=q_f, q(t_i)=q_i} \mathcal{D}q e^{-\frac{i}{\hbar} \int_{t_i}^{t_f} dt L(q, \dot{q})}$$



$$\langle q_f | e^{-\frac{\tau_f - \tau_i}{\hbar} \hat{H}} | q_i \rangle = \int_{q(\tau_f)=q_f, q(\tau_i)=q_i} \mathcal{D}q e^{-\frac{1}{\hbar} \int_{\tau_i}^{\tau_f} d\tau L_E(q, \frac{dq}{d\tau})}$$

If $V(q) \rightarrow +\infty$ as $|q| \rightarrow \infty$



then $S_E[q] \rightarrow \infty$ at ∞ of $\{q(\tau)\}$.

The path-integral $\int \mathcal{D}q e^{-\frac{1}{\hbar} S_E[q]}$

is **well-behaved**.

Partition function

$$\text{Tr}_{\mathcal{H}} \left(e^{-\frac{T}{\hbar} \hat{H}} \right) = \int dq \langle q | e^{-\frac{T}{\hbar} \hat{H}} | q \rangle$$

$$= \int dq \int_{\substack{\mathcal{D}q \\ q(\tau) = q = q(0)}} e^{-\frac{1}{\hbar} \int_0^T d\tau L_E \left(q, \frac{dq}{d\tau} \right)}$$

$$= \int_{\substack{\mathcal{D}q \\ q(\tau) = q(0)}} e^{-\frac{1}{\hbar} \int_0^T d\tau L_E \left(q, \frac{dq}{d\tau} \right)}$$

$q(\tau) = q(0) \Leftrightarrow q(\tau)$ is periodic under $\tau \rightarrow \tau + T$.

Partition function $\text{Tr} e^{-\frac{T}{\hbar} \hat{H}}$

= Euclidean path-integral over configurations on the circle $S_T^1 = \mathbb{R}/T\mathbb{Z}$ of circumference T

$$Z(S_T^1) = \int \mathcal{D}q e^{-\frac{1}{\hbar} \int_{S_T^1} d\tau L_E(q, \frac{dq}{d\tau})}$$

Note: This is **well-behaved** if the energy spectrum

$\{E_n\}_{n=0}^{\infty}$ (ie, eigenvalues of \hat{H})

is bounded below and $E_n \rightarrow \infty$ as $n \rightarrow \infty$

fast enough.

Symmetry and Ward identity

Consider a "QFT" with fields $\phi = (\phi_1, \dots, \phi_n)$,

measure $d^n\phi = d\phi_1 \dots d\phi_n$

& action $S_E(\phi) = S_E(\phi_1, \dots, \phi_n)$

Focus of interest :

$$Z = \int d^n\phi e^{-S_E(\phi)} \quad \text{Partition function}$$

$$\langle f \rangle = \frac{1}{Z} \int d^n\phi e^{-S_E(\phi)} f(\phi) \quad \text{Correlation function}$$

A symmetry of the theory is a transformation

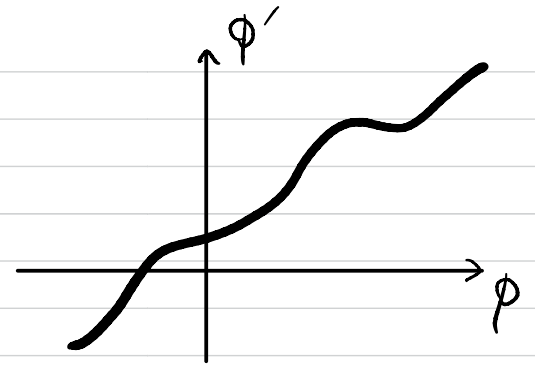
$$\phi = (\phi_1, \dots, \phi_n) \mapsto g(\phi) = (g_1(\phi), \dots, g_n(\phi))$$

that leaves $d^n\phi e^{-S_E(\phi)}$ invariant.

$$\text{i.e.} \quad \det\left(\frac{\partial g_i(\phi)}{\partial \phi_j}\right) e^{-S_E(g(\phi))} = e^{-S_E(\phi)}$$

Change of integration variables :

Single variable case: $\phi \rightsquigarrow \phi' = g(\phi)$



$$\int_{-\infty}^{\infty} d\phi F(\phi) = \int_{-\infty}^{\infty} d\phi' F(\phi') = \int_{-\infty}^{\infty} d\phi g'(\phi) F(g(\phi)).$$

Likewise

$$\int d^n\phi e^{-S_E(\phi)} f(\phi) = \int d^n g(\phi) e^{-S_E(g(\phi))} f(g(\phi))$$

|| \leftarrow if g is a symmetry.

$$d^n\phi e^{-S_E(\phi)}$$

\therefore If g is a symmetry, correlation functions satisfy

$$\langle f \rangle = \langle f \circ g \rangle$$

Ward identity

Infinitesimal form :

$\{ g_\alpha \}_{\alpha \in \mathbb{R}}$: 1-parameter group of transformations.

$$\phi \mapsto \phi + \delta\phi \quad ; \quad \delta\phi = \left. \frac{d}{d\alpha} g_\alpha(\phi) \right|_{\alpha=0}.$$

-- infinitesimal transformation.

If $\{ g_\alpha \}_{\alpha \in \mathbb{R}}$ is a 1-parameter group of symmetries,

Ward identity : $\langle f \rangle = \langle f \circ g_\alpha \rangle \quad \forall \alpha$

$$\Rightarrow \left. \frac{d}{d\alpha} \right|_{\alpha=0} :$$

$$0 = \langle \delta f \rangle$$

(infinitesimal form of)

Ward identity

where $\delta f(\phi) := \left. \frac{d}{d\alpha} f(g_\alpha(\phi)) \right|_{\alpha=0}$

There are Ward identities even for non-symmetries:

$$\int d^n \phi e^{-S_E(\phi)} f(\phi) = \int d^n g_\alpha(\phi) e^{-S_E(g_\alpha(\phi))} f(g_\alpha(\phi))$$

① Suppose $d^n \phi$ is invariant but $S_E(\phi)$ is not.

$$\rightarrow 0 = \int d^n \phi e^{-S_E(\phi)} \left(-\delta S_E(\phi) f(\phi) + \delta f(\phi) \right)$$

$$\langle \delta f \rangle = \langle \delta S_E \cdot f \rangle$$

② Suppose $S_E(\phi)$ is invariant but $d^n \phi$ is not,

and the change is known: $d^n g_\alpha(\phi) = d^n \phi e^{\alpha Q(\phi)}$

(called anomalous symmetry with anomaly a)

$$\rightarrow 0 = \int d^n \phi e^{-S_E(\phi)} \left(a(\phi) \cdot f(\phi) + \delta f(\phi) \right)$$

$$\langle \delta f \rangle = -\langle a \cdot f \rangle$$

anomalous
Ward identity

Path-integral \rightarrow Operator

$$Z(t_f, q_f; t_i, q_i) = \int \mathcal{D}q \, e^{\frac{i}{\hbar} \int_{t_i}^{t_f} dt L(q, \dot{q})}$$
$$q(t_f) = q_f, \quad q(t_i) = q_i$$

\leadsto An operator \hat{Z}_{t_f, t_i} on the space \mathcal{H} of functions on q :

$$\Psi(q) \mapsto (\hat{Z}_{t_f, t_i} \Psi)(q) = \int dq' Z(t_f, q; t_i, q') \Psi(q').$$

- \hat{Z}_{t_f, t_i} depends only on $t_f - t_i$, so can be written as

$$\hat{Z}_{t_f, t_i} = \hat{Z}_{t_f - t_i} \quad t_3 > t_2 > t_1$$

- $\int dq_2 Z(t_3, q_3; t_2, q_2) Z(t_2, q_2; t_1, q_1) = Z(t_3, q_3; t_1, q_1)$.

$$\therefore \hat{Z}_{t_3 - t_2} \circ \hat{Z}_{t_2 - t_1} = \hat{Z}_{t_3 - t_1}$$

\leadsto One can write $\hat{Z}_T = e^{-\frac{i}{\hbar} T \hat{H}}$

for some operator \hat{H} on \mathcal{H}

\uparrow

At this moment, we don't know the relation to Hamiltonian. (We'll see it later.)

Let \mathcal{O} be an expression of $q, \dot{q}, \ddot{q}, \dots$

e.g. $\mathcal{O} = q^3, \dot{q}^2, q\dot{q}, \dot{q}^2 + q^2, \dots$ "local observable"

For $t_i < t < t_f$, write

$$Z(t_f, q_f; \mathcal{O}(t); t_i, q_i) = \int_{q(t_i)=q_i, q(t_f)=q_f} \mathcal{D}q e^{\frac{i}{\hbar} \int_{t_i}^{t_f} dt L(q, \dot{q})} \mathcal{O}(t)$$

\rightsquigarrow an operator on \mathcal{H} :

$$\Psi(q) \mapsto \left(\hat{Z}_{t_f, t_i}(\mathcal{O}(t)) \Psi \right)(q) = \int dq' Z(t_f, q; \mathcal{O}(t); t_i, q') \Psi(q')$$

$$\bullet \hat{Z}_{t_f + \Delta t_f, t_i - \Delta t_i}(\mathcal{O}(t)) = \hat{Z}_{\Delta t_f} \circ \hat{Z}_{t_f, t_i}(\mathcal{O}(t)) \circ \hat{Z}_{\Delta t_i}$$

$$\bullet \hat{Z}_{t_f + \Delta t, t_i + \Delta t}(\mathcal{O}(t + \Delta t)) = \hat{Z}_{t_f, t_i}(\mathcal{O}(t))$$

Define $\hat{\mathcal{O}} := \lim_{\substack{t_f \rightarrow t \\ t_i \rightarrow t}} \hat{Z}_{t_f, t_i}(\mathcal{O}(t))$... indep of t .

Then,

$$\hat{Z}_{t_f, t_i}(\mathcal{O}(t)) = \hat{Z}_{t_f, t} \circ \hat{\mathcal{O}} \circ \hat{Z}_{t, t_i} = e^{-\frac{i}{\hbar}(t_f - t)\hat{H}} \circ \hat{\mathcal{O}} \circ e^{-\frac{i}{\hbar}(t - t_i)\hat{H}}$$

For $t_1 < t_2 < t_f$ and local observables O_1 & O_2 ,

$$Z(t_f, q_f; O_1(t_1) O_2(t_2); t_i, q_i) = \int_{q(t_f)=q_f, q(t_i)=q_i} \mathcal{D}q e^{\frac{i}{\hbar} \int_{t_i}^{t_f} dt L(q, \dot{q})} O_1(t_1) O_2(t_2)$$

\leadsto Operator $\hat{Z}_{t_f, t_i}(O_1(t_1) O_2(t_2))$ on \mathcal{H}

is defined in the similar way.

$$Z(t_f, q_f; O_1(t_1) O_2(t_2); t_i, q_i)$$

$$= \begin{cases} \int dq' Z(t_f, q_f; O_2(t_2); t', q') Z(t', q'; O_1(t_1); t_i, q_i) \\ \text{if } t_2 > t_1 \text{ (for any } t' \in (t_1, t_2) \text{)} \\ \\ \int dq' Z(t_f, q_f; O_1(t_1); t', q') Z(t', q'; O_2(t_2); t_i, q_i) \\ \text{if } t_1 > t_2 \text{ (for any } t' \in (t_2, t_1) \text{)} \end{cases}$$

$$\hat{\Sigma}_{t_f, t_i}(O_1(t_1)O_2(t_2))$$

$$= \begin{cases} e^{-\frac{i}{\hbar}(t_f-t_2)\hat{H}} \hat{O}_2 e^{-\frac{i}{\hbar}(t_2-t_1)\hat{H}} \hat{O}_1 e^{-\frac{i}{\hbar}(t_1-t_i)\hat{H}} & \text{if } t_2 > t_1, \\ e^{-\frac{i}{\hbar}(t_f-t_1)\hat{H}} \hat{O}_1 e^{-\frac{i}{\hbar}(t_1-t_2)\hat{H}} \hat{O}_2 e^{-\frac{i}{\hbar}(t_2-t_i)\hat{H}} & \text{if } t_1 > t_2. \end{cases}$$

Correlation function of product of local observables corresponds to the time ordered product of the corresponding operators.

Symmetry in classical mechanics (in Lagrangian)

Suppose \exists a symmetry $q \mapsto q + \delta q$ ($\delta q = \epsilon u(q, \dot{q})$)

$$\delta L(q, \dot{q}) = \epsilon \frac{d}{dt} (\dots) \quad \text{total derivative}$$

Allow variational parameter ϵ to depend on time, $\epsilon(t)$,
s.t. $\epsilon(t_f) = \epsilon(t_i) = 0$:

$$\delta S = \delta \int_{t_i}^{t_f} dt L(q, \dot{q}) = \int_{t_i}^{t_f} dt \dot{\epsilon}(t) \underbrace{Q(q, \dot{q})}$$

This $Q = Q(q, \dot{q})$ is called the Noether charge.

Noether's theorem Q is conserved. I.e. it is

time-independent for a solution to equation of motion.

proof A solution is s.t. $\delta S = 0$ for $\forall \delta q$ s.t. $\delta q|_{t_f, t_i} = 0$.

For $\forall \epsilon(t)$ s.t. $\epsilon(t_f) = \epsilon(t_i) = 0$, under $q \rightarrow q + \epsilon(t) u(q, \dot{q})$,

$$0 = \delta S = \int_{t_i}^{t_f} dt \dot{\epsilon}(t) Q = - \int_{t_i}^{t_f} dt \epsilon(t) \frac{dQ}{dt}$$

$$\therefore \frac{dQ}{dt} = 0 \quad \underline{\text{Q.E.D.}}$$

A preview of the unfinished part:

① $\widehat{\delta U} = i[\widehat{Q}, \widehat{U}]$ Operator form of
Ward identity

“Noether charge generates the symmetry transformation.”

② $\widehat{H} = \widehat{H(p, q)} + \text{C-number}$
 \widehat{H} indeed corresponds to Hamiltonian.

③ $[\widehat{q}, \widehat{p}] = i\hbar$

Can be derived as a consequence of
Ward identity (of not necessarily symmetry).