Renormalization group

Choices of renormalization conditions :

"On shell", "intermediate", "Another (μ)", ...

All these originate from the same classical Lagrangian

$\longrightarrow$ same physics.

But we need a dictionary :

renormalization condition I $\rightarrow$ $\phi_I, \lambda_I, \cdots$

renormalization condition II $\rightarrow$ $\phi_{II}, \lambda_{II}, \cdots$

relation ?

e.g. in 4d ϕ^4 theory

"another R.C." parametrized by a mass scale μ

renormalization point

$$\begin{cases} \Gamma(-p,p)\big|_{p^2=\mu^2} = \mu^2 + m^2 \\ \dfrac{d}{dp^2}\Gamma(-p,p)\Big|_{p^2=\mu^2} = 1 \\ \Gamma(p_1,p_2,p_3,p_4)\Big|_{p_i\cdot p_j = \begin{cases} \mu^2 & i=j \\ -\mu^2/3 & i\neq j \end{cases}} = \lambda \end{cases}$$

at μ : ϕ, m, λ

at μ' : ϕ', m', λ'

To describe the same physics,

<u>how are they related?</u>

Answer: so that the bare fields/couplings are the same.

$$Z_0^{\frac{1}{2}}(m,\lambda;\mu,\Lambda)\,\phi = \phi_0 = Z_0^{\frac{1}{2}}(m',\lambda';\mu',\Lambda)\,\phi'$$

$$m_0(m,\lambda;\mu,\Lambda) = m_0 = m_0(m',\lambda';\mu',\Lambda)$$

$$\lambda_0(m,\lambda;\mu,\Lambda) = \lambda_0 = \lambda_0(m',\lambda';\mu',\Lambda)$$

$$\Gamma_0(\phi_0, m_0, \lambda_0;\Lambda)$$

$$\Gamma(\phi,m,\lambda;\mu,\Lambda) \quad \stackrel{\star}{=} \quad \Gamma(\phi',m',\lambda';\mu',\Lambda)$$

The change $(\phi,m,\lambda)\to(\phi',m',\lambda')$ for $\mu\to\mu'$ is called the renormalization group (RG) transformation, and the equality $\star$ is called the RG equation.

The relation between the renormalized fields/couplings has a limit as $\Lambda\to\infty$, and

$\Gamma(\phi,m,\lambda;\mu) := \lim_{\Lambda\to\infty}\Gamma(\phi,m,\lambda;\mu,\Lambda)$ satisfies

$$\Gamma(\phi,m,\lambda;\mu) = \Gamma(\phi',m',\lambda';\mu').$$

The RG transformation may be written as

$$\phi' = Z^{-\frac{1}{2}}(m, \lambda; \mu', \mu)\,\phi,$$

$$m' = R^{m}(m, \lambda; \mu', \mu),$$

$$\lambda' = R^{\lambda}(m, \lambda; \mu', \mu).$$

Put

$$\mu' \frac{\partial}{\partial \mu'} Z^{\frac{1}{2}}(m, \lambda; \mu', \mu)\Big|_{\mu'=\mu} =: \gamma(m, \lambda; \mu),$$

$$\mu' \frac{\partial}{\partial \mu'} R^{m}(m, \lambda; \mu', \mu)\Big|_{\mu'=\mu} =: -\gamma_m(m, \lambda; \mu)\, m,$$

$$\mu' \frac{\partial}{\partial \mu'} R^{\lambda}(m, \lambda; \mu', \mu)\Big|_{\mu'=\mu} =: \beta(m, \lambda; \mu).$$

Then, the <u>infinitesimal RG transformation</u> (<u>RG flow</u>) is

$$\mu \frac{d}{d\mu} \phi = -\gamma(m, \lambda; \mu)\,\phi,$$

$$\mu \frac{d}{d\mu} m = -\gamma_m(m, \lambda; \mu)\, m,$$

$$\mu \frac{d}{d\mu} \lambda = \beta(m, \lambda; \mu),$$

and the <u>infinitesimal RG equation</u> is

$$\left[-\gamma \phi \cdot \frac{\delta}{\delta \phi} - \gamma_m\, m \frac{\partial}{\partial m} + \beta \frac{\partial}{\partial \lambda} + \mu \frac{\partial}{\partial \mu} \right] \Gamma(\phi, m, \lambda; \mu) = 0.$$

Computation in 4d ϕ^4 theory

Recall $$Z_0 = 1 + \hbar\lambda a_1 + \hbar^2\lambda^2 a_2 + \cdots$$

$$Z_0 m_0^2 = m^2 + \hbar\lambda b_1 + \hbar^2\lambda^2 b_2 + \cdots$$

$$Z_0^2\lambda_0 = \lambda + \hbar\lambda^2 c_1 + \hbar^2\lambda^3 c_2 + \cdots$$

$$a_1 = 0$$

$$b_1 = -\frac{1}{2(4\pi)^2}\left[\Lambda^2 - m^2\left(\log\left(\frac{\Lambda^2}{m^2}\right) + 1 - \gamma\right) + m^2 O\left(\frac{m^2}{\Lambda^2}\right)\right]$$

$$c_1 = \frac{3}{2(4\pi)^2}\left[\log\left(\frac{\Lambda^2}{2m^2}\right) - \gamma - 1 - \int_0^1 dx \log\left(1 + x(1-x)\frac{4\mu^2}{3m^2}\right) + O\left(\frac{m^2}{\Lambda^2}\right)\right]$$

Thus, to 1-loop, $Z_0 = 1 + O(\hbar^2)$, and

$$0 = \mu\frac{d}{d\mu}\phi_0 = \mu\frac{d}{d\mu}\phi + O(\hbar^2)$$

$$0 = \mu\frac{d}{d\mu}m_0^2 = \mu\frac{d}{d\mu}m^2 + \hbar\,\mu\frac{d}{d\mu}\lambda b_1 + O(\hbar^2)$$

$$0 = \mu\frac{d}{d\mu}\lambda_0 = \mu\frac{d}{d\mu}\lambda + \hbar\left(\mu\frac{d}{d\mu}\lambda^2 c_1 + \lambda^2\mu\frac{d}{d\mu}c_1\right) + O(\hbar^2)$$

$$\Rightarrow \quad \mu\frac{d}{d\mu}\phi = O(\hbar^2)$$

$$\mu\frac{d}{d\mu}\lambda = -\hbar\lambda^2\mu\frac{d}{d\mu}c_1 + O(\hbar^2)$$

$$\mu\frac{d}{d\mu}m = O(\hbar^2)$$

$\therefore \quad \gamma(m,\lambda;\mu) = O(\hbar^2)$

$\gamma_m(m,\lambda;\mu) = O(\hbar^2)$

$$\beta(m,\lambda;\mu) = -\hbar\lambda^2 \mu \frac{d}{d\mu} C_1$$

$$= \frac{3\hbar\lambda^2}{2(4\pi)^2}\int_0^1 dx \, \frac{x(1-x)\frac{8\mu^2}{3m^2}}{1+x(1-x)\frac{4\mu^2}{3m^2}} + O(\hbar^2)$$

$$= \begin{cases} \frac{3\hbar\lambda^2}{(4\pi)^2} + O(\hbar^2) & \mu \gg m \\ \frac{2\hbar\lambda^2}{3(4\pi)^2}\frac{\mu^2}{m^2} + O(\hbar^2) & \mu \ll m \end{cases}$$

Let us consider the limit $m \to 0$.

As λ & Z are dimensionless, β & γ are functions of λ only:

$$\beta = \beta(\lambda), \quad \gamma = \gamma(\lambda).$$

Indeed, in this limit,

$$\gamma = 0, \quad \beta = \frac{3\lambda^2}{(4\pi)^2} \quad \text{at 1-loop.}$$

The RG flow : $\mu \frac{d}{d\mu} \lambda = \beta(\lambda)$

$$\mu \frac{d}{d\mu} \phi = -\gamma(\lambda)\, \phi$$

The RG eqn :

$$\left[\mu \frac{\partial}{\partial \mu} + \beta(\lambda) \frac{\partial}{\partial \lambda} - \gamma(\lambda)\, \phi \cdot \frac{\delta}{\delta \phi} \right] \Gamma(\phi, \lambda; \mu) = 0.$$

Instead of μ, use $t = \log(\mu/\mu_0)$ or $\mu = e^t \mu_0$.

Then, the RG flow takes the form

$$\lambda = \bar{\lambda}(t) \quad \left(\leftarrow \text{a solution to } \frac{d}{dt}\lambda = \beta(\lambda) \right)$$

$$\phi = \bar{\phi}(t) = \bar{\phi}(0) \cdot e^{-\int_0^t dt' \, \gamma(\bar{\lambda}(t'))}$$

and the RGE :

$\Gamma(\bar{\phi}(t), \bar{\lambda}(t); e^t \mu_0)$ is t-independent.

Write

$$\Gamma = \sum_{n=0}^{\infty} \frac{1}{n!} \int \prod_{i=1}^{n} \frac{d^4 p_i}{(2\pi)^4} \cdot (2\pi)^4 \delta(p_1 + \cdots + p_n)$$

$$\Gamma(p_1, \cdots, p_n, \lambda; \mu)\, \phi(p_1) \cdots \phi(p_n)$$

RGE :

$$e^{-n\int_0^t dt'\,\gamma(\bar{\lambda}(t'))}\,\Gamma(p_1,\dots,p_n,\bar{\lambda}(t);e^t\mu_0) \quad \text{is } t\text{-independent}$$

$$= \Gamma(p_1,\dots,p_n,\bar{\lambda}(0);\mu_0)$$

On the other hand, the canonical dimensions are

$$[\mu]=1,\ [\phi]=1,\ [\lambda]=0,\ [\Gamma]=0$$

$$\therefore\ [\Gamma(p_1,\dots,p_n,\lambda;\mu)] = 4-n \quad \Rightarrow$$

$$\Gamma(e^t p_1,\dots,e^t p_n,\lambda;e^t\mu) = e^{(4-n)t}\,\Gamma(p_1,\dots,p_n,\lambda;\mu).$$

"dimensional analysis"

Combining,

$$\boxed{\begin{aligned}
&\Gamma(e^t p_1,\dots,e^t p_n,\bar{\lambda}(0);\mu_0)\\
&\overset{\text{RGE}}{=} e^{-n\int_0^t dt'\,\gamma(\bar{\lambda}(t'))}\,\Gamma(e^t p_1,\dots,e^t p_n,\bar{\lambda}(t);e^t\mu_0)\\
&\overset{\text{dim. an.}}{=} e^{4t-n\int_0^t dt'\,(1+\gamma(\bar{\lambda}(t')))}\,\Gamma(p_1,\dots,p_n,\bar{\lambda}(t);\mu_0)
\end{aligned}}$$

This means

① If we uniformly rescale the momenta as

$$p_i \to e^t p_i,$$

the coupling λ effectively changes as

$$\bar{\lambda}(0) \to \bar{\lambda}(t).$$

$\bar{\lambda}(t)$ is the "<u>effective coupling constant</u>."

② The dimension of ϕ has also changed as

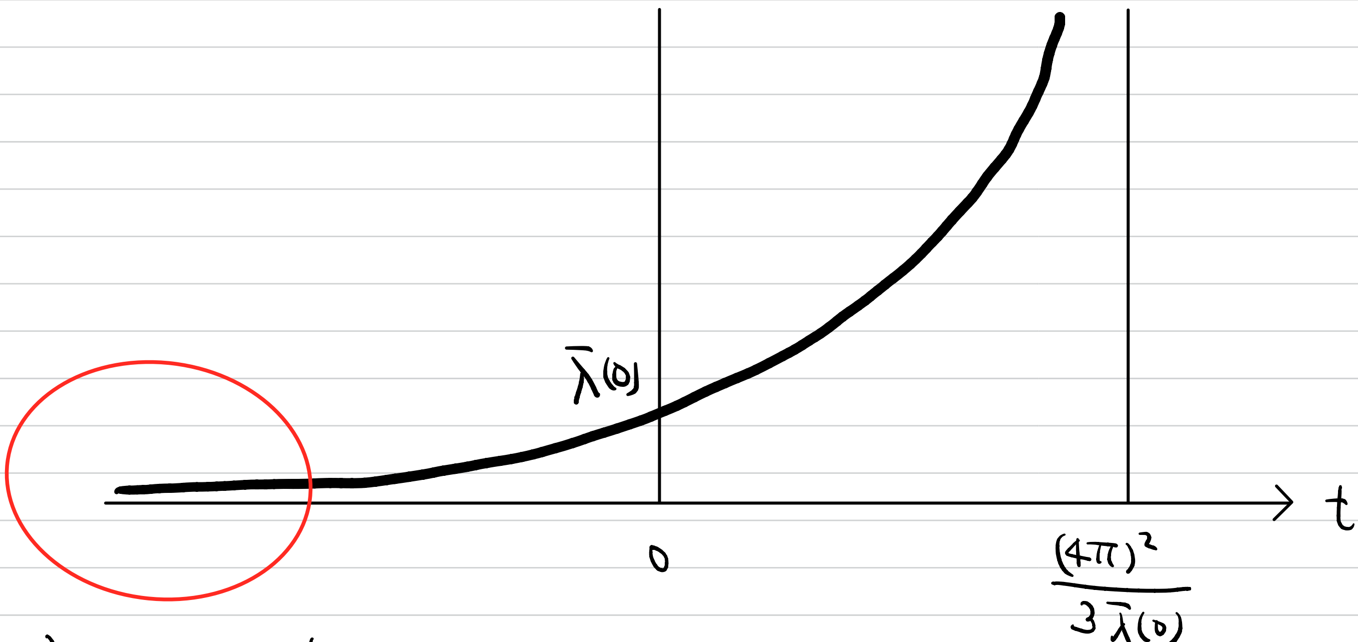
$$1 \to 1 + \gamma(\bar{\lambda}(t))$$

$\gamma(\lambda)$ is the "<u>anomalous dimension</u>" of ϕ.

At 1-loop, $\dfrac{d\lambda}{dt} = \dfrac{3\lambda^2}{(4\pi)^2}$.

$$\int \frac{d\lambda}{\lambda^2} = \int \frac{3dt}{(4\pi)^2} \sim -\frac{1}{\bar{\lambda}(t)} + \frac{1}{\bar{\lambda}(0)} = \frac{3t}{(4\pi)^2}$$

$$\boxed{\bar{\lambda}(t) = \frac{\bar{\lambda}(0)}{1 - \frac{3t}{(4\pi)^2}\bar{\lambda}(0)}}$$



$\lambda \to 0$ as $t \to -\infty$

$\mu = e^t \mu_0 \to 0$

The coupling is weaker at lower energies

or stronger at higher energies.

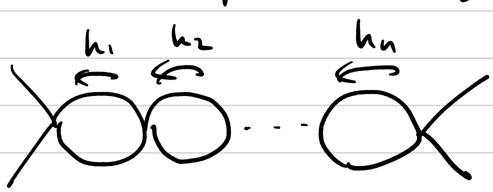
- $$\lambda(\mu) = \frac{\lambda(\mu_0)}{1 - \frac{3\lambda(\mu_0)}{(4\pi)^2} \log(\mu/\mu_0)}$$

is valid for $\mu \ll \mu_0$ even if $|\log(\mu/\mu_0)|$ may be large.

- The series expansion

$$\lambda(\mu) = \sum_{n=0}^{\infty} \lambda(\mu_0) \left(\frac{3\lambda(\mu_0)}{(4\pi)^2} \log(\mu/\mu_0) \right)^n$$

has a Feynman diagram interpretation:

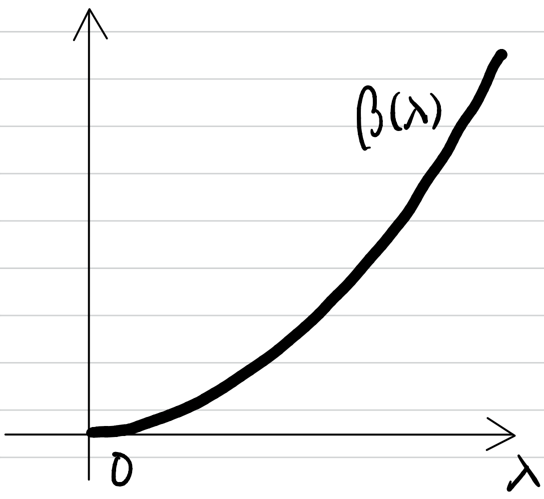


$$\sim \lambda^{n+1} \int_{\mu}^{\mu_0} \frac{d^4 k_1}{(k_1^2)^2} \cdots \int_{\mu}^{\mu_0} \frac{d^4 k_n}{(k_n^2)^2}$$

$$\sim \lambda^{n+1} \left(\log \mu_0/\mu \right)^n$$

"RG sums up a series of Feynman diagrams"

Various possibilities



$\lambda \to 0$ in the IR limit

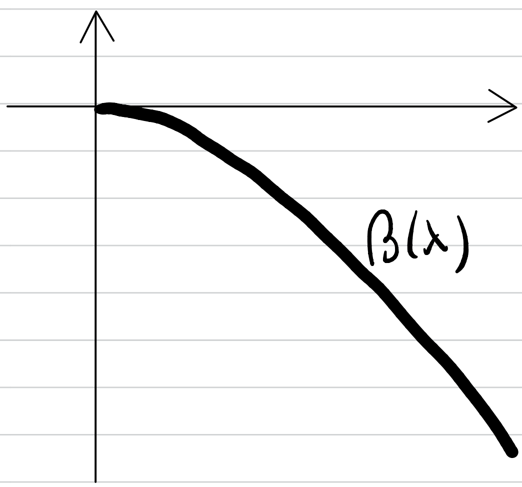
infra-red free theory

We've just seen / We'll see next

e.g. 4d ϕ^4 theory, QED_4

QCD_4 with large # of flavors

next semester



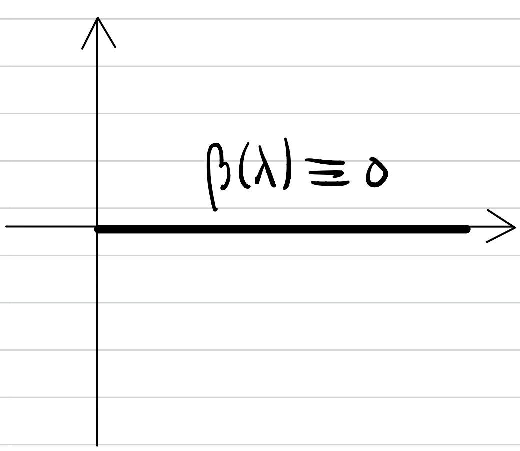
$\lambda \to 0$ in the UV limit

asymptotically free theory

e.g. 4d Yang-Mills theory

QCD_4 with small # of flavors

next semester



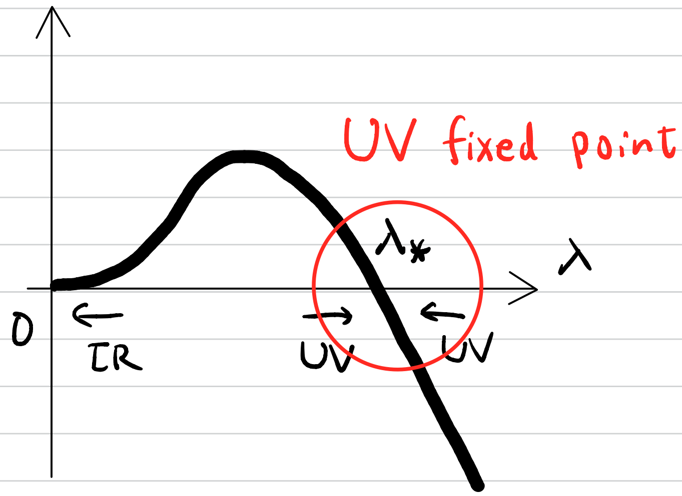
λ does not run!

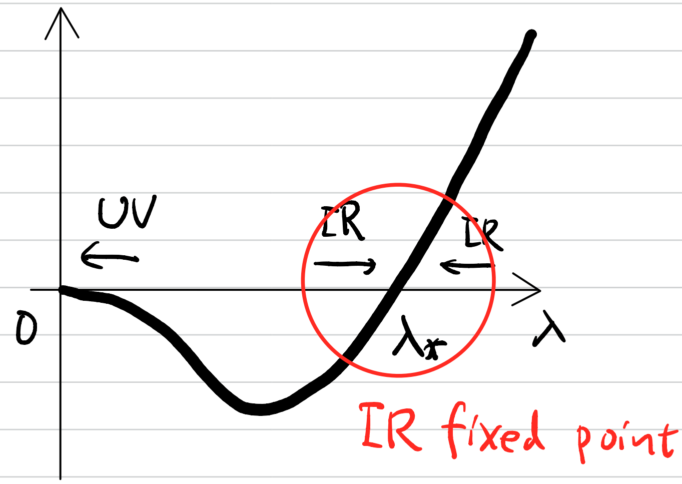
finite theory

e.g. 4d $\mathcal{N}=4$ supersymmetric Yang-Mills

next semester

Other possibilities :





$\therefore$ $\exists$ <u>non-trivial fixed point of RG flow</u>.

At such a point λ_*, with $\gamma_* := \gamma(\lambda_*)$,

$$\Gamma(e^t p_1, \cdots, e^t p_n, \lambda_* ; \mu_0)$$

$$= e^{(4-n(1+\gamma_*))t} \, \Gamma(p_1, \cdots, p_n, \lambda_* ; \mu_0)$$

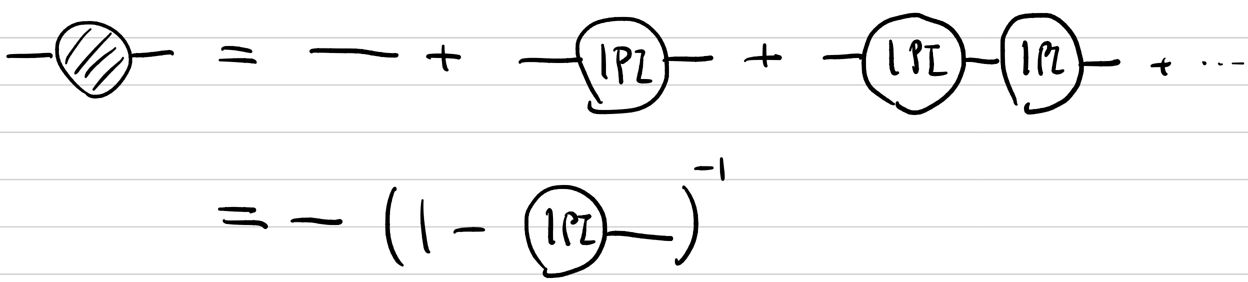
Correlation functions scales in a simple way.

"scale invariant theory".

e.g. $\Gamma(-p, p) = \text{const} \cdot (p^2)^{1-\gamma_*}$

$$\Gamma(\phi) = \frac{1}{2}\phi_i A_{ij} \phi_j - \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{i_1 \cdots i_n} \lambda_{1PI}^{i_1 \cdots i_n} \phi_{i_1} \cdots \phi_{i_n}$$

$$\Gamma(-p,p) = p^2 - \lambda^{(2)}_{1PI}(p^2)$$



$$= \frac{1}{p^2}\left(1 - \lambda^{(2)}_{1PI}(p^2)\frac{1}{p^2}\right)^{-1} = \frac{1}{p^2 - \lambda^{(2)}_{1PI}(p^2)} = \frac{1}{\Gamma(-p,p)}$$

$$\langle \phi(x)\phi(0) \rangle = \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ipx}}{\Gamma(-p,p)} \propto (p^2)^{1-\gamma_*}$$

$$\propto \frac{1}{|x|^{2+2\gamma_*}}$$

An application: RG improvement of $U_{eff}(\phi)$

In 4d ϕ^4 theory, we obtained 1-loop effective potential

$$U_{eff}(\phi) = \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4$$

$$+ \frac{1}{4(4\pi)^2}\left[\left(m^2 + \frac{\lambda\phi^2}{2}\right)^2 \log\left(1 + \frac{\lambda\phi^2}{2m^2}\right) - \frac{\lambda}{2}m^2\phi^2 - \left(\frac{3}{2} + K\right)\left(\frac{\lambda\phi^2}{2}\right)^2\right]$$

$$; K = \int_0^1 dx \log\left(1 + x(1-x)\frac{4\mu^2}{3m^2}\right) \qquad \text{in "Another RC."}$$

$$= \int_0^1 dx \log\left(\frac{m^2}{\mu^2} + x(1-x)\frac{4}{3}\right) - \log\left(\frac{m^2}{\mu^2}\right)$$

$$= \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4$$

$$+ \frac{1}{4(4\pi)^2}\left[\left(m^4 + \lambda m^2\phi^2\right)\log\left(1 + \frac{\lambda\phi^2}{2m^2}\right) - \frac{\lambda}{2}m^2\phi^2\right]$$

$$+ \frac{1}{4(4\pi)^2}\left(\frac{\lambda\phi^2}{2}\right)^2\left[\log\left(\frac{m^2}{\mu^2} + \frac{\lambda\phi^2}{2\mu^2}\right) - \frac{3}{2} - \int_0^1 dx \log\left(\frac{m^2}{\mu^2} + x(1-x)\frac{4}{3}\right)\right]$$

$\exists\, m \to 0$ limit :

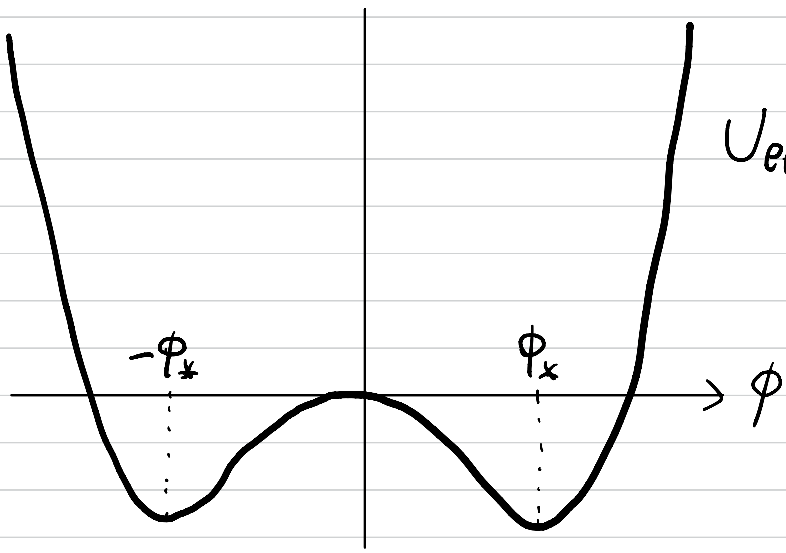
$$U_{eff}(\phi) = \frac{\lambda}{4!}\phi^4 + \frac{1}{4(4\pi)^2}\left(\frac{\lambda\phi^2}{2}\right)^2\left(\log\left(\frac{\lambda\phi^2}{2\mu^2}\right) + C\right)$$

$$C = -\frac{3}{2} - \int_0^1 dx \log\left(x(1-x)\frac{4}{3}\right) = \frac{1}{2} - \log\frac{4}{3}$$

$$U_{eff}'(\phi) = 4\phi^3 \underbrace{\left(\frac{\lambda}{4!} + \frac{1}{4(4\pi)^2}\left(\frac{\lambda}{2}\right)^2 \left(\log\left(\frac{\lambda\phi^2}{2\mu^2}\right) + C + \frac{1}{2} \right)\right)}_{\circledast}$$

$$\overset{!}{=} 0 \iff \phi = 0\,,\ \pm\phi_* \leftarrow \text{zero of } \circledast$$

$$U_{eff}(0) = 0\,, \quad U_{eff}(\pm\phi_*) = -\frac{1}{8(4\pi)^2}\left(\frac{\lambda\phi_*^2}{2}\right)^2 < 0$$



$\phi = \pm\phi_*$ are the minimum. Are these the VEV of ϕ?

$Z_2 : \phi \to -\phi$ is spontaneously broken?

$\circledast = 0$: Cancellation of tree and 1-loop correction.

$$\Rightarrow \ |\text{tree}| = |\text{1-loop}|$$

$$\boxed{\begin{array}{l}\text{Perturbation theory is invalid.} \\ \text{You cannot trust this conclusion.}\end{array}}$$

But then, what is the real conclusion?

$\mathbb{Z}_2$ symmetry is spontaneously broken or not?

→ <u>We can answer to this using RG!</u>

RGE on $\Gamma(\phi, \lambda; \mu)$ ⇒

$$\left(\mu \frac{\partial}{\partial \mu} + \beta(\lambda) \frac{\partial}{\partial \lambda} - \gamma(\lambda) \phi \frac{\partial}{\partial \phi} \right) U_{eff}(\phi, \lambda; \mu) = 0$$

If $\bar{\lambda}(t)$, $\bar{\phi}(t)$ solve $\frac{d\lambda}{dt} = \beta(\lambda)$, $\frac{d\phi}{dt} = -\gamma(\lambda)\phi$,

RGE ⟺ $U_{eff}(\bar{\phi}(t), \bar{\lambda}(t); e^t \mu)$ is t-independent.

By dimensional analysis, $U_{eff}(\phi, \lambda; \mu) = \phi^4 u(\phi/\mu, \lambda)$.

At 1-loop, $\beta = \frac{3\lambda^2}{(4\pi)^2}$, $\gamma = 0$,

$$u(\phi/\mu, \lambda) = \frac{\lambda}{4!} + \frac{1}{4(4\pi)^2} \left(\frac{\lambda}{2}\right)^2 \left(\log\left(\frac{1}{2}\left(\frac{\phi}{\mu}\right)^2\right) + C \right).$$

⇝ $\bar{\lambda}(t) = \dfrac{\lambda}{1 - \frac{3t}{(4\pi)^2}\lambda}$, $\bar{\phi}(t) = \phi$ for $(\bar{\lambda}(0), \bar{\phi}(0)) = (\lambda, \phi)$.

Let t_ϕ be s.t. $\frac{1}{2}\left(\frac{\phi}{e^{t_\phi}\mu}\right)^2 = 1$. i.e. $t_\phi = \frac{1}{2}\log\left(\frac{\phi^2}{2\mu^2}\right)$.

Then $$\bar{\lambda}(t_\phi) = \frac{\lambda}{1 - \frac{3\lambda}{2(4\pi)^2}\log\left(\frac{\phi^2}{2\mu^2}\right)} =: \lambda(\phi)$$

$$U_{eff}(\phi, \lambda; \mu) = U_{eff}\big(\underbrace{\bar{\phi}(t_\phi)}_{\phi}, \underbrace{\bar{\lambda}(t_\phi)}_{\lambda(\phi)}; e^{t_\phi}\mu\big)$$

$$= \phi^4\, u\left(\frac{\phi}{e^{t_\phi}\mu}, \lambda(\phi)\right)$$

$$= \phi^4 \left[\frac{\lambda(\phi)}{4!} + \frac{1}{4(4\pi)^2}\left(\frac{\lambda(\phi)}{2}\right)^2\left(\log\left(\frac{\lambda(\phi)}{2}\underbrace{\left(\frac{\phi}{e^{t_\phi}\mu}\right)^2}_{=1}\right) + C\right) + O(\lambda(\phi)^3)\right]$$

$$= \frac{\lambda(\phi)}{4!}\phi^4 + \frac{1}{4(4\pi)^2}\left(\frac{\lambda(\phi)\phi^2}{2}\right)^2\left(\log\lambda(\phi) + C\right) + \cdots$$

This is better and better as $|\phi/\mu| \to 0$ where $\lambda(\phi) \to 0$

[In contrast to $U_{eff} = \frac{\lambda}{4!}\phi^4 + \frac{1}{4(4\pi)^2}\left(\frac{\lambda\phi^2}{2}\right)^2\left(\log\frac{\lambda\phi^2}{2\mu^2} + C\right)$

which breaks down as $|\phi/\mu| \to 0$.]

RG improvement of perturbation theory !

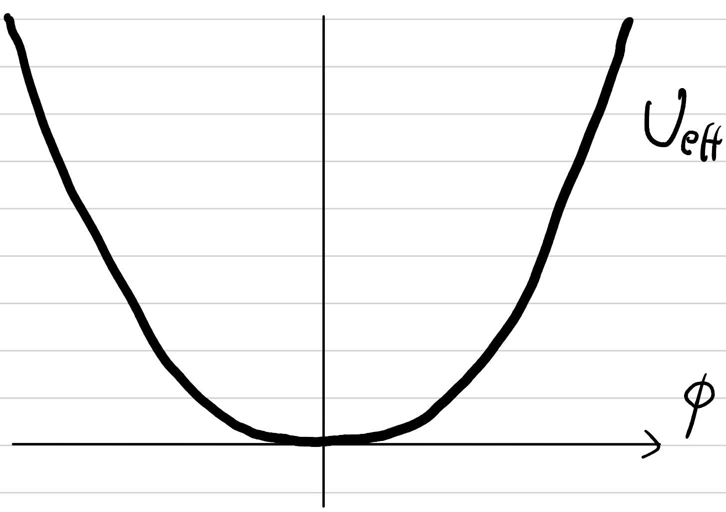
Where is the minimum?

$$0 = U'_{eff} = 4\phi^3 \left[\frac{\lambda(\phi)}{4!} + \frac{1}{4(4\pi)^2}\left(\frac{\lambda(\phi)}{2}\right)^2 \left(\log \lambda(\phi) + C\right) + O(\lambda(\phi)^3) \right]$$

$$+ \ \phi^4 \left(\frac{1}{4!} + O(\lambda(\phi)) \right) \underbrace{\frac{d\lambda(\phi)}{d\phi}}_{\frac{3\lambda(\phi)^2}{(4\pi)^2}\phi^{-1}}$$

$$= \phi^3 \lambda(\phi) \underbrace{\left[\frac{1}{3!} + \frac{\lambda(\phi)}{4(4\pi)^2} \underbrace{\left(\log \lambda(\phi) + C + \frac{1}{2} \right)}_{\log\left(\frac{3\lambda(\phi)}{4}\right) + 1} + O(\lambda(\phi)^2) \right]}_{\text{positive } \forall \lambda(\phi) \geq 0}$$

$\phi = 0$ is the unique minimum! (at least in the range $\lambda(\phi) \ll 1$)



$\exists$ a unique $\mathbb{Z}_2$ symmetric vacuum.

No spontaneous $\mathbb{Z}_2$ symmetry breaking!