Recap

$$S_\Lambda = \left[\int d^4x \left(\frac{1}{2}(\partial\phi_0)^2 + \frac{m_0(\Lambda)^2}{2}\phi_0^2 + \frac{\lambda_0(\Lambda)}{4!}\phi_0^4 \right) \right]_\Lambda$$

regularization

$\Lambda \leftarrow$ UV cutoff

$$\phi_0 = \sqrt{Z_0(\Lambda)}\,\phi$$

$$= \left[\int d^4x \left(\frac{1}{2} Z_0(\Lambda)(\partial\phi)^2 + \frac{m_0(\Lambda)^2}{2} Z_0(\Lambda)\phi^2 + \frac{\lambda_0(\Lambda)}{4!} Z_0(\Lambda)^2 \phi^4 \right) \right]_\Lambda$$

$$\begin{cases} Z_0(\Lambda) = 1 + \lambda a_1(\Lambda) + \lambda^2 a_2(\Lambda) + \cdots \\ m_0(\Lambda)^2 Z_0(\Lambda) = m^2 + \lambda b_1(\Lambda) + \lambda^2 b_2(\Lambda) + \cdots \\ \lambda_0(\Lambda) Z_0(\Lambda)^2 = \lambda + \lambda^2 C_1(\Lambda) + \lambda^3 C_2(\Lambda) + \cdots \end{cases}$$

Determine $a_n(\Lambda)$, $b_n(\Lambda)$, $C_n(\Lambda)$ so that

$$\Gamma_0(\phi_0, m_0(\Lambda), \lambda_0(\Lambda); \Lambda) = \Gamma(\phi, m, \lambda; \Lambda)$$

is finite as a functional of ϕ, m, λ as $\Lambda \to \infty$.

To fix ambiguity, we need to impose renormalization condition. (R.C.)

$$\Gamma(\phi) \doteq \Gamma(\phi, m, \lambda; \Lambda)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \int \prod_{i=1}^{n} \frac{d^4 p_i}{(2\pi)^4} \tilde{\phi}(p_i) \cdot (2\pi)^4 \delta^{(4)}(p_1 + \cdots + p_n)\, \Gamma(p_1, \cdots, p_n)$$

Examples of renormalization conditions:

<u>On shell R.C.</u>

$$\begin{cases} \Gamma(-p,p)\big|_{p^2=-m^2} = 0 \\ \dfrac{d}{dp^2}\Gamma(-p,p)\big|_{p^2=-m^2} = 1 \end{cases}$$

$$\Gamma(p_1,p_2,p_3,p_4)\Big|_{p_i\cdot p_j = \begin{cases} -m^2 & i=j \\ m^2/3 & i\neq j \end{cases}} = \lambda$$

<u>Intermediate R.C.</u>

$$\begin{cases} \Gamma(-p,p)\big|_{p^2=0} = m^2 \\ \dfrac{d}{dp^2}\Gamma(-p,p)\big|_{p^2=0} = 1 \\ \Gamma(p_1,\cdots,p_4)\big|_{p_i\cdot p_j=0} = \lambda \end{cases}$$

<u>another R.C.</u>

$$\begin{cases} \Gamma(-p,p)\big|_{p^2=\mu^2} = \mu^2+m^2 \\ \dfrac{d}{dp^2}\Gamma(-p,p)\big|_{p^2=\mu^2} = 1 \\ \Gamma(p_1,\cdots,p_4)\Big|_{p_i\cdot p_j = \begin{cases} \mu^2 & i=j \\ -\mu^2/3 & i\neq j \end{cases}} = \lambda \end{cases}$$

Solution to R.C. at 1-loop

① <u>momentum cut-off</u>

$$\Gamma_1(-p,p) = p^2 + m^2 + \frac{\lambda m^2}{2(4\pi)^2}\left(\frac{\Lambda^2}{m^2} - \log\left(\frac{\Lambda^2}{m^2}\right) - 1 + \gamma + O\left(\frac{m^2}{\Lambda^2}\right)\right)$$

$$+ \lambda a_1(\Lambda) p^2 + \lambda b_1(\Lambda)$$

$$\Gamma_1(p_1,\dots,p_4) = \lambda - \frac{\lambda^2}{2(4\pi)^2}\left[\log\left(\frac{\Lambda^2}{2m^2}\right) - \gamma - 1 - \int_0^1 dx \log\left(1 + x(1-x)\frac{p_{12}^2}{m^2}\right) + O\left(\frac{p_{12}^2}{\Lambda^2}, \frac{m^2}{\Lambda^2}\right)\right]$$

$$- (2 \leftrightarrow 3) - (2 \leftrightarrow 4)$$

$p_{12} = p_1 + p_2$ etc

$$+ \lambda^2 c_1(\Lambda)$$

$$a_1(\Lambda) = 0$$

$$b_1(\Lambda) = \frac{m^2}{2(4\pi)^2}\left(-\frac{\Lambda^2}{m^2} + \log\left(\frac{\Lambda^2}{m^2}\right) + 1 - \gamma + O\left(\frac{m^2}{\Lambda^2}\right)\right]$$

$$c_1(\Lambda) = \frac{3}{2(4\pi)^2}\left[\log\left(\frac{\Lambda^2}{2m^2}\right) - 1 - \gamma - K + O\left(\frac{m^2}{\Lambda^2}\right)\right]$$

$$K = \begin{cases} \int_0^1 dx \log\left(1 - \frac{4}{3}x(1-x)\right) & \underline{\text{On shell R.C.}} \\ 0 & \underline{\text{intermediate R.C.}} \\ \int_0^1 dx \log\left(1 + x(1-x)\frac{4\mu^2}{3m^2}\right) & \underline{\text{another R.C.}} \end{cases}$$

3 dim reg

$$\Gamma_1(-p,p) = p^2 + m^2 - \frac{\lambda m^2}{2(4\pi)^2}\left(\frac{2}{\epsilon} + \log\left(\frac{4\pi\mu_{DR}^2}{m^2}\right) + 1 - \gamma + O(\epsilon)\right)$$

$$+ \lambda a_1(\epsilon)p^2 + \lambda b_1(\epsilon)$$

$$\Gamma_1(p_1,\dots,p_4) = \lambda - \frac{\lambda^2}{2(4\pi)^2}\left[\frac{2}{\epsilon} + \log\left(\frac{4\pi\mu_{DR}^2}{m^2}\right) - \gamma - \int_0^1 dx \log\left(1 + x(1-x)\frac{p_{12}^2}{m^2}\right) + O(\epsilon)\right]$$

$$- (2\leftrightarrow 3) - (2\leftrightarrow 4)$$

$$+ \lambda^2 c_1(\epsilon)$$

$$a_1(\epsilon) = 0$$

$$b_1(\epsilon) = \frac{m^2}{2(4\pi)^2}\left[\frac{2}{\epsilon} + \log\left(\frac{4\pi\mu_{DR}^2}{m^2}\right) + 1 - \gamma + O(\epsilon)\right]$$

$$c_1(\epsilon) = \frac{3}{2(4\pi)^2}\left[\frac{2}{\epsilon} + \log\left(\frac{4\pi\mu_{DR}^2}{m^2}\right) - \gamma - K + O(\epsilon)\right]$$

$$K = \begin{cases} \int_0^1 dx \log\left(1 - \frac{4}{3}x(1-x)\right) & \underline{\text{on shell R.C}} \\ 0 & \underline{\text{intermediate}} \\ \int_0^1 dx \log\left(1 + x(1-x)\frac{4\mu^2}{3m^2}\right) & \underline{\text{another}} \end{cases}$$

Same as in 1

Computation of effective potential

The effective potential $U_{eff}(\phi)$ is defined by

$$\Gamma(\phi) = \int d^4x \, U_{eff}(\phi) \quad \text{for constant } \phi.$$

Let us compute it to the 1-loop level. Recall

$$\Gamma(\phi) = S_0(\phi) + S_1(\phi) + \frac{1}{2} \log \det S_0''(\phi) + \text{2-loop \& higher}$$

For constant ϕ,

$$S_0(\phi) = \int d^4x \, U(\phi) \,; \qquad U(\phi) = \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4$$

$$S_1(\phi) = \int d^4x \, \delta_1 U(\phi) \,; \qquad \delta_1 U(\phi) = \frac{\lambda b_1}{2}\phi^2 + \frac{\lambda^2 c_1}{4!}\phi^4$$

$$\frac{1}{2} \log \det S''(\phi) = \frac{1}{2} \operatorname{Tr} \log\left(-\partial^2 + U''(\phi)\right)$$

$$= \frac{1}{2} \int d^4x \sum_k \varphi_k(x)^* \log\left(-\partial^2 + U''(\phi)\right) \varphi_k(x) \leftarrow \varphi_k(x) \sim e^{ikx}$$

$$= \int d^4x \, \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \log\left(k^2 + U''(\phi)\right)$$

$$\therefore U_{eff}(\phi) = U(\phi) + \delta_1 U(\phi) + \underbrace{\frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \log\left(k^2 + U''(\phi)\right)}_{=: \, U_1(\phi)} + \text{2 loop \& higher}$$

Let us compute $U_1(\phi)$ via ① momentum cutoff

& ③ dimensional regularization.

① momentum cut-off

$$\frac{k^2+m^2}{e^{-(k^2+m^2)/\Lambda^2}}$$

$$U_1 = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4} \underbrace{\log\left(k^2+m^2+\frac{\lambda\phi^2}{2}\right)}_{\log\left(\frac{k^2+m^2}{e^{-(k^2+m^2)/\Lambda^2}}\right) + \log\left(1+\frac{e^{-(k^2+m^2)/\Lambda^2}}{k^2+m^2}\frac{\lambda\phi^2}{2}\right)}$$

constant → drop

$$U_1^{①} = \frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\log\left(1+\frac{e^{-\frac{k^2+m^2}{\Lambda^2}}}{k^2+m^2}\frac{\lambda\phi^2}{2}\right)$$

$$\left[\ \frac{\mathrm{Vol}(S^3)}{2(2\pi)^4} = \frac{1}{(4\pi)^2}, \quad k^2+m^2 =: \Lambda^2 t \right.$$

$\Delta := m^2/\Lambda^2$

$$= \frac{m^4}{2(4\pi)^2}\int_\Delta^\infty\left(\frac{t}{\Delta^2}-\frac{1}{\Delta}\right)dt\,\log\left(1+\frac{e^{-t}}{t}\Delta\cdot\underbrace{\frac{\lambda\phi^2}{2m^2}}_{=:A}\right)$$

$\vdots$

$$= \frac{m^4}{2(4\pi)^2}\left[\frac{1}{2}(1+A)^2\log(1+A) + \left(\frac{1}{\Delta}+\log\Delta+\gamma-\frac{3}{2}\right)A\right.$$

$$\left.+\frac{1}{2}\left(\log(2\Delta)+\gamma-\frac{1}{2}\right)A^2\ \right]$$

Using the expressions for $b_1(\Lambda)$ & $c_1(\Lambda)$:

$$\delta_1 U^{①} = \frac{\lambda m^2}{2} \cdot \frac{1}{2(4\pi)^2}\left(-\frac{\Lambda^2}{m^2} + \log\left(\frac{\Lambda^2}{m^2}\right) + 1 - \gamma + O\left(\frac{m^2}{\Lambda^2}\right)\right]\phi^2$$

$$+ \frac{\lambda^2}{4!}\,\frac{3}{2(4\pi)^2}\left(\log\left(\frac{\Lambda^2}{2m^2}\right) - 1 - \gamma - K + O\left(\frac{m^2}{\Lambda^2}\right)\right)\phi^4$$

$$= -\frac{m^4}{2(4\pi)^2}\left\{\left(\frac{1}{\Delta} + \log\Delta - 1 + \gamma + O(\Delta)\right)A\right. \qquad \left(\begin{array}{l}\Delta = \frac{m^2}{\Lambda^2} \\ A = \frac{\lambda\phi^2}{2m^2}\end{array}\right.$$

$$\left. + \frac{1}{2}\left(\log(2\Delta) + 1 + \gamma + K + O(\Delta)\right)A^2\right\}$$

$$U_{\text{eff}} = U + \underbrace{\delta_1 U^{①} + U_1^{①}}_{\text{divergence cancel!}} + \text{higher loop}$$

$$\xrightarrow{\Lambda\to\infty} U + \frac{m^4}{2(4\pi)^2}\left\{\frac{1}{2}(1+A)^2\log(1+A) - \frac{1}{2}A - \frac{1}{2}\left(\frac{3}{2}+K\right)A^2\right\}$$

+ higher loop

$$= \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4$$

$$+ \frac{1}{4(4\pi)^2}\left\{\left(m^2 + \frac{\lambda\phi^2}{2}\right)^2\log\left(1 + \frac{\lambda\phi^2}{2m^2}\right)\right.$$

$$\left. - \frac{1}{2}\lambda m^2\phi^2 - \left(\frac{3}{2}+K\right)\frac{1}{4}\lambda^2\phi^4\right\}$$

+ higher loop

③ Dimensional regularization

$$U_1^{③} = \frac{\mu_{DR}^{4-d}}{2}\int \frac{d^d k}{(2\pi)^d} \log\left(k^2 + U''(\varphi)\right) \qquad U''(\varphi) = m^2 + \frac{\lambda\varphi^2}{2}$$

$$\vdots$$

$$= -\frac{\mu_{DR}^{4-d}}{2(4\pi)^{d/2}} \left(U''(\phi)\right)^{\frac{d}{2}} \Gamma\left(-\frac{d}{2}\right)$$

$d = 4-\epsilon$

$$= -\frac{\left(U''(\varphi)\right)^2}{4(4\pi)^2}\left(\frac{2}{\epsilon} - \log\left(\frac{U''(\varphi)}{4\pi\mu_{DR}^2}\right) + \frac{3}{2} - \gamma + O(\epsilon)\right)$$

$$= \frac{\left(m^2 + \frac{\lambda\varphi^2}{2}\right)^2}{4(4\pi)^2}\left(-\frac{2}{\epsilon} + \log\left(1 + \frac{\lambda\varphi^2}{2m^2}\right) - \log\left(\frac{4\pi\mu_{DR}^2}{m^2}\right) - \frac{3}{2} + \gamma + O(\epsilon)\right)$$

Using the expressions for $b_1(\epsilon)$ & $C_1(\epsilon)$

$$\delta_1 U^{③} = \frac{\lambda m^2}{2}\cdot\frac{1}{2(4\pi)^2}\left(\frac{2}{\epsilon} + \log\left(\frac{4\pi\mu_{DR}^2}{m^2}\right) + 1 - \gamma + O(\epsilon)\right)\varphi^2$$

$$+ \frac{\lambda^2}{4!}\frac{3}{2(4\pi)^2}\left(\frac{2}{\epsilon} + \log\left(\frac{4\pi\mu_{DR}^2}{m^2}\right) - \gamma - K + O(\epsilon)\right)\varphi^4$$

$$= \frac{1}{4(4\pi)^2}\left(\lambda m^2\varphi^2 + \frac{\lambda^2\varphi^4}{4}\right)\left(\frac{2}{\epsilon} + \log\left(\frac{4\pi\mu_{DR}^2}{m^2}\right) - \gamma + O(\epsilon)\right)$$

$$+ \frac{1}{4(4\pi)^2}\left(\lambda m^2\varphi^2 - \frac{K}{4}\lambda^2\varphi^4\right)$$

$$U_{eff} = U + \delta_1 U^{③} + U_1^{③} + \text{higher loop}$$

$$= \frac{m^4}{4(4\pi)^2}\left(-\frac{2}{\epsilon} - \log\left(\frac{4\pi M_{DR}^2}{m^2}\right) - \frac{3}{2} + \gamma + O(\epsilon)\right)\Bigg]$$ Constant → drop

$$+ U + \frac{1}{4(4\pi)^2}\Bigg[\left(m^2 + \frac{\lambda\phi^2}{2}\right)^2 \log\left(1 + \frac{\lambda\phi^2}{2m^2}\right)$$

$$- \frac{1}{2}\lambda m^2\phi^2 - \left(\frac{3}{2} + K\right)\frac{\lambda^2}{4}\phi^4 + O(\epsilon)\Bigg]$$

+ higher loop

After dropping the constant term

$$\xrightarrow{\epsilon \to 0} \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4$$

$$+ \frac{1}{4(4\pi)^2}\Bigg[\left(m^2 + \frac{\lambda\phi^2}{2}\right)^2 \log\left(1 + \frac{\lambda\phi^2}{2m^2}\right)$$

$$- \frac{1}{2}\lambda m^2\phi^2 - \left(\frac{3}{2} + K\right)\frac{\lambda^2}{4}\phi^4\Bigg]$$

+ higher loop.

The results for ① & ③ agree

for each renormalization condition.

Physical meaning of the effective potential

Hamiltonian density

FACT $\quad U_{eff}(\varphi) = \langle\psi|\mathcal{H}|\psi\rangle$ for a state $|\psi\rangle$

that extremizes $\langle\psi|\mathcal{H}|\psi\rangle$ under the condition

① $\langle\psi|\psi\rangle = 1$

② $\langle\psi|\phi|\psi\rangle = \varphi$

☺ in Q.M. for simplicity.

Extremization problem :

$$f(|\psi\rangle, \lambda_1, \lambda_2) := \langle\psi|H|\psi\rangle - \lambda_1(\langle\psi|\psi\rangle - 1) - \lambda_2(\langle\psi|\phi|\psi\rangle - \varphi)$$

$$\partial_{\lambda_1} f = \partial_{\lambda_2} f = 0 \quad \Rightarrow \quad ① \ \& \ ②$$

$$\delta_{|\psi\rangle} f = 0 \quad \Rightarrow \quad (H - \lambda_1 - \lambda_2\phi)|\psi\rangle = 0$$

Recall the def of $\Gamma[\boldsymbol{\varphi}]$ (Notation here : $\boldsymbol{\varphi} = (\varphi(t))$ etc)

only here

$$e^{-W[\mathbf{J}]} = \int \mathcal{D}\boldsymbol{\phi}\, e^{-S[\boldsymbol{\phi}] + \int dt\, J(t)\phi(t)}$$

$$-\frac{\delta}{\delta J(t)} W[\mathbf{J}] \stackrel{!}{=} \varphi(t) \quad \xrightarrow{\text{soln}} \quad \mathbf{J} = \mathbf{J}[\boldsymbol{\varphi}]$$

$$\Gamma[\boldsymbol{\varphi}] := W[\mathbf{J}(\boldsymbol{\varphi})] + \int dt\, J[\boldsymbol{\varphi}](t)\,\varphi(t)$$

Consider the case $-\frac{T}{2} \leq t \leq \frac{T}{2}$, $T \to \infty$.

For $J(t) \equiv J$ constant (write: $\mathbf{J} \equiv J$)

$$W[\mathbf{J}]\Big|_{\mathbf{J}\equiv J} = T \cdot E(J) \quad \text{where}$$

$E(J)$ is the energy of the ground state $|0\rangle_J$ of the system with Hamiltonian $H - J\phi$

$$(H - J\phi)|0\rangle_J = E(J)|0\rangle_J .$$

$$\Rightarrow \quad E(J) = {}_J\langle 0|(H - J\cdot\phi)|0\rangle_J$$

$$\frac{\partial}{\partial J}E(J) = -{}_J\langle 0|\phi|0\rangle_J + \frac{\partial}{\partial J}{}_J\langle 0|\underbrace{(H - J\cdot\phi)|0\rangle_J}_{E(J)|0\rangle_J} + \underbrace{{}_J\langle 0|(H - J\cdot\phi)}_{{}_J\langle 0|E(J)}\frac{\partial}{\partial J}|0\rangle_J$$

$$= -{}_J\langle 0|\phi|0\rangle_J + \underbrace{E(J)\frac{\partial}{\partial J}\underbrace{{}_J\langle 0|0\rangle_J}_{=1}}_{=0}$$

$$= -{}_J\langle 0|\phi|0\rangle_J$$

If $J(\varphi)$ is a solution to $\frac{\partial}{\partial J}E(J) = -\varphi$,

the state $|\psi\rangle = |0\rangle_{J(\varphi)}$ solves the extremization problem

with $\lambda_1 = E(J(\varphi))$, $\lambda_2 = J(\varphi)$.

The extremum is

$$ {}_{J(\varphi)}\langle 0 | H | 0 \rangle_{J(\varphi)} = {}_{J(\varphi)}\langle 0 | \left(E(J(\varphi)) + J(\varphi)\,\phi \right) | 0 \rangle_{J(\varphi)} $$

$$ = E(J(\varphi)) + J(\varphi) \cdot \varphi . $$

On the other hand,

$$ \frac{\delta}{\delta J(t)} W[\mathbf{J}] \Big|_{\mathbf{J} \equiv J} = \frac{\partial}{\partial J} E(J) \overset{\overset{J = J(\varphi)}{\downarrow}}{=} -\varphi $$

$$ \Rightarrow \quad J(\varphi) = \mathbf{J}[\boldsymbol{\varphi}] \big|_{\boldsymbol{\varphi} \equiv \varphi} $$

$$ \Gamma[\boldsymbol{\varphi}] \big|_{\boldsymbol{\varphi} \equiv \varphi} = \underbrace{W[\mathbf{J}[\boldsymbol{\varphi}]] \big|_{\boldsymbol{\varphi} \equiv \varphi}}_{\substack{W[\mathbf{J}] |_{\mathbf{J} \equiv J(\varphi)} \\ = T\, E(J(\varphi))}} + \underbrace{\int dt\, \mathbf{J}[\boldsymbol{\varphi}](t)\, \varphi(t) \Big|_{\boldsymbol{\varphi} \equiv \varphi}}_{T\, J(\varphi)\, \varphi} $$

$$ \| $$

$$ T\, U_{eff}(\varphi) $$

$$ \therefore \quad U_{eff}(\varphi) = E(J(\varphi)) + J(\varphi) \cdot \varphi $$

Combining,

$$ U_{eff}(\varphi) = {}_{J(\varphi)}\langle 0 | H | 0 \rangle_{J(\varphi)} , \text{ the extremum.} $$

//

<u>Claim</u> "extremize" can be replaced by "minimize":

$$\boxed{\begin{aligned} &U_{eff}(\varphi) \overset{?}{\doteq} \text{minimum of } \langle\psi|H|\psi\rangle \\ &\qquad\qquad \text{for } \langle\psi|\psi\rangle = 1, \ \langle\psi|\phi|\psi\rangle = \varphi. \end{aligned}}$$

This is indeed the case for $\varphi = \langle 0|\phi|0\rangle = \phi_*$, the VEV.

$$e^{-W[0]} = \int \mathcal{D}\phi \, e^{-S[\phi]} = e^{-TE_0}$$

$J[\phi_*] = 0 \rightarrow \ \|$

$$e^{-\Gamma[\phi_*]} = e^{-TU_{eff}(\phi_*)}$$

$\therefore \ U_{eff}(\phi_*) = E_0 =$ the ground state energy.

And we know for $\forall |\psi\rangle$ s.t. $\langle\psi|\psi\rangle = 1$,

$$\langle\psi|H|\psi\rangle \geq E_0. \qquad //$$

By continuity, this holds as φ is moved from ϕ_* as long as level crossing does not occur.

Anyhow, the VEU $\langle 0|\phi|0\rangle = \phi_*$ minimizes $U_{eff}(\varphi)$

and $U_{eff}(\varphi_*)$ is the ground state energy.

(energy density in QFT)

In other words, $U_{eff}(\varphi)$ can be used to find

the VEV of ϕ.