Renormalization of QED

(We work in Euclidean signature throughout. Suppress "E")

$$\mathcal{L} = \frac{1}{4e^2} F^{\mu\nu} F_{\mu\nu} + \bar{\psi}(-i\not{D}_A + m)\psi + \frac{1}{2e^2}(\partial\cdot A)^2 + \bar{c}\,\partial^2 c$$

decouple

↓

drop.

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\not{D}_A \psi = \gamma^\mu D_\mu \psi = \gamma^\mu(\partial_\mu + iA_\mu)\psi$$

$\exists$ gauge symmetry $\quad \delta A_\mu = -\partial_\mu \alpha, \quad \delta\psi = i\alpha\psi, \quad \delta\bar{\psi} = \bar{\psi}(-i\alpha)$

which is broken just by the gauge fixing term :

$$\delta\mathcal{L} = \frac{1}{e^2}(-\partial^2\alpha)\,\partial\cdot A.$$

Let us rescale the variables $A_\mu \to e A_\mu$.

$$\mathcal{L} = \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \bar{\psi}(-i\not{\partial} + m)\psi + e\bar{\psi}\not{A}\psi + \frac{1}{2}(\partial\cdot A)^2.$$

$$\delta A_\mu = -\frac{1}{e}\partial_\mu \alpha, \quad \delta\psi = i\alpha\psi, \quad \delta\bar{\psi} = \bar{\psi}(-i\alpha)$$

$$\Rightarrow \quad \delta\mathcal{L} = -\frac{1}{e}\partial^2\alpha\,\partial\cdot A.$$

Note:

$$S = \int d^4x \Big(\underbrace{\tfrac{1}{4} F^{\mu\nu} F_{\mu\nu} + \tfrac{1}{2} (\partial \cdot A)^2 + \bar{\psi}(-i\not{\partial} + m)\psi}_{\text{free part}} + \underbrace{e \bar{\psi} \not{A} \psi}_{\text{interaction}} \Big)$$

$$A_\mu(x) = \int \frac{d^4p}{(2\pi)^4} e^{-ipx} A_\mu(p)$$

$$\psi(x) = \int \frac{d^4p}{(2\pi)^4} e^{-ipx} \psi(p), \quad \bar{\psi}(x) = \int \frac{d^4x}{(2\pi)^4} e^{-ipx} \bar{\psi}(p)$$

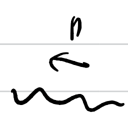
$$= \int \frac{d^4p}{(2\pi)^4} \Big[\tfrac{1}{2} A_\mu(-p) (\delta^{\mu\nu} p^2 - \cancel{p^\mu p^\nu}) A_\nu(p) + \tfrac{1}{2} A_\mu(-p) \cancel{p^\mu p^\nu} A_\nu(p)$$

$$+ \bar{\psi}(-p)(-\not{p} + m)\psi(p)$$

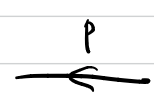
$$+ \int \frac{d^4p}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} \bar{\psi}(-p-q)\, e \gamma^\mu \psi(p) A_\nu(q)$$

Free propagators:

$$\overbrace{A_\mu(x) A_\nu(y)} = \int \frac{d^4p}{(2\pi)^4} \frac{\delta_{\mu\nu}}{p^2} e^{-ip(x-y)}$$



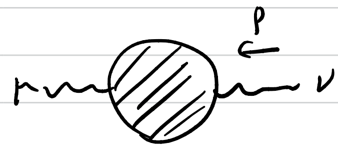
$$\overbrace{\psi(x) \bar{\psi}(y)} = \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip(x-y)}}{-\not{p} + m} = \int \frac{d^4p}{(2\pi)^4} \frac{\not{p} + m}{p^2 + m^2} e^{-ip(x-y)}$$



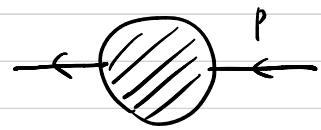
cf $\{\gamma^\mu, \gamma^\nu\} = -2\delta^{\mu\nu}$

Full propagators

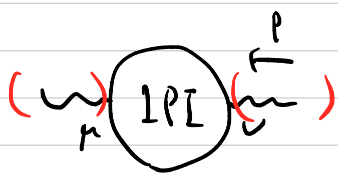
$$\langle A_\mu(x) A_\nu(y) \rangle = \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} G_{\mu\nu}(p)$$



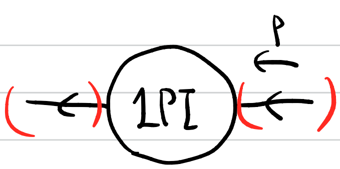
$$\langle \psi(x) \bar{\psi}(y) \rangle = \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} S(p)$$



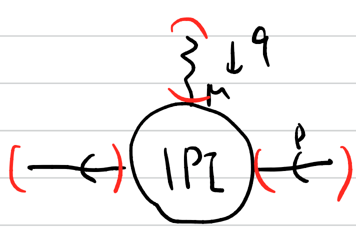
Special 1PI vertices



$$=: \Pi^{\mu\nu}(p)$$

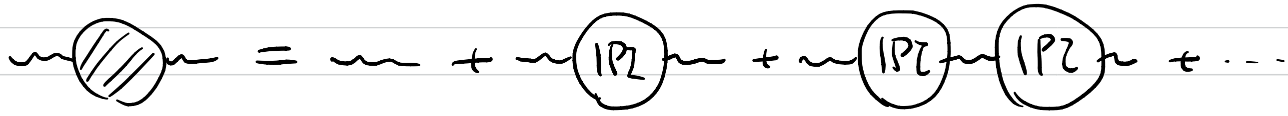


$$=: -\Sigma(p)$$



$$=: -e\Gamma^\mu(p,q)$$

Relationship

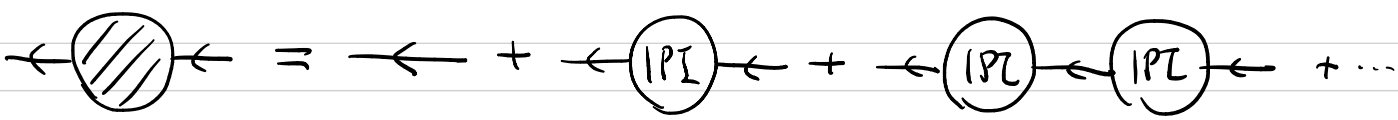


$$= \sim \left(1 - \text{1PI} \sim\right)^{-1} = \frac{1}{p^2}\left(1 - \Pi(p)\frac{1}{p^2}\right)^{-1}$$

$$= \left[\left(1 - \Pi(p)\frac{1}{p^2}\right)p^2\right]^{-1} = \left(p^2 - \Pi(p)\right)^{-1}$$

$$\therefore \quad G(p) = \left(p^2 - \Pi(p)\right)^{-1}$$

i.e. $\left(\delta^{\mu\nu}p^2 - \Pi^{\mu\nu}(p)\right)G_{\nu\rho}(p) = \delta^{\mu}_{\rho}$

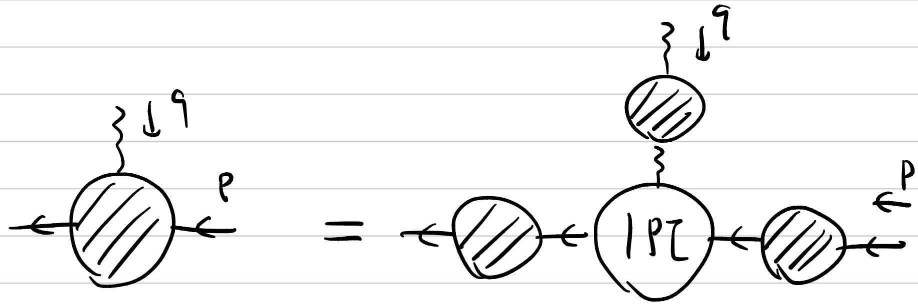


$$= \leftarrow \left(1 - \text{1PI} \leftarrow\right)^{-1} = \frac{1}{-\not{p}+m}\left(1 + \Sigma(p)\frac{1}{-\not{p}+m}\right)^{-1}$$

$$= \left[\left(1 + \Sigma(p)\frac{1}{-\not{p}+m}\right)(-\not{p}+m)\right]^{-1} = \left(-\not{p}+m+\Sigma(p)\right)^{-1}$$

$$\therefore \quad S(p) = \left(-\not{p}+m+\Sigma(p)\right)^{-1}$$

i.e. $S(p)^{-1} = -\not{p}+m+\Sigma(p)$.



$$\Rightarrow \langle \psi(x) \bar{\psi}(y) A_\nu(z) \rangle$$

$$= \int \frac{d^4p}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} \, e^{-i(p+q)x + ipy + iqz} \times$$

$$S(p+q)\left(-e\Gamma^\mu(p,q)\right) S(p) \, G_{\mu\nu}(q)$$

Ward identities

For gauge symmetry $\delta A_\mu = -\frac{1}{e}\partial_\mu \alpha$, $\delta \psi = i\alpha\psi$, $\delta\bar\psi = \bar\psi(-i\alpha)$,

$$\delta S = \int d^4x \, \frac{1}{e}(-\partial^2\alpha)\,\partial\cdot A \,.$$

• $$0 = \frac{1}{Z}\int \delta\left(\mathcal{D}\text{fields}\; e^{-S} A_\mu(x)\right)$$

$$= \int d^4y \, \frac{1}{e}\partial^2\alpha(y)\langle \partial\cdot A(y)\, A_\mu(x)\rangle - \frac{1}{e}\partial_\mu\alpha(x)$$

$$\alpha(x) = e^{-iqx}$$

$$= \int d^4y\, \frac{1}{e}\left(-q^2 e^{-iqy}\right)\underbrace{\langle \partial\cdot A(y)\, A_\mu(x)\rangle}_{\int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)}\, ip^\nu G_{\mu\nu}(p)} + \frac{1}{e}\, i q_\mu e^{-iqx}$$

$$= \frac{1}{e}(-q^2)\, e^{-iqx}\, i q^\nu G_{\mu\nu}(q) + \frac{1}{e}\, i q_\mu e^{-iqx}$$

$$\boxed{q^2 G_{\mu\nu}(q)\, q^\nu = q_\mu}$$

$$\left(\delta^{\mu\nu}q^2 - \Pi^{\mu\nu}(q)\right)G_{\nu\rho}(q) = \delta^\mu_\rho$$

$$\boxed{\Pi^{\mu\nu}(q)\, q_\nu = 0}$$

By Euclidean symmetry, $\Pi^{\mu\nu}(q) = \delta^{\mu\nu} X(q^2) + q^\mu q^\nu Y(q^2)$.

$$\Rightarrow \quad q^\mu X(q^2) + q^\mu q^2 Y(q^2) = 0$$

$$\Rightarrow \quad \boxed{\Pi^{\mu\nu}(q) = \left(\delta^{\mu\nu} q^2 - q^\mu q^\nu\right) \Pi(q^2)}$$

- $$0 = \frac{1}{Z} \int \delta\left(\mathcal{D}\text{fields } e^{-S} A_{\mu_1}(x_1) A_{\mu_2}(x_2) A_{\mu_3}(x_3)\right)$$

$$= \int d^4y \, \frac{1}{e} \partial^2 \alpha(y) \left\langle \partial \cdot A(y) A_{\mu_1}(x_1) A_{\mu_2}(x_2) A_{\mu_3}(x_3) \right\rangle$$

$$- \frac{1}{e} \partial_{\mu_1} \alpha(x_1) \left\langle A_{\mu_2}(x_2) A_{\mu_3}(x_3) \right\rangle - 2 \text{ other terms}$$

decomposition into connected parts

$$= \int d^4y \, \frac{1}{e} \partial^2 \alpha(y) \Big\{ \left\langle \partial \cdot A(y) A_{\mu_1}(x_1) A_{\mu_2}(x_2) A_{\mu_3}(x_3) \right\rangle_{\text{conn}}$$

$$+ \left\langle \partial \cdot A(y) A_{\mu_1}(x_1) \right\rangle \left\langle A_{\mu_2}(x_2) A_{\mu_3}(x_3) \right\rangle + 2 \text{ other permutations} \Big\}$$

$$- \frac{1}{e} \partial_{\mu_1} \alpha(x_1) \left\langle A_{\mu_2}(x_2) A_{\mu_3}(x_3) \right\rangle - 2 \text{ other terms}$$

$$\left[\quad \int d^4y \, \partial^2 \alpha(y) \left\langle \partial \cdot A(y) A_\mu(x) \right\rangle = \partial_\mu \alpha(x) \right.$$

$$= \int d^4y \, \frac{1}{e} \partial^2 \alpha(y) \left\langle \partial \cdot A(y) A_{\mu_1}(x_1) A_{\mu_2}(x_2) A_{\mu_3}(x_3) \right\rangle_{\text{conn}}$$

$$\boxed{\begin{array}{l} \therefore \quad \langle \partial\cdot A(y)\, A_{\mu_1}(x_1)\, A_{\mu_2}(x_2)\, A_{\mu_3}(x_3) \rangle_{\text{conn}} = 0 \\ \text{Similarly } \langle \partial\cdot A(x_1)\, A_{\mu_2}(x_2) \cdots A_{\mu_s}(x_s) \rangle_{\text{conn}} = 0 \quad \forall \text{ even } s. \end{array}}$$

- Charge conjugation symmetry

S is invariant under

$$A_\mu \to -A_\mu$$

$$\psi \to C\bar{\psi}^T, \quad \bar{\psi} \to \psi^T(-C^{-1})$$

for $C^{-1}\gamma^\mu C = -(\gamma^\mu)^T$

$$\rightsquigarrow \boxed{\langle A_{\mu_1}(x_1) \cdots A_{\mu_s}(x_s) \rangle = 0 \quad \text{if } s \text{ is odd.}}$$

- Fermion number symmetry

S is invariant under $A_\mu \to A_\mu$, $\psi \to e^{i\alpha}\psi$, $\bar{\psi} \to \bar{\psi} e^{-i\alpha}$

$$\rightsquigarrow \boxed{\langle \psi_{\alpha_1}(x_1) \cdots \psi_{\alpha_s}(x_s)\, \bar{\psi}^{\beta_1}(y_1) \cdots \bar{\psi}^{\beta_t}(y_t) \rangle = 0 \quad \text{if } s \neq t}$$

• Back to gauge symmetry :

$$0 = \frac{1}{Z}\int \delta\left(\mathcal{D}\text{fields}\, e^{-S}\, \psi(x)\,\bar{\psi}(y)\right)$$

$$= \int d^4z\, \frac{1}{e}\, \partial^2\alpha(z)\, \langle \partial\cdot A(z)\, \psi(x)\, \bar{\psi}(y)\rangle$$

$$+\, i\left(\alpha(x) - \alpha(y)\right)\langle \psi(x)\bar{\psi}(y)\rangle$$

$\alpha(x) = e^{-iqx}$

$\swarrow$

$$= \int d^4z\, \frac{1}{e}\left(-q^2 e^{-iqz}\right)\langle \partial\cdot A(z)\, \psi(x)\, \bar{\psi}(y)\rangle$$

$$+\, i\left(e^{-iqx} - e^{-iqy}\right)\overbrace{\langle \psi(x)\bar{\psi}(y)\rangle}^{\int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} S(p)}$$

$$\int \frac{d^4p}{(2\pi)^4}\frac{d^4k}{(2\pi)^4}\, e^{-i(p+k)x + ikz + ipy}\, S(p+k)\left(-e\,\Gamma^\mu(p,k)\right)S(p)\, G_{\mu\nu}(k)\, ik^\nu$$

$$= q^2 \int \frac{d^4p}{(2\pi)^4}\, e^{-i(p+q)x + ipy}\, S(p+q)\,\Gamma^\mu(p,q)\, S(p)\, G_{\mu\nu}(q)\, iq^\nu$$

$$+\, i\int \frac{d^4p}{(2\pi)^4}\, e^{-i(p+q)x + ipy}\left(S(p) - S(p+q)\right)$$

As $q^2 G_{\mu\nu}(q)\, q^\nu = q_\mu$, this yields

$$S(p+q)\,\Gamma^\mu(p,q)\, S(p)\, q_\mu = S(p+q) - S(p).$$

Multiplying $S(p+q)^{-1}$ from the left & $S(p)^{-1}$ from the right,

$$q_\mu \Gamma^\mu(p,q) = S(p)^{-1} - S(p+q)^{-1}$$

$$= \left(-\cancel{\not p}+\cancel{m}+\Sigma(p)\right) - \left(-(\cancel{\not p}+\not q)+\cancel{m}+\Sigma(p+q)\right)$$

$$\boxed{\therefore \quad q_\mu \Gamma^\mu(p,q) = \not q + \Sigma(p) - \Sigma(p+q).}$$

Power counting in QED

E_ψ = # external $\psi, \bar{\psi}$ lines, I_ψ = # internal $\overbrace{\psi\bar{\psi}}$'s

E_A = # external A lines, I_A = # internal $\overbrace{AA}$'s

V = # vertices, L = # loops

Superficial degree of divergence

$$D = 4L - I_\psi - 2 I_A$$

$$L = I - V + 1 = I_\psi + I_A - V + 1$$

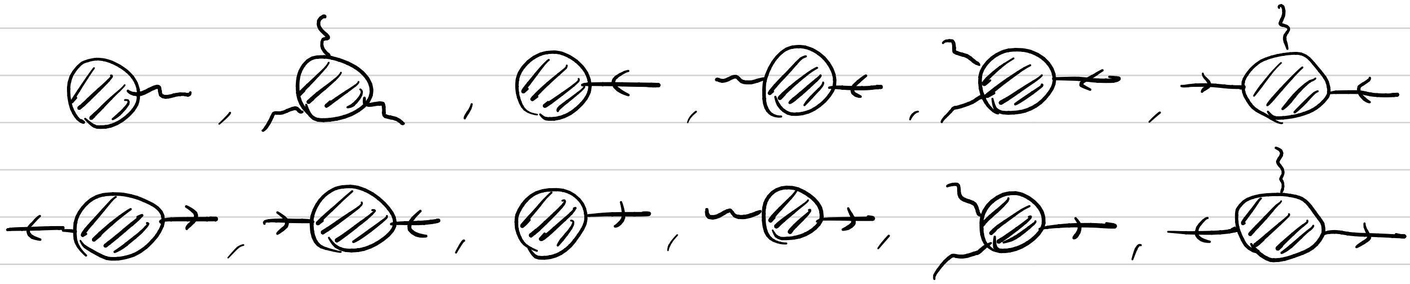
$$V = 2 I_A + E_A = \frac{1}{2}(2 I_\psi + E_\psi)$$

$$\begin{aligned} D &= 4(I_\psi + I_A - V + 1) - I_\psi - 2 I_A \\ &= 3 I_\psi + 2 I_A - 4V + 4 \\ &= 3\left(V - \tfrac{1}{2} E_\psi\right) + (V - E_A) - 4V + 4 \\ &= 4 - \frac{3}{2} E_\psi - E_A \end{aligned}$$

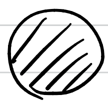
Which amplitudes are superficially divergent?

$E_\psi \leq 2$, $E_A \leq 4$, ...

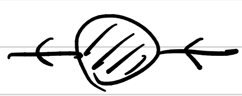
Charge conjugation & fermion # symmetry exclude

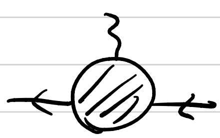
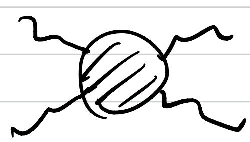


The superficially divergent amplitudes are :

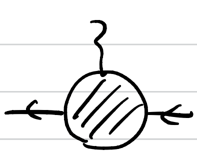
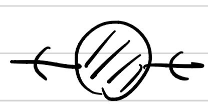
D = 4 : 

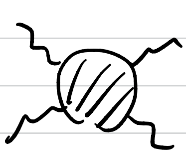
D = 2 : 

D = 1 : 

D = 0 :  , 

 : vacuum energy shift. ~ omit.

 : related to  by Ward identity.

 : not divergent by Ward identity.

see below.

To control the divergence, we regularize the system.

The regularization must respect the gauge symmetry.

For example, a naive momentum cut-off, such as

$$\frac{1}{-\not{p}+m} \rightsquigarrow \frac{1}{-\not{p}+m} e^{-\frac{p^2+m^2}{\Lambda^2}}$$

$$\Leftrightarrow \quad \bar{\psi}(-i\not{\partial}+m)\psi \rightsquigarrow \bar{\psi}(-i\not{\partial}+m)\, e^{\frac{\partial^2+m^2}{\Lambda^2}}\psi$$

breaks the gauge symmetry and is NOT GOOD.

Examples of good regularizations :

- Pauli-Villars regularization

  Introduce regulator fields and interaction so that the original gauge symmetry extends to the regularized system.

  (We'll use this later.)

- dimensional regularization

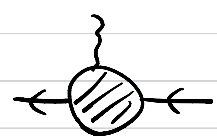
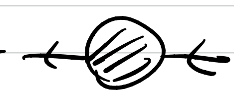
Once you have a gauge invariant regularized system, the gauge fixing can be done as usual.

Then, the gauge symmetry is broken only by the gauge fixing term,

$$\delta S_{reg} = \int d^4x \, \frac{1}{e} (-\partial^2 \alpha) \, \partial \cdot A .$$

⇝ the Ward identities we derived remains to hold, and have important constraints on divergences:

- $q_\mu \Gamma^\mu(p, q) = \not{q} + \Sigma(p) - \Sigma(p+q)$.

  ⇝ the divergent (as $\Lambda \to \infty$) part of  is expressed by the divergent part of  .

- $\langle \partial \cdot A(y) \, A_{\mu_1}(x_1) \, A_{\mu_2}(x_2) \, A_{\mu_3}(x_3) \rangle_{conn} = 0$

$$\langle A_{\mu_1}(x_1) \cdots A_{\mu_4}(x_4) \rangle_{conn} = \int \prod_{i=1}^{4} \frac{d^4 p_i}{(2\pi)^4} \, e^{-i p_i x_i} \, G^{conn}_{\mu_1 \cdots \mu_4}(p_1, \cdots, p_4)$$

$$\Rightarrow \quad p_1^{\mu_1} \, G^{conn}_{\mu_1 \mu_2 \mu_3 \mu_4}(p_1, p_2, p_3, p_4) = 0 .$$

By Euclidean symmetry,

$$p^2 = \{ p_i \cdot p_j \}_{1 \le i,j \le 4}$$

$$
\begin{aligned}
G^{\text{conn}}_{\mu_1 \cdots \mu_4}(p_1, \cdots, p_4)
&= \delta_{\mu_1\mu_2}\delta_{\mu_3\mu_4} G^{(12)(34)}(p^2) + 2 \text{ other permutations} \\
&\quad + \sum_{1 \le i,j \le 4} \left(p_{i\mu_1} p_{j\mu_2} \delta_{\mu_3\mu_4} G^{(i1)(j2)}(p^2) + 5 \text{ other permutations} \right) \\
&\quad + \sum_{1 \le i,j,k,l \le 4} p_{i\mu_1} p_{j\mu_2} p_{k\mu_3} p_{l\mu_4} G^{ijkl}(p^2).
\end{aligned}
$$

By dimensional analysis, only $G^{(12)(34)}(p^2)$ can be divergent,

$G^{(ia)(jb)}(p^2)$ & $G^{ijkl}(p^2)$ are finite.

Ward identity:

$$
\begin{aligned}
0 &= p_1^{\mu_1} G^{\text{conn}}_{\mu_1 \cdots \mu_4}(p_1, \cdots, p_4) \\
&= \left.\begin{aligned}
& p_{1\mu_2}\delta_{\mu_3\mu_4} G^{(12)(34)}(p^2) + p_{1\mu_3}\delta_{\mu_2\mu_4} G^{(13)(24)}(p^2) + p_{1\mu_4}\delta_{\mu_2\mu_3} G^{(14)(23)}(p^2) \\
& + \sum_{i,j} \left(p_1 \cdot p_i \, p_{j\mu_2} \delta_{\mu_3\mu_4} G^{(i1)(j2)}(p^2) + 2 \text{ others} \right)
\end{aligned}\right] = 0 \\
&\quad \left.\begin{aligned}
& + \sum_{i,j} \left(p_{i\mu_2} p_{j\mu_3} p_{1\mu_4} G^{(i2)(j3)}(p^2) + 2 \text{ others} \right) \\
& + \sum_{i,j,k,l} (p_1 \cdot p_i) \, p_{j\mu_2} p_{k\mu_3} p_{l\mu_4} G^{ijkl}(p^2)
\end{aligned}\right] = 0
\end{aligned}
$$

$\mu_3 = \mu_4 \neq \mu_2$:

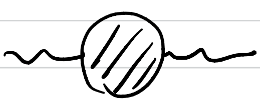
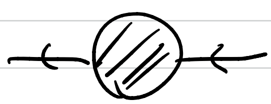
$$0 = p_{1\mu_2} G^{(12)(34)}(p^2) + \sum_{i,j} p_1 \cdot p_i \, p_{j\mu_2} G^{(i1)(j2)}(p^2) = 0$$

$$\Rightarrow \quad G^{(12)(34)}(p^2) = - \sum_i p_1 \cdot p_i \, G^{(i,1)(1,2)}(p^2) \Leftarrow \text{finite.}$$

Similarly, $G^{(13)(24)}(p^2)$ and $G^{(14)(23)}(p^2)$ are also finite.

$\therefore$ $G^{\text{conn}}_{\mu_1 \cdots \mu_4}(p_1, \cdots, p_4)$ are finite.

Thus, only

 & 

have independent divergences.

One-loop computation

$$\langle A_\rho(x)\, A_\lambda(y)\rangle = A_\rho(x)\, A_\lambda(y)$$

$$\underbrace{+\frac{1}{2}(-e)^2 \int d^4z_1\, d^4z_2 \left\langle A_\rho(x) (\bar\psi \not{A} \psi)(z_1)(\bar\psi \not{A}\psi)(z_2)\, A_\lambda(y) \right\rangle_{free}}_{\circledast} + \cdots$$

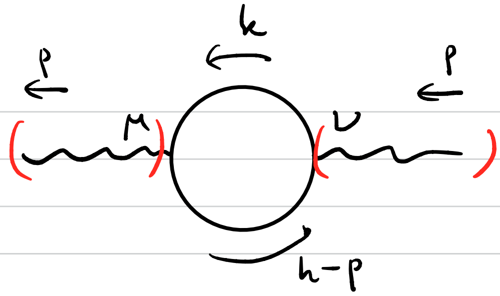
$$\circledast = \frac{e^2}{2}\int d^4z_1\, d^4z_2\, A_\rho(x)\, A_\mu(z_1)\, (\bar\psi \gamma^\mu \psi)(z_1)\, A_\nu(z_2)\, (\bar\psi\gamma^\nu\psi)(z_2)\, A_\lambda(y) \times 2$$

move

$$= e^2 \int d^4z_1\, d^4z_2\, \underbrace{A_\rho(x)\, A_\mu(z_1)}_{\int \frac{d^4p_1}{(2\pi)^4} \frac{\delta_{\rho\mu}\, e^{-ip_1(x-z_1)}}{p_1^2}}\, (-\mathrm{tr})\left(\gamma^\mu \underbrace{\psi(z_1)\, \bar\psi(z_2)}_{\int \frac{d^4k_1}{(2\pi)^4} \frac{e^{-ik_1(z_1-z_2)}}{-\not{k}_1+m}}\, \gamma^\nu \underbrace{\psi(z_2)\, \bar\psi(z_1)}_{\int \frac{d^4k_2}{(2\pi)^4} \frac{e^{-ik_2(z_2-z_1)}}{-\not{k}_2+m}}\right) \underbrace{A_\nu(z_2)\, A_\lambda(y)}_{\int \frac{d^4p_2}{(2\pi)^4} \frac{\delta_{\nu\lambda}\, e^{-ip_2(z_2-y)}}{p_2^2}}$$

$$\left[\begin{array}{l} \int d^4z_1 \Rightarrow (2\pi)^4 \delta(p_1 - k_1 + k_2), \quad \int d^4z_2 \Rightarrow (2\pi)^4 \delta(k_1 - k_2 - p_2) \\ \quad \rightsquigarrow (2\pi)^8 \delta(p_1 - p_2)\, \delta(k_2 - (k_1 - p)) \end{array}\right.$$

$$= \int \frac{d^4p}{(2\pi)^4} \frac{\delta_{\rho\mu}\, e^{-ipx}}{p^2}\; \underbrace{e^2 \int \frac{d^4k}{(2\pi)^4} (-\mathrm{tr})\left(\gamma^\mu \frac{1}{-\not{k}+m}\, \gamma^\nu \frac{1}{-(\not{k}-\not{p})+m}\right)}_{=:\, \Pi_2^{\mu\nu}(p)}\; \frac{\delta_{\nu\lambda}\, e^{ipy}}{p^2}$$

$\Pi_2^{\mu\nu}(p) =$ 

$$= -e^2\int\frac{d^4k}{(2\pi)^4}\,\mathrm{tr}\left(\gamma^\mu\,\frac{1}{-\not{k}+m}\,\gamma^\nu\,\frac{1}{-(\not{k}-\not{p})+m}\right)$$

$$= -e^2\int\frac{d^4k}{(2\pi)^4}\,\mathrm{tr}\left(\gamma^\mu\,\frac{\not{k}+m}{k^2+m^2}\,\gamma^\nu\,\frac{\not{k}-\not{p}+m}{(k-p)^2+m^2}\right)$$

$$\mathrm{tr}\left(\gamma^\mu(\not{k}+m)\gamma^\nu((\not{k}-\not{p})+m)\right)$$

$$= \mathrm{tr}(\gamma^\mu\gamma^\nu)\,m^2 + \mathrm{tr}(\gamma^\mu\not{k}\gamma^\nu(\not{k}-\not{p}))$$

$$\cdot\ \mathrm{tr}(\gamma^\mu\gamma^\nu) = \frac{1}{2}\mathrm{tr}\{\gamma^\mu,\gamma^\nu\} = -\mathrm{tr}\,\delta^{\mu\nu} = -4\,\delta^{\mu\nu}$$

$$\cdot\ \mathrm{tr}(\gamma^\mu\gamma^\rho\gamma^\nu\gamma^\lambda) = \frac{1}{2}\mathrm{tr}(\underbrace{\gamma^\mu\gamma^\rho\gamma^\nu\gamma^\lambda}) + \frac{1}{2}\mathrm{tr}(\cancel{\gamma^\rho\gamma^\nu}\gamma^\lambda\gamma^\mu)$$

$$\{\gamma^\mu,\gamma^\rho\}\gamma^\nu\gamma^\lambda - \gamma^\rho\{\gamma^\mu,\gamma^\nu\}\gamma^\lambda + \gamma^\rho\gamma^\nu\{\gamma^\mu,\gamma^\lambda\} - \cancel{\gamma^\rho\gamma^\nu\gamma^\lambda\gamma^\mu}$$

$$= -\delta^{\mu\rho}\,\mathrm{tr}(\gamma^\nu\gamma^\lambda) + \delta^{\mu\nu}\,\mathrm{tr}(\gamma^\rho\gamma^\lambda) - \delta^{\mu\lambda}\,\mathrm{tr}(\gamma^\rho\gamma^\nu)$$

$$= 4\delta^{\mu\rho}\delta^{\nu\lambda} - 4\delta^{\mu\nu}\delta^{\rho\lambda} + 4\delta^{\mu\lambda}\delta^{\rho\nu}$$

$$= -4\delta^{\mu\nu}m^2 + 4k^\mu(k-p)^\nu - 4\delta^{\mu\nu}\,k\cdot(k-p) + 4k^\nu(k-p)^\mu$$

$$= 4\left(-\delta^{\mu\nu}(m^2 + k\cdot(k-p)) + k^\mu(k-p)^\nu + k^\nu(k-p)^\mu\right)$$

$$\Pi_2^{\mu\nu}(p) = -4e^2 \int \frac{d^4k}{(2\pi)^4} \frac{-\delta^{\mu\nu}(m^2 + k\cdot(k-p)) + k^\mu (k-p)^\nu + k^\nu (k-p)^\mu}{(k^2+m^2)((k-p)^2+m^2)}$$

- $$\frac{1}{(k^2+m^2)((k-p)^2+m^2)} = \int_0^1 \frac{dx}{\big(\underbrace{(1-x)(k^2+m^2)+x((k-p)^2+m^2)}\big)^2}$$

$$k^2 - 2xpk + xp^2 + m^2$$

$$= (\underbrace{k - xp}_{\ell})^2 + \underbrace{x(1-x)p^2 + m^2}_{\Delta}$$

- numerator $= -\delta^{\mu\nu}\left(m^2 + (\ell + xp)\cdot(\ell + (x-1)p)\right)$

$$+ (\ell+xp)^\mu(\ell+(x-1)p)^\nu + (\ell+xp)^\nu(\ell+(x-1)p)^\mu$$

$$= -\delta^{\mu\nu}\left(m^2 - x(1-x)p^2 + \ell^2\right) + 2\ell^\mu\ell^\nu - 2x(1-x)p^\mu p^\nu$$

$+ \ell$-linear terms

$$\ell^\mu \ell^\nu \simeq \frac{1}{4}\delta^{\mu\nu}\ell^2$$

equality after $\int d^4\ell$ (provided convergent).

$$\simeq -\delta^{\mu\nu}\Big(\underbrace{m^2 + x(1-x)p^2}_{\Delta} + \frac{\ell^2}{2}\Big) + 2x(1-x)\left(\delta^{\mu\nu}p^2 - p^\mu p^\nu\right)$$

If the integral were convergent, $\Pi_2^{\mu\nu}(p)$ would be

$$-4e^2 \int_0^1 dx \int \frac{d^4\ell}{(2\pi)^4} \frac{-\delta^{\mu\nu}\left(\Delta + \frac{\ell^2}{2}\right) + 2x(1-x)\left(\delta^{\mu\nu}p^2 - p^\mu p^\nu\right)}{(\ell^2+\Delta)^2},$$

but this is quadratically & logarithmically divergent.

Pauli-Villars regularization:

above $=: \Pi_2^{\mu\nu}(p,m)$

$$\Pi_{2\,PV}^{\mu\nu}(p) = \sum_i C_i \, \Pi_2^{\mu\nu}(p, M_i) \qquad \left(C_0 = 1,\ M_0 = m \right)$$

so that the integral is convergent.

$\Leftrightarrow$ introduce regulator fields $(\psi_i, \bar{\psi}_i)$ with Lagrangian

$$\Delta \mathcal{L} = \sum_i \bar{\psi}_i \left(-i\not{\partial} + \sqrt{C_i}\, e \not{A} + M_i \right) \psi_i$$

The system preserves gauge invariance: the gauge transformation

$$\delta A_\mu = -\frac{1}{e} \partial_\mu \alpha, \quad \delta \psi = i\alpha\psi, \quad \delta\bar{\psi} = \bar{\psi}(-i\alpha),$$

also does $\delta\psi_i = i\sqrt{C_i}\,\alpha\,\psi_i, \quad \delta\bar{\psi}_i = \bar{\psi}_i\left(-i\sqrt{C_i}\,\alpha\right).$

Then,

$$\Pi_{2\,PV}^{\mu\nu}(p) = -4e^2 \int_0^1 dx \int \frac{d^4 l}{(2\pi)^4} \sum_i C_i \frac{-\delta^{\mu\nu}\left(\Delta_i + \frac{l^2}{2}\right) + 2x(1-x)\left(\delta^{\mu\nu} p^2 - p^\mu p^\nu\right)}{\left(l^2 + \Delta_i\right)^2}$$

with $\Delta_i = x(1-x)p^2 + M_i^2$

$$\frac{1}{(4\pi)^2}$$

$$\int \frac{d^4\ell}{(2\pi)^4} f(\ell^2) = \frac{Vol(S^3)}{2(2\pi)^4} \int_0^\infty \ell^2 d\ell^2 f(\ell^2) = \frac{1}{(4\pi)^2} \int_0^\infty t\,dt\, f(t)\,.$$

$$\Pi^{\mu\nu}_{2,PV}(p) = -\frac{4e^2}{(4\pi)^2} \int_0^1 dx \int_0^\infty t\,dt \sum_i C_i \frac{-\delta^{\mu\nu}\left(\Delta_i + \frac{t}{2}\right) + 2x(1-x)(\delta^{\mu\nu}p^2 - p^\mu p^\nu)}{(t+\Delta_i)^2}$$

- $$\frac{dt\, t^2}{(t+\Delta_i)^2} = dt \frac{(t+\Delta_i - \Delta_i)^2}{(t+\Delta_i)^2} = dt\left(1 - \frac{2\Delta_i}{t+\Delta_i} + \frac{\Delta_i^2}{(t+\Delta_i)^2}\right)$$
$$= d\left(t - 2\Delta_i \log(t+\Delta_i) - \frac{\Delta_i^2}{t+\Delta_i}\right)$$

- $$\frac{dt\, t}{(t+\Delta_i)^2} = \frac{dt}{t+\Delta_i} - \frac{dt\,\Delta_i}{(t+\Delta_i)^2} = d\left(\log(t+\Delta_i) + \frac{\Delta_i}{t+\Delta_i}\right)$$

$$= -\frac{4e^2}{(4\pi)^2}\int_0^1 dx \Bigg[-\frac{1}{2}\delta^{\mu\nu}\sum_i C_i\left(t - 2\Delta_i \log(t+\Delta_i) - \frac{\Delta_i^2}{t+\Delta_i}\right)$$
$$+ \sum_i C_i\left(-\delta^{\mu\nu}\Delta_i + 2x(1-x)(\delta^{\mu\nu}p^2 - p^\mu p^\nu)\right)\left(\log(t+\Delta_i) + \frac{\Delta_i}{t+\Delta_i}\right)\Bigg]_0^\infty$$

The integral is convergent provided

$$\sum_i C_i = 0 \quad \text{and} \quad \sum_i C_i M_i^2 = 0.$$

This is possible with two regulator fields : $i = 0, 1, 2$.

Then,

$$\Pi^{\mu\nu}_{2PV}(p) = -\frac{4e^2}{(4\pi)^2}\int_0^1 dx \Big[\frac{1}{2}\delta^{\mu\nu}\sum_i C_i \left(-2\Delta_i \cancel{\log\Delta_i} - \cancel{\Delta_i}\right)$$

$$- \sum_i C_i \left(-\delta^{\mu\nu}\cancel{\Delta_i} + 2x(1-x)(\delta^{\mu\nu}p^2 - p^\mu p^\nu)\right)\left(\log\Delta_i + \cancel{X}\right) \Big]$$

$$= \frac{8e^2}{(4\pi)^2}\left(\delta^{\mu\nu}p^2 - p^\mu p^\nu\right)\int_0^1 dx\ x(1-x)\sum_i C_i \log\Delta_i$$

$$\sum_i C_i \log\Delta_i = \log\left(m^2 + x(1-x)p^2\right) + \sum_{i=1,2} C_i \log\left(M_i^2 + x(1-x)p^2\right)$$

$$= \log\left(m^2 + x(1-x)p^2\right) + \sum_{i=1,2} C_i \log M_i^2 + O\left(\frac{p^2}{M_1^2}, \frac{p^2}{M_2^2}\right)$$

Define $M := M_1^{-C_1} M_2^{-C_2}$ (mass scale ☺ $-C_1 - C_2 = C_0 = 1$).

Then $\sum_i C_i \log\Delta_i = \log\left(\frac{m^2 + x(1-x)p^2}{M^2}\right) + O\left(\frac{p^2}{M_1^2}, \frac{p_2^2}{M_2^2}\right)$

$$\Pi^{\mu\nu}_{2PV}(p) = \left(\delta^{\mu\nu}p^2 - p^\mu p^\nu\right)\Pi_{2PV}(p^2) \ ;$$

$$\Pi_{2PV}(p^2) = -\frac{8e^2}{(4\pi)^2}\int_0^1 dx\ x(1-x)\log\left(\frac{M^2}{m^2 + x(1-x)p^2}\right)$$