

Regularization and Renormalization

Divergences

$$\mathcal{L}_E = \frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 \quad \text{in } d=4.$$

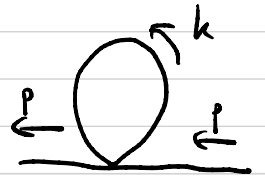
$$\langle \phi(x_1)\phi(x_2) \rangle_{\text{PI}} = \text{loop} + \text{tadpole} + \text{self-energy} + \dots$$

$$\text{loop} = -\frac{\lambda}{2} \int d^4y \overbrace{\phi(x_1)\phi(y)} \overbrace{\phi(y)\phi(y)} \overbrace{\phi(y)\phi(x_2)}$$

$$= \frac{(\lambda\pi)^4 \delta^4(p_1-p_2)}{(2\pi)^4} \int \frac{d^4p_1}{(2\pi)^4} \frac{e^{-ip_1(x_1-y)}}{p_1^2+m^2} \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik(y-y)}}{k^2+m^2} \int \frac{d^4p_2}{(2\pi)^4} \frac{e^{-ip_2(y-x_2)}}{p_2^2+m^2}$$

$$= \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ipx_1}}{p^2+m^2} \left(-\frac{\lambda}{2} \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2+m^2} \right) \frac{e^{ipx_2}}{p^2+m^2}$$

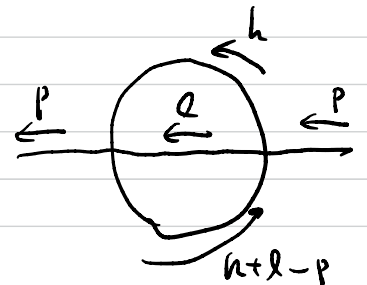
quadratically divergent



$$\text{tadpole} = \frac{\lambda^2}{6} \int d^4y_1 d^4y_2 \overbrace{\phi(x_1)\phi(y_1)} \overbrace{\phi(y_1)\phi(y_2)}^3 \overbrace{\phi(y_2)\phi(x_2)}$$

$$= \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ipx_1}}{p^2+m^2} \left(\frac{\lambda^2}{6} \int \frac{d^4k}{(2\pi)^4} \frac{d^4l}{(2\pi)^4} \frac{1}{k^2+m^2} \frac{1}{l^2+m^2} \frac{1}{(k+l-p)^2+m^2} \right) \frac{e^{ipx_2}}{p^2+m^2}$$

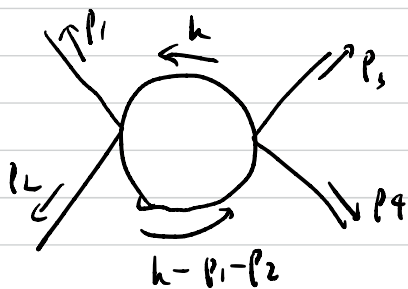
quadratically divergent



$$\langle \phi(x_1) \phi(x_2) \phi(x_3) \phi(x_4) \rangle_{1PI} = \begin{array}{c} 1 \quad 3 \\ \diagdown \quad \diagup \\ 2 \quad 4 \end{array} + \begin{array}{c} 1 \quad 3 \\ \diagdown \quad \diagup \\ \text{circle} \\ \diagup \quad \diagdown \\ 2 \quad 4 \end{array} + \begin{array}{c} 1 \quad 3 \\ \diagdown \quad \diagup \\ \text{circle} \\ \diagup \quad \diagdown \\ 2 \quad 4 \end{array} + \begin{array}{c} 1 \quad 3 \\ \diagdown \quad \diagup \\ \text{circle} \\ \diagup \quad \diagdown \\ 2 \quad 2 \end{array} + \dots$$

$$= \int \prod_{a=1}^4 \frac{d^4 p_a}{(2\pi)^4} \frac{e^{-i p_a x_a}}{p_a^2 + m^2} (2\pi)^4 \delta^4(p_1 + p_2 + p_3 + p_4) \times \left\{ -\lambda \right.$$

$$\left. + \frac{\lambda^2}{2} \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2 + m^2} \frac{1}{(k - p_1 - p_2)^2 + m^2} + (2 \leftrightarrow 3) + (2 \leftrightarrow 4) + \dots \right\}$$



logarithmically divergent

The integral over the loop momenta k 's can be divergent at $|k| \rightarrow \infty$

..... ultra-violet (= short distance) divergence

superficial degree of divergence D

\equiv power of momenta k of the integral

$$= (\text{power in numerator}) - (\text{power in denominator})$$

\uparrow \uparrow
e.g. from $d^d k$, vertex... from propagator

$E = \#$ external lines, $I = \#$ internal lines, $V = \#$ vertices,

$$L = \# \text{ loops} = I - V + 1 = \text{net } \# \text{ of momentum integrals}$$

Lecture 10


Theory of scalar ϕ in d -dimensions : $D = \underbrace{dL}_{d^d k} - \underbrace{2I}_{\frac{1}{k^2 + m^2}}$


If $\mathcal{L}_{int} \propto \phi^4$: $2I + E = 4V$

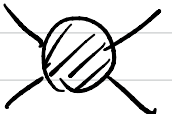
$$D = d(I - V + 1) - 2I$$


$$\stackrel{\phi^4}{=} d(-V + 1) + (d - 2) \frac{4V - E}{2} \stackrel{d=4}{=} 4 - E$$

4d ϕ^4 theory

$E = 0$  : $D = 4$ quartic divergence

$E = 2$  : $D = 2$ quadratic divergence

$E = 4$  : $D = 0$ logarithmic divergence

$E \geq 6$  : $D < 0$ (superficially) convergent

For $E=0, 2, 4$, the divergence occurs for any number V of vertices,
i.e. at all orders in perturbative expansion

$$\phi^4 \text{ theory in other } d : D = d + (d-4)V - \frac{d-2}{2} E$$

$d < 4$ $D < 0$ for large enough V .

Only a finite number of Feynman diagrams are
(superficially) divergent.

$d > 4$ For each E , $D > 0$ for large enough V .

Any correlator is (superficially) divergent
at sufficiently high orders in perturbative expansion.

How do we deal with such divergences?

— At least, we need a

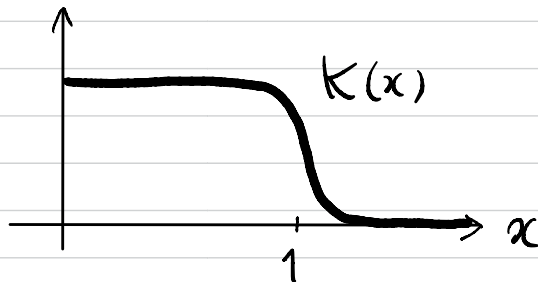
regularization:

a systematic change of the theory
so that the loop integrals are all finite.

Regularizations

① Change of propagator $\frac{1}{p^2+m^2} \rightsquigarrow \frac{K(p^2/\Lambda^2)}{p^2+m^2}$

$$K(x) = \begin{cases} 1 & x \ll 1 \\ 0 & x \gg 1 \end{cases}$$



The propagator remains the same as the original at low $|p|$ compared to Λ , but is significantly modified at $|p| \gtrsim \Lambda$.

Λ : ultra-violet cut-off (UV cut-off)

e.g. $\frac{1}{p^2+m^2} \rightarrow \begin{cases} \frac{1}{p^2+m^2} & p^2 < \Lambda^2 \\ 0 & p^2 > \Lambda^2 \end{cases}$ sharp cut-off

e.g. $\frac{1}{p^2+m^2} = \int_0^\infty d\alpha e^{-\alpha(p^2+m^2)}$
 $\rightarrow \int_{1/\Lambda^2}^\infty d\alpha e^{-\alpha(p^2+m^2)} = \frac{e^{-\frac{p^2+m^2}{\Lambda^2}}}{p^2+m^2}$

\leftrightarrow change of Lagrangian:

$$\mathcal{L}_{E,\Lambda} = \frac{1}{2} \phi (-\partial^2+m^2) \underbrace{e^{-\frac{\partial^2+m^2}{\Lambda^2}}}_{K(-\partial^2/\Lambda^2)} \phi + \frac{\lambda}{4!} \phi^4$$

$K(-\partial^2/\Lambda^2)^{-1}$ more generally

①' Pauli-Villars regularization (⊂ ①)

$$\frac{1}{p^2+m^2} \rightarrow \frac{1}{p^2+m^2} - \frac{1}{p^2+\Lambda^2} = \frac{\Lambda^2-m^2}{(p^2+m^2)(p^2+\Lambda^2)}, \text{ or}$$

$$\frac{1}{p^2+m^2} \rightarrow \frac{1}{p^2+m^2} - \frac{\alpha_1}{p^2+\Lambda_1^2} - \frac{\alpha_2}{p^2+\Lambda_2^2} - \dots = \frac{\text{Const}}{(p^2)^N + \text{lower}}$$

One can choose $\Lambda_1, \alpha_1, \Lambda_2, \alpha_2, \dots$ to make the power $2N$ of denominator as large as possible.

↔ introduce new field variables Φ_1, Φ_2, \dots (regulators) and consider the system with Lagrangian

$$\mathcal{L}_{E, \text{reg}} = \frac{1}{2}(\partial\Phi)^2 + \frac{m^2}{2}\Phi^2 + \sum_{i=1,2,\dots} \left[\frac{1}{2}(\partial\phi_i)^2 + \frac{\Lambda_i^2}{2}\phi_i^2 \right] \text{ Free part}$$

$$+ \frac{\lambda}{4!} \left(\Phi + \sum_{i=1,2,\dots} \sqrt{-\alpha_i} \phi_i \right)^4 \quad \text{interaction}$$

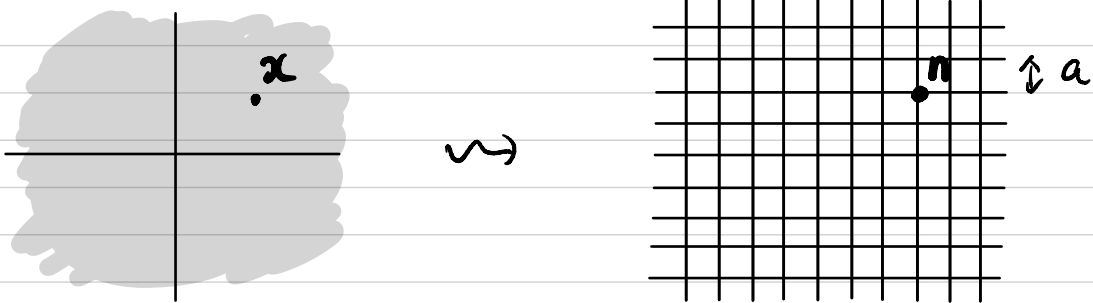
The internal propagators are only for $\Phi = \Phi + \sum_i \sqrt{-\alpha_i} \phi_i$:

$$\overline{\Phi(x)\Phi(y)} = \overline{\Phi(x)\Phi(y)} + \sum_i (-\alpha_i) \overline{\phi_i(x)\phi_i(y)}$$

$$= \int \frac{d^d k}{(2\pi)^d} e^{-ik(x-y)} \left(\frac{1}{k^2+m^2} - \sum_i \frac{\alpha_i}{k^2+\Lambda_i^2} \right)$$

② Lattice

$$x \in \mathbb{R}^d \mapsto \phi(x) \rightsquigarrow n \in \mathbb{Z}^d \mapsto \phi_n$$



$$S_{E, \text{reg}} = \sum_n a^d \left(\frac{1}{2} \sum_\mu \left(\frac{\phi_{n+\mu} - \phi_n}{a} \right)^2 + \frac{m^2}{2} \phi_n^2 + \frac{\lambda}{4!} \phi_n^4 \right)$$

Advantage: momentum integral is over compact space

$$\phi_n = \int_0^{2\pi/a} \dots \int_0^{2\pi/a} \frac{d^d p}{(2\pi)^d} e^{-ipna} \phi(p)$$

$$\overbrace{\phi_n \phi_{n'}} = \int_0^{2\pi/a} \dots \int_0^{2\pi/a} \frac{d^d p}{(2\pi)^d} \frac{e^{-ip(na - n'a)}}{\sum_\mu \left(\frac{e^{-i\mu na} - 1}{a} \right) \left(\frac{e^{i\mu n'a} - 1}{a} \right) + m^2}$$

③ Dimensional regularization

dimension $d \in \mathbb{Z}$ say 4 $\rightsquigarrow d \in \mathbb{C}$

$$\int_{\mathbb{R}^d} \frac{d^d k}{(2\pi)^d} f(k^2) \rightsquigarrow M_{\text{DR}}^{4-d} \int_{\mathbb{R}^d} \frac{d^d k}{(2\pi)^d} f(k^2)$$

M_{DR} : a parameter of mass dimension 1

$$= M_{\text{DR}}^{4-d} \frac{\text{Vol}(S^{d-1})}{(2\pi)^d} \int_0^\infty k^{d-1} f(k^2) dk^2 = \frac{1}{2} \int_0^\infty (k^2)^{\frac{d}{2}-1} dk^2 f(k^2)$$

$$\left(\int_{\mathbb{R}} \frac{dx}{\sqrt{\pi}} e^{-x^2} \right)^d = \int_{\mathbb{R}^d} \frac{d^d x}{(2\pi)^d} e^{-\|x\|^2} = \frac{\text{Vol}(S^{d-1})}{(2\pi)^d} \int_0^\infty r^{d-1} dr e^{-r^2}$$

$$\left(\frac{1}{\sqrt{\pi}} \sqrt{\pi} \right)^d = \frac{1}{(4\pi)^{d/2}}$$

$$\frac{1}{2} \int_0^\infty (r^2)^{\frac{d}{2}-1} dr^2 e^{-r^2} = \frac{1}{2} \Gamma\left(\frac{d}{2}\right)$$

$$\therefore \frac{\text{Vol}(S^{d-1})}{2(2\pi)^d} = \frac{1}{(4\pi)^{d/2} \Gamma(d/2)}$$

$$= \frac{M_{\text{DR}}^{4-d}}{(4\pi)^{d/2} \Gamma(d/2)} \int_0^\infty (k^2)^{\frac{d}{2}-1} dk^2 f(k^2)$$

This makes sense also for $d \in \mathbb{C}$

e.g. $I = \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2 + m^2}$ & $V = \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2 + m^2} \frac{1}{(k-p)^2 + m^2}$

Via ① $\frac{1}{p^2 + m^2} \sim \int_{\frac{1}{\Lambda^2}}^{\infty} d\alpha e^{-\alpha(p^2 + m^2)}$ & ③ dim reg: $4 \rightarrow d = 4 - \epsilon$

$$I_{\text{①}} = \frac{1}{(4\pi)^2} \left[\Lambda^2 - m^2 \left(\log \left(\frac{\Lambda^2}{m^2} \right) + 1 - \gamma \right) + m^2 O\left(\frac{m^2}{\Lambda^2}\right) \right]$$

$\gamma := \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \frac{1}{k} - \log n \right) = 0.57721 \dots$ Euler's constant

$$I_{\text{③}} = \frac{M_{\text{DR}} m^{d-2}}{(4\pi)^{d/2}} \Gamma\left(1 - \frac{d}{2}\right) \dots \text{divergent for } d=4, \text{ but for } d=4-\epsilon:$$

$$= -\frac{m^2}{(4\pi)^2} \left[\frac{2}{\epsilon} + \log\left(\frac{4\pi M_{\text{DR}}^2}{m^2}\right) + 1 - \gamma + O(\epsilon) \right]$$

$$V_{\text{①}} = \frac{1}{(4\pi)^2} \left[\log\left(\frac{\Lambda^2}{2m^2}\right) - \gamma - 1 - \int_0^1 dx \log\left(1 + x(1-x)\frac{p^2}{m^2}\right) + O\left(\frac{m^2}{\Lambda^2}, \frac{p^2}{\Lambda^2}\right) \right]$$

$$V_{\text{③}} = \frac{M_{\text{DR}}^{4-d} \Gamma\left(2 - \frac{d}{2}\right)}{(4\pi)^{d/2}} \int_0^1 dx \left(x(1-x)p^2 + m^2\right)^{\frac{d}{2} - 2}$$

... divergent for $d=4$, but for $d=4-\epsilon$:

$$= \frac{1}{(4\pi)^2} \left[\frac{2}{\epsilon} + \log\left(\frac{4\pi M_{\text{DR}}^2}{m^2}\right) - \gamma - \int_0^1 dx \log\left(1 + x(1-x)\frac{p^2}{m^2}\right) + O(\epsilon) \right]$$

⑦ Exercise.

Renormalization

After regularization, we let the couplings to depend on the cut-off (Λ in ①, a in ②, (ϵ, μ_{DR}) in ③)

so that the correlation function of properly normalized fields are finite, as we remove the cut-off ($\Lambda \rightarrow \infty$; $a \rightarrow 0$; $\epsilon \rightarrow 0$).

$$S_\Lambda = \left[\int d^4x \left(\frac{1}{2} (\partial \phi_0)^2 + \frac{m_0(\Lambda)^2}{2} \phi_0^2 + \frac{\lambda_0(\Lambda)}{4!} \phi_0^4 \right) \right]_\Lambda$$

regularization ↙
↘ cutoff

$$\phi_0 = \sqrt{Z_0(\Lambda)} \phi$$
$$= \left[\int d^4x \left(\frac{1}{2} Z_0(\Lambda) (\partial \phi)^2 + \frac{m_0(\Lambda)^2}{2} Z_0(\Lambda) \phi^2 + \frac{\lambda_0(\Lambda)}{4!} Z_0(\Lambda)^2 \phi^4 \right) \right]_\Lambda$$

Choose $Z_0(\Lambda)$, $m_0(\Lambda)$, $\lambda_0(\Lambda)$ so that

$\langle \phi(x_1) \dots \phi(x_n) \rangle$ are all finite as Λ is removed

We do this order by order in perturbation theory.

$$Z_0(\Lambda) = 1 + \lambda a_1(\Lambda) + \lambda^2 a_2(\Lambda) + \dots$$

$$Z_0(\Lambda) m_0(\Lambda)^2 = m^2 + \lambda b_1(\Lambda) + \lambda^2 b_2(\Lambda) + \dots$$

$$Z_0(\Lambda)^2 \lambda_0(\Lambda) = \lambda + \lambda^2 c_1(\Lambda) + \lambda^3 c_2(\Lambda) + \dots$$

$$\mathcal{L} = \mathcal{L}_0 + \underbrace{\mathcal{L}_1 + \mathcal{L}_2 + \dots}_{\text{counter terms}}$$

$$\mathcal{L}_0 = \frac{1}{2} (\partial\phi)^2 + \frac{m^2}{2} \phi^2 + \frac{\lambda}{4!} \phi^4$$

$$\mathcal{L}_1 = \frac{1}{2} \lambda a_1(\Lambda) (\partial\phi)^2 + \frac{1}{2} \lambda b_1(\Lambda) \phi^2 + \frac{\lambda^2}{4!} c_1(\Lambda) \phi^4$$

$$\mathcal{L}_2 = \frac{1}{2} \lambda^2 a_2(\Lambda) (\partial\phi)^2 + \frac{1}{2} \lambda^2 b_2(\Lambda) \phi^2 + \frac{\lambda^3}{4!} c_2(\Lambda) \phi^4$$

⋮

Do perturbation theory with

$$\mathcal{L}_{\text{free}} = \frac{1}{2} (\partial\phi)^2 + \frac{m^2}{2} \phi^2 ; \mathcal{L}_{\text{int}} = \frac{\lambda}{4!} \phi^4 + \mathcal{L}_1 + \mathcal{L}_2 + \dots$$

$\mathcal{L}_0 \leftrightarrow$ tree

Find $a_n(\Lambda)$, $b_n(\Lambda)$, $c_n(\Lambda)$ recursively

$\mathcal{L}_1 \leftrightarrow$ 1-loop

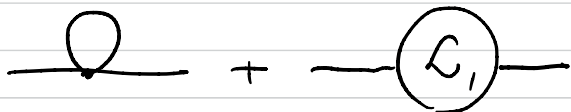
so that the correlation functions of ϕ 's

$\mathcal{L}_2 \leftrightarrow$ 2-loop

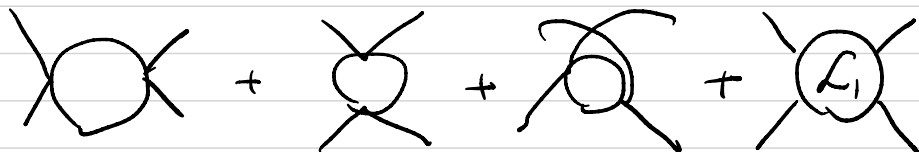
are finite at each order.

⋮

② 1-loop



$$= -\frac{\lambda}{2} \frac{1}{(4\pi)^2} \left[\Lambda^2 - m^2 \log\left(\frac{\Lambda^2}{m^2}\right) + \text{finite} \right] - \lambda a_1(\Lambda) p^2 - \lambda b_1(\Lambda)$$



$$\frac{\lambda}{2} \frac{1}{(4\pi)^2} \log\left(\frac{\Lambda^2}{2m^2}\right) \times 3 + \text{finite} - \lambda^2 C_1(\Lambda)$$

Can these be made finite?

Yes,

$$a_1(\Lambda) = \text{finite}$$

$$b_1(\Lambda) = -\frac{1}{2(4\pi)^2} \left(\Lambda^2 - m^2 \log\left(\frac{\Lambda^2}{m^2}\right) \right) + \text{finite}$$

$$C_1(\Lambda) = \frac{3}{(4\pi)^2} \log\left(\frac{\Lambda^2}{2m^2}\right) + \text{finite}$$

will do the job!

Claim For each $n \geq 1$, it is possible to find

$a_n(\Lambda), b_n(\Lambda), c_n(\Lambda)$ so that $L \leq n$ loop contributions to all the correlation functions of ϕ are finite.

Such a theory is said to be renormalizable.

$\phi_0 / m_0(\Lambda) / \lambda_0(\Lambda)$: bare field / mass / coupling

$\phi / m / \lambda$: renormalized field / mass / coupling

Claim A theory is renormalizable when the superficial degree of divergence D is ≥ 0 only for a finite number of correlation functions.

Eg. ϕ^4 theory

$d \leq 4$: Yes \Rightarrow renormalizable

$\left(\begin{array}{l} d < 4 : \text{No divergence at high enough loops} \\ \Rightarrow \text{superrenormalizable} \end{array} \right)$

$d > 4$: No \Rightarrow not renormalizable.

Criterion: mass dimension of couplings

$$S = \int d^d x \mathcal{L} = \int d^d x \left(\frac{1}{2} (\partial\phi)^2 + \frac{m^2}{2} \phi^2 + \frac{\lambda}{4!} \phi^4 \right)$$

Mass-dimension of $S = 0$ so that e^{-S} makes sense.

$$[S] = 0. \quad [d^d x] = -d \quad \therefore [\mathcal{L}] = d.$$

$$[\partial_\mu] = 1 \quad \Rightarrow \quad [\phi] = \frac{d-2}{2}$$

$$[m^2] = 2$$

$$[\lambda] = d - 4 \left(\frac{d-2}{2} \right) = 4-d.$$

The theory is

$$\text{renormalizable} \Leftrightarrow [\text{coupling}] \geq 0$$

$$\text{superrenormalizable} \Leftrightarrow [\text{coupling}] > 0$$

$$\text{not renormalizable} \Leftrightarrow [\text{coupling}] < 0.$$

Recall: any diagram is a tree diagram with LPI vertices.

So, to carry out renormalization, it is enough

to make the LPI effective action finite

as a function of renormalized fields/masses/couplings

as the cut-off is removed.

e.g. $\Gamma_0(\phi_0, m_0(\Lambda), \lambda_0(\Lambda); \Lambda) = \Gamma(\phi, m, \lambda; \Lambda)$

is finite as a function of ϕ, m, λ as $\Lambda \rightarrow \infty$.

Now, an important point:

Even when this is possible, there is an ambiguity

in the choice of renormalized fields/masses/couplings.

e.g. $a_1(\Lambda) = \underline{\text{finite}}$

$b_1(\Lambda) = \dots + \underline{\text{finite}}$

$c_1(\Lambda) = \dots + \underline{\text{finite}}$

To fix the ambiguity, impose renormalization condition :

For example

$$\Gamma(\phi) = \Gamma(\phi, m, \lambda; \Lambda)$$

$$= \sum_{\substack{n=0 \\ \text{even}}}^{\infty} \frac{1}{n!} \int \frac{d^4 p_1}{(2\pi)^4} \dots \frac{d^4 p_n}{(2\pi)^4} (2\pi)^4 \delta^{(4)}(p_1 + \dots + p_n)$$

$$\Gamma(p_1, \dots, p_n) \tilde{\phi}(p_1) \dots \tilde{\phi}(p_n)$$

$$\left\{ \begin{array}{l} \Gamma(-p, p) \Big|_{p^2 = -m^2} = 0 \\ \frac{d}{dp^2} \Gamma(-p, p) \Big|_{p^2 = -m^2} = 1 \\ \Gamma(p_1, \dots, p_4) \Big|_{p_i \cdot p_j = \begin{cases} -m^2 & i=j \\ m^2/3 & i \neq j \end{cases}} = \lambda \end{array} \right. \quad \text{"On shell renormalization"}$$

or

$$\left\{ \begin{array}{l} \Gamma(-p, p) \Big|_{p^2 = 0} = m^2 \\ \frac{d}{dp^2} \Gamma(-p, p) \Big|_{p^2 = 0} = 1 \\ \Gamma(p_1, \dots, p_4) \Big|_{p_i \cdot p_j = 0} = \lambda \end{array} \right. \quad \text{"intermediate renormalization"}$$

or ($\mu = \text{some mass scale}$)

$$\left\{ \begin{array}{l} \Gamma(-p, p) \Big|_{p^2 = \mu^2} = \mu^2 + m^2 \\ \frac{d}{dp^2} \Gamma(-p, p) \Big|_{p^2 = \mu^2} = 1 \\ \Gamma(p_1, \dots, p_4) \Big|_{p_i \cdot p_j = \begin{cases} \mu^2 & i=j \\ -\mu^2/3 & i \neq j \end{cases}} = \lambda \end{array} \right. \quad \text{"another R.C."}$$

When the renormalization condition is imposed,
the ambiguity is completely fixed.

Let us confirm this at 1-loop

$$\Gamma_i(-p, p) = p^2 + m^2 - \left(\text{loop diagram 1} + \text{loop diagram 2} \right)$$

$$\Gamma_i(p_1, \dots, p_4) = - \left(\text{tree diagram 1} + \text{tree diagram 2} + \text{tree diagram 3} + \text{tree diagram 4} + \text{loop diagram 1} \right)$$

For ① momentum cut-off $\frac{1}{p^2+m^2} \rightarrow \frac{e^{-\frac{p^2+m^2}{\Lambda^2}}}{p^2+m^2}$

$$\Gamma_1(-p, p) = p^2 + m^2 + \frac{\lambda m^2}{2(4\pi)^2} \left(\frac{\Lambda^2}{m^2} - \left(\log \left(\frac{\Lambda^2}{m^2} \right) + 1 - \gamma \right) + O\left(\frac{m^2}{\Lambda^2}\right) \right) + \lambda a_1(\Lambda) p^2 + \lambda b_1(\Lambda)$$

$$\Gamma_1(p_1, \dots, p_4) = \lambda - \frac{\lambda^2}{2(4\pi)^2} \left[\log \left(\frac{\Lambda^2}{2m^2} \right) - \gamma - 1 - \int_0^1 dx \log \left(1 + x(1-x) \frac{p_{12}^2}{m^2} \right) + O\left(\frac{p_{12}^2}{\Lambda^2}, \frac{m^2}{\Lambda^2}\right) \right] p_{12} = p_1 + p_2 \text{ etc} - (2 \leftrightarrow 3) - (2 \leftrightarrow 4) + \lambda^2 c_1(\Lambda)$$

For ② dimensional regularization

$$\Gamma_1(-p, p) = p^2 + m^2 - \frac{\lambda m^2}{2(4\pi)^2} \left(\frac{2}{\epsilon} + \log \left(\frac{4\pi M_{DR}^2}{m^2} \right) + 1 - \gamma + O(\epsilon) \right) + \lambda a_1(\epsilon) p^2 + \lambda b_1(\epsilon)$$

$$\Gamma_1(p_1, \dots, p_4) = \lambda - \frac{\lambda^2}{2(4\pi)^2} \left[\frac{2}{\epsilon} + \log \left(\frac{4\pi M_{DR}^2}{m^2} \right) - \gamma - \int_0^1 dx \log \left(1 - x(1-x) \frac{p_{12}^2}{m^2} \right) + O(\epsilon) \right] - (2 \leftrightarrow 3) - (2 \leftrightarrow 4) + \lambda^2 c_1(\epsilon)$$

$a_i(\Lambda), b_i(\Lambda), c_i(\Lambda)$ or $a_i(\epsilon), b_i(\epsilon), c_i(\epsilon)$

are determined uniquely by the renormalization condition.

On shell renormalization

momentum cut-off:

$$a_i(\Lambda) = 0, \quad b_i(\Lambda) = \frac{m^2}{2(4\pi)^2} \left[-\frac{\Lambda^2}{m^2} + \log\left(\frac{\Lambda^2}{m^2}\right) + 1 - \gamma + O\left(\frac{m^2}{\Lambda^2}\right) \right]$$

$$c_i(\Lambda) = \frac{3}{2(4\pi)^2} \left[\log\left(\frac{\Lambda^2}{2m^2}\right) - \gamma - 1 - \kappa + O\left(\frac{m^2}{\Lambda^2}\right) \right]$$

dimensional regularization:

$$a_i(\epsilon) = 0, \quad b_i(\epsilon) = \frac{m^2}{2(4\pi)^2} \left[\frac{2}{\epsilon} + \log\left(\frac{4\pi\mu_{DR}^2}{m^2}\right) + 1 - \gamma + O(\epsilon) \right]$$

$$c_i(\epsilon) = \frac{3}{2(4\pi)^2} \left[\frac{2}{\epsilon} + \log\left(\frac{4\pi\mu_{DR}^2}{m^2}\right) - \gamma - \kappa + O(\epsilon) \right]$$

where $\kappa = \int_0^1 dx \log\left(1 - \frac{4}{3}x(1-x)\right)$

Exercise Determine $a_i(\Lambda), b_i(\Lambda), c_i(\Lambda); a_i(\epsilon), b_i(\epsilon), c_i(\epsilon)$

also for intermediate renormalization

and "another R.C." ← especially this!