$$\langle \psi(x)\bar{\psi}(y)\rangle_{conn} = \psi(x)\bar{\psi}(y)$$

$$+ \underbrace{\frac{1}{2}(-e)^2\int d^4z_1\, d^4z_2 \left\langle \psi(x)\,(\bar{\psi}\not{A}\psi)(z_1)\,(\bar{\psi}\not{A}\psi)(z_2)\,\bar{\psi}(y)\right\rangle_{free}}_{\circledast} + \cdots$$

$$\circledast = \frac{e^2}{2}\int d^4z_1\, d^4z_2\; \psi(x)\,(\bar{\psi}\not{A}\psi)(z_1)\,(\bar{\psi}\not{A}\psi)(z_2)\,\bar{\psi}(y) \times 2$$

$$= e^2\int d^4z_1\, d^4z_2\; \underbrace{\psi(x)\bar{\psi}(z_1)}_{\int \frac{d^4p_1}{(2\pi)^4}\frac{e^{-ip_1(x-z_1)}}{-\not{p}_1+m}}\gamma^\mu \underbrace{\psi(z_1)\bar{\psi}(z_2)}_{\int \frac{d^4k_1}{(2\pi)^4}\frac{e^{-ik_1(z_1-z_2)}}{-\not{k}_1+m}}\gamma^\nu \underbrace{\psi(z_2)\bar{\psi}(y)}_{\int \frac{d^4p_2}{(2\pi)^4}\frac{e^{-ip_2(z_2-y)}}{-\not{p}_2+m}} \cdot \underbrace{A_\mu(z_1)A_\nu(z_2)}_{\int \frac{d^4k_2}{(2\pi)^4}\frac{\delta_{\mu\nu}e^{-ik_2(z_1-z_2)}}{k_2^2}}$$

$$(2\pi)^4\delta(p_1-k_1-k_2)\,(2\pi)^4\delta(k_1-p_2+k_2)$$

$$= (2\pi)^8\delta(p_1-p_2)\,\delta(k_2+k_1-p_2)$$

$$= \int \frac{d^4p}{(2\pi)^4}\frac{e^{-ipx}}{-\not{p}+m}\; \underbrace{e^2\int \frac{d^4k}{(2\pi)^4}\gamma^\mu \frac{1}{-\not{k}+m}\gamma^\nu \frac{\delta_{\mu\nu}}{(k-p)^2}}_{=:\, -\Sigma_2(p)}\; \frac{e^{ipy}}{-\not{p}+m}$$

$$\Sigma_2(p) = -e^2 \int \frac{d^4k}{(2\pi)^4} \gamma^\mu \frac{1}{-\not{k}+m} \gamma^\nu \frac{\delta_{\mu\nu}}{(k-p)^2}$$

$$= -e^2 \int \frac{d^4k}{(2\pi)^4} \frac{\gamma^\mu(\not{k}+m)\gamma_\mu}{(k^2+m^2)(k-p)^2}$$

$$\left[\gamma^\mu(\not{k}+m)\gamma_\mu = 2\not{k} - 4m \right.$$

$$\left(\begin{aligned} &\gamma^\mu\gamma_\mu = -4 \\ &\gamma^\mu\gamma^\rho\gamma_\mu = \{\gamma^\mu,\gamma^\rho\}\gamma_\mu - \gamma^\rho\gamma^\mu\gamma_\mu \\ &\qquad = -2\delta^{\mu\rho}\gamma_\mu + 4\gamma^\rho = 2\gamma^\rho \end{aligned} \right)$$

$$= e^2 \int \frac{d^4k}{(2\pi)^4} \frac{-2\not{k}+4m}{(k^2+m^2)(k-p)^2}$$

Pauli-Villars regularization

$$\frac{1}{(k-p)^2} \longrightarrow \frac{1}{(k-p)^2} - \frac{1}{(k-p)^2+\Lambda^2}$$

$\Leftrightarrow$ introduce a regulator field B_μ with Lagrangian

$$\Delta\mathcal{L} = \frac{1}{2} B^\mu(-\partial^2+\Lambda^2)B_\mu + ie\bar{\Psi}\not{B}\Psi$$

gauge invariance is preserved with $\delta B = 0$.

- $$\frac{1}{(k^2+m^2)((k-p)^2+\Lambda^2)} = \int_0^1 \frac{dx}{(\underbrace{(1-x)(k^2+m^2)+x((k-p)^2+\Lambda^2)}_{})^2}$$

$$k^2 - 2xpk + xp^2 + (1-x)m^2 + x\Lambda^2$$

$$= (\underbrace{k-xp}_{\ell})^2 + \underbrace{x(1-x)p^2 + (1-x)m^2 + x\Lambda^2}_{\Delta}$$

- denominator $= -2(\cancel{\ell} + x\cancel{p}) + 4m \simeq -2x\cancel{p} + 4m$

$$\Sigma_{2,PV}(p) = \sum_{i=0}^{1} C_i \Sigma_2(p, \Lambda_i) \;;\; (C_0, \Lambda_0) = (1, 0),\ (C_1, \Lambda_1) = (-1, \Lambda)$$

$$= 2e^2 \int_0^1 dx \int \frac{d^4\ell}{(2\pi)^4} \sum_i C_i \frac{-x\cancel{p} + 2m}{(\ell^2 + \Delta_i)^2}$$

$$= \frac{2e^2}{(4\pi)^2} \int_0^1 dx \int_0^\infty t\,dt \sum_i C_i \frac{-x\cancel{p} + 2m}{(t + \Delta_i)^2}$$

$$= \frac{2e^2}{(4\pi)^2} \int_0^1 dx \left[\sum_i C_i \left(\log(t+\Delta_i) + \frac{\Delta_i}{t+\Delta_i} \right) \right]_0^\infty (-x\cancel{p} + 2m)$$

$$= -\frac{2e^2}{(4\pi)^2} \int_0^1 dx \sum_i C_i \left(\log\Delta_i + \cancel{1} \right) (-x\cancel{p} + 2m)$$

$$= -\frac{2e^2}{(4\pi)^2} \int_0^1 dx \log\left(\frac{\Delta_0}{\Delta_1} \right) (-x\cancel{p} + 2m)$$

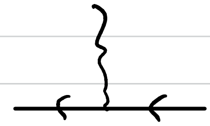
$$\log \frac{\Delta_0}{\Delta_1} = \log\left(\frac{x(1-x)p^2 + (1-x)m^2}{x(1-x)p^2 + (1-x)m^2 + x\Lambda^2} \right)$$

$$= \log\left(\frac{x(1-x)p^2 + (1-x)m^2}{x\Lambda^2} \right) + O\left(\frac{p^2}{\Lambda^2}, \frac{m^2}{\Lambda^2}\right)$$

$$\Sigma_{2,PV}(p) = \frac{2e^2}{(4\pi)^2} \int_0^1 dx \, (-x\not{p} + 2m) \log\left(\frac{x\Lambda^2}{x(1-x)p^2 + (1-x)m^2} \right)$$

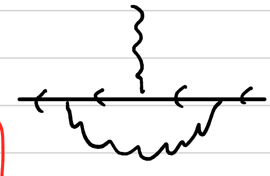
$$\langle \psi(x)\,\bar\psi(y)\,A_\nu(z)\rangle_{1PI}$$

$$= -e\int d^4w\; \psi(x)\bar\psi(w)\,\gamma^\mu A_\mu(w)\,\psi(w)\bar\psi(y)\,A_\nu(z)$$

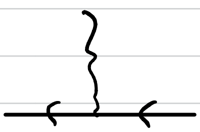


$$+\frac{1}{3!}(-e)^3\int d^4w_1\, d^4w_2\, d^4w_3$$

$$\psi(x)\,(\bar\psi \not A \psi)(w_1)\,(\bar\psi \not A \psi)(w_2)\,(\bar\psi \not A \psi)(w_3)\,\bar\psi(y)\,A_\nu(z)\times 3!$$

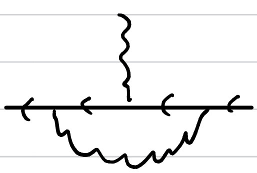


$+\;\cdots$



$$= -e\int d^4w\int \frac{d^4p_1}{(2\pi)^4}\,\frac{e^{-ip_1(x-w)}}{-\not p_1+m}\,\gamma^\mu\int\frac{d^4p_2}{(2\pi)^4}\,\frac{e^{-ip_2(w-y)}}{-\not p_2+m}\int d^4q\,\frac{e^{-iq(w-z)}}{q^2}\,\delta_{\mu\nu}$$

$$=\int\frac{d^4p}{(2\pi)^4}\,\frac{e^{-i(p+q)x}}{-(\not p+\not q)+m}\,(-e\gamma^\mu)\,\frac{e^{ipy}}{-\not p+m}\,\frac{\delta_{\mu\nu}\,e^{iqz}}{q^2}$$



$$=(-e)^3\int d^4w_1\,d^4w_2\,d^4w_3\;\psi(x)\bar\psi(w_1)\gamma^\rho A_\rho(w_1)\,\psi(w_1)\cdots$$

$$\cdots\bar\psi(w_2)\gamma^\mu A_\mu(w_2)\,\psi(w_2)\;\bar\psi(w_3)\gamma^\lambda A_\lambda(w_3)\,\psi(w_3)\cdots$$

$$\cdots\bar\psi(y)\;A_\nu(z)$$

$$= (-e)^3 \int d^4w_1\, d^4w_2\, d^4w_3 \int \frac{d^4p_1}{(2\pi)^4} \frac{e^{-ip_1(x-w_1)}}{-\not{p}_1+m} \gamma^\rho \int \frac{d^4k_1}{(2\pi)^4} \frac{e^{-ik_1(w_1-w_2)}}{-\not{k}_1+m}$$

$$\gamma^\mu \int \frac{d^4k_2}{(2\pi)^4} \frac{e^{-ik_2(w_2-w_3)}}{-\not{k}_2+m} \gamma^\lambda \int \frac{d^4p_2}{(2\pi)^4} \frac{e^{-ip_2(w_2-y)}}{-\not{p}_2+m}$$

$$\int \frac{d^4k_3}{(2\pi)^4} \frac{e^{-ik_3(w_1-w_3)}}{k_3^2} \delta_{\rho\lambda} \int \frac{d^4q}{(2\pi)^4} \frac{e^{-iq(w_2-z)}}{q^2} \delta_{\mu\nu}$$

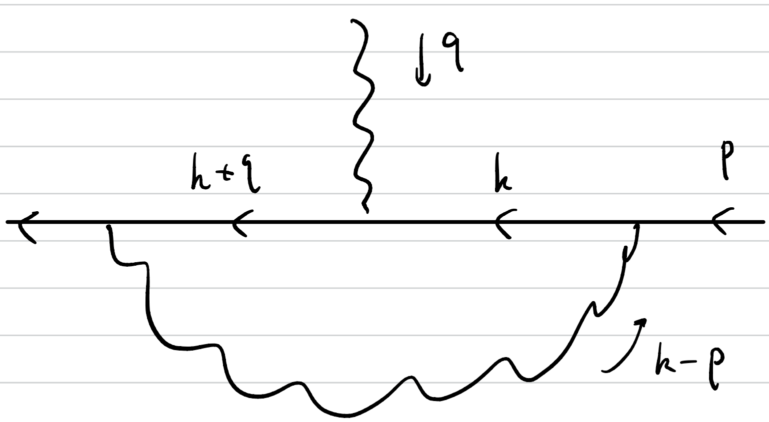
$$\int d^4w_1\, d^4w_2\, d^4w_3 \Rightarrow (2\pi)^4\delta(p_1-k_1-k_3)\,(2\pi)^4\delta(k_1-k_2-q)\,(2\pi)^4\delta(k_2-p_2+k_3)$$

$$= (2\pi)^{4\cdot 3}\,\delta(p_1-p_2-q)\,\delta(k_1-k_2-q)\,\delta(k_2-p_2+k_3)$$

$$= \int \frac{d^4p}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} \frac{e^{-i(p+q)x}}{-(\not{p}+\not{q})+m} \left(-e\,\Gamma_2^\mu(p,q)\right) \frac{e^{ipy}}{-\not{p}+m} \frac{e^{iqz}\,\delta_{\mu\nu}}{q^2},$$

where

$$\Gamma_2^\mu(p,q) = e^2 \int \frac{d^4k}{(2\pi)^4} \gamma^\rho \frac{1}{-(\not{k}+\not{q})+m} \gamma^\mu \frac{1}{-\not{k}+m} \gamma^\lambda \frac{\delta_{\rho\lambda}}{(k-p)^2}$$



$$\Gamma_2^{\mu}(p,q) = e^2 \int \frac{d^4k}{(2\pi)^4} \frac{\gamma^{\rho}\left((\not{k}+\not{q})+m\right)\gamma^{\mu}(\not{k}+m)\gamma_{\rho}}{\left((k+q)^2+m^2\right)\left(k^2+m^2\right)(k-p)^2}$$

$$\gamma^{\rho}\gamma^{\mu}\gamma_{\rho} = \underbrace{\{\gamma^{\rho},\gamma^{\mu}\}}_{-2\delta^{\rho\mu}}\gamma_{\rho} - \gamma^{\mu}\underbrace{\gamma^{\rho}\gamma_{\rho}}_{-4} = 2\gamma^{\mu}$$

$$\gamma^{\rho}\gamma^{\nu}\gamma^{\mu}\gamma_{\rho} = \underbrace{\{\gamma^{\rho},\gamma^{\nu}\}}\gamma^{\mu}\gamma_{\rho} - \gamma^{\nu}\underbrace{\{\gamma^{\rho},\gamma^{\mu}\}}\gamma_{\rho} + \gamma^{\nu}\gamma^{\mu}\underbrace{\gamma^{\rho}\gamma_{\rho}}$$

$$= -2\delta^{\rho\nu}\gamma^{\mu}\gamma_{\rho} + 2\delta^{\rho\mu}\gamma^{\nu}\gamma_{\rho} - 4\gamma^{\nu}\gamma^{\mu}$$

$$= -2\gamma^{\mu}\gamma^{\nu} - 2\gamma^{\nu}\gamma^{\mu} = 4\delta^{\mu\nu}$$

$$\gamma^{\rho}\gamma^{\nu}\gamma^{\mu}\gamma^{\sigma}\gamma_{\rho} = \gamma^{\rho}\gamma^{\nu}\gamma^{\mu}\underbrace{\{\gamma_{\rho},\gamma^{\sigma}\}}_{-2\delta^{\sigma}_{\rho}} - \underbrace{\gamma^{\rho}\gamma^{\nu}\gamma^{\mu}\gamma_{\rho}}_{4\delta^{\mu\nu}}\gamma^{\sigma}$$

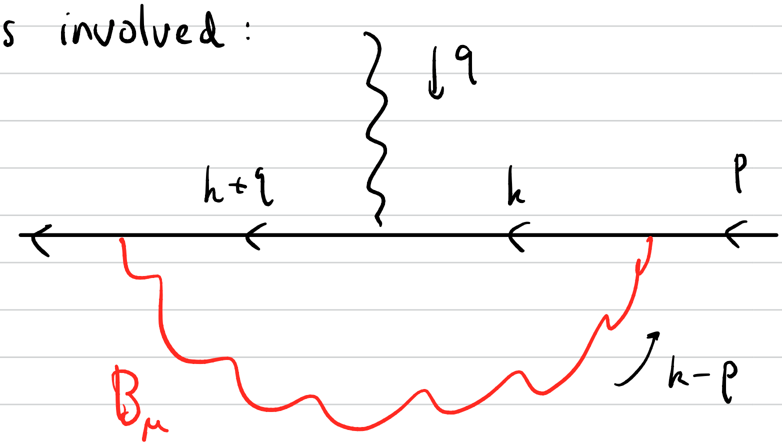
$$= -2\gamma^{\sigma}\underbrace{\gamma^{\nu}\gamma^{\mu}} - 4\delta^{\mu\nu}\gamma^{\sigma} = 2\gamma^{\sigma}\gamma^{\mu}\gamma^{\nu}$$

$\therefore$ nemerator $= 2\not{k}\gamma^{\mu}(\not{k}+\not{q}) + 4m(k+q)^{\mu} + 4m k^{\mu} + 2m^2\gamma^{\mu}$

$$= 2\left[\not{k}\gamma^{\mu}(\not{k}+\not{q}) + 2m(2k+q)^{\mu} + m^2\gamma^{\mu}\right]$$

$$\Gamma_2^{\mu}(p,q) = 2e^2 \int \frac{d^4k}{(2\pi)^2} \frac{\not{k}\gamma^{\mu}(\not{k}+\not{q}) + 2m(2k+q)^{\mu} + m^2\gamma^{\mu}}{\left((k+q)^2+m^2\right)\left(k^2+m^2\right)(k-q)^2}$$

In the Pauli-Villars regularization we have been using, there is another 1-loop diagram where the regulator field B_μ is involved:



As the coupling is $ie\bar{\Psi}\not{B}\Psi$, this will produce $i^2\,\Gamma^\mu(p,q;\Lambda)$ where

$$\Gamma_2^\mu(p,q;\Lambda) := 2e^2\int\frac{d^4k}{(2\pi)^2}\,\frac{\not{k}\gamma^\mu\cancel{(\not{k}+\not{q})}+2m(2k+q)^\mu+m^2\gamma^\mu}{((k+q)^2+m^2)(k^2+m^2)((k-q)^2+\Lambda^2)}$$

In total,

$$\begin{aligned}\Gamma^\mu_{2,PV}(p,q) &= \Gamma_2^\mu(p,q;0)-\Gamma_2^\mu(p,q;\Lambda)\\ &= \sum_{i=0}^{1} c_i\,\Gamma_2^\mu(p,q;\Lambda_i)\end{aligned}$$

where $(c_0,\Lambda_0)=(1,0)$ and $(c_1,\Lambda_1)=(-1,\Lambda)$.

Let us evaluate the integral.

- $$\frac{1}{((k+q)^2+m^2)(k^2+m^2)((k-q)^2+\Lambda_i^2)}$$

$$= 2\int_0^\infty \frac{dx\, dy\, dz\, \delta(1-x-y-z)}{[\underbrace{x(k^2+m^2)+y((k+q)^2+m^2)+z((k-p)^2+\Lambda_i^2)}]^3}$$

$$k^2+2yqk-2zpk+yq^2+zp^2+(x+y)m^2+z\Lambda_i^2$$

$$= (\underbrace{k+yq-zp}_{\ell})^2 + \underbrace{y(1-y)q^2+z(1-z)p^2+2yz\,pq+(1-z)m^2+z\Lambda_i^2}_{\Delta_i}$$

- numerator

$$(\not{\ell}-y\not{q}+z\not{p})\gamma^\mu(\not{\ell}+(1-y)\not{q}+z\not{p}) + 2m(2\ell+(1-2y)q+2zp)^\mu + m^2\gamma^\mu$$

$$= \not{\ell}\gamma^\mu\not{\ell} + (-y\not{q}+z\not{p})\gamma^\mu((1-y)\not{q}+z\not{p}) + 2m((1-2y)q+2zp)^\mu + m^2\gamma^\mu$$

$$+ \ell\text{-linear}$$

$$\left[\ \not{\ell}\gamma^\mu\not{\ell} \simeq \frac{1}{4}\ell^2\gamma^\rho\gamma^\mu\gamma_\rho = \frac{1}{2}\ell^2\gamma^\mu\right.$$

$$\simeq \frac{1}{2}\ell^2\gamma^\mu + \underbrace{(-y\not{q}+z\not{p})\gamma^\mu((1-y)\not{q}+z\not{p}) + 2m((1-2y)q+2zp)^\mu + m^2\gamma^\mu}_{=:\, X^\mu}$$

$$\therefore\ \Gamma^{\mu}_{2PV}(p,q) = 4e^2 \int_0^\infty dx\,dy\,dz\,\delta(1-x-y-z) \underbrace{\int \frac{d^4 l}{(2\pi)^4} \sum_i c_i \frac{\frac{1}{2} l^2 \gamma^\mu + X^\mu}{(l^2+\Delta_i)^3}}_{\text{depends only on } y,z}$$

$$= 4e^2 \int_{\triangle} dy\,dz \int \frac{d^4 l}{(2\pi)^4} \sum_i c_i \frac{\frac{1}{2} l^2 \gamma^\mu + X^\mu}{(l^2+\Delta_i)^3}$$

where $\triangle := \left\{ (y,z) \in \mathbb{R}^2 \;\middle|\; y \geq 0,\ z \geq 0,\ y+z \leq 1 \right\}$

- $$\int \frac{d^4 l}{(2\pi)^4} \frac{1}{(l^2+\Delta)^3} = \frac{1}{(4\pi)^2} \int_0^\infty \frac{t\,dt}{(t+\Delta)^3} = \frac{1}{(4\pi)^2} \int_0^\infty d\left(\frac{-1}{t+\Delta} + \frac{1}{2} \frac{\Delta}{(t+\Delta)^2} \right)$$

$$= \frac{1}{(4\pi)^2} \left(\frac{1}{\Delta} - \frac{1}{2} \frac{\Delta}{\Delta^2} \right) = \frac{1}{2(4\pi)^2} \frac{1}{\Delta} .$$

- $$\int \frac{d^4 l}{(2\pi)^4} \sum_i c_i \frac{l^2}{(l^2+\Delta_i)^3} = \frac{1}{(4\pi)^2} \int_0^\infty \underbrace{t\,dt \sum_i c_i \frac{t}{(t+\Delta_i)^3}}_{d\left(\sum_i c_i \log(t+\Delta_i) + 2\sum_i \frac{c_i \Delta_i}{t+\Delta_i} - \frac{1}{2} \sum_i \frac{c_i \Delta_i^2}{(t+\Delta_i)^2} \right)}$$

$$= \frac{1}{(4\pi)^2} \left(-\sum_i c_i \log \Delta_i \right) = \frac{1}{(4\pi)^2} \log\left(\frac{\Delta_1}{\Delta_0} \right) .$$

$$\therefore \Gamma^{\mu}_{2PV}(p,q) = 4e^2 \int_{\triangle} dy\,dz \left[\frac{1}{(4\pi)^2} \log\left(\frac{\Delta_1}{\Delta_0}\right) \frac{1}{2}\gamma^{\mu} + \frac{1}{2(4\pi)^2}\left(\frac{X^{\mu}}{\Delta_0} - \frac{X^{\mu}}{\Delta_1}\right)\right]$$

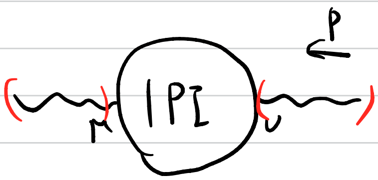
$$= \frac{2e^2}{(4\pi)^2} \int_{\triangle} dy\,dz \left[\log\left(\frac{z\Lambda^2}{\Delta_0}\right)\gamma^{\mu} + \frac{X^{\mu}}{\Delta_0} + O\left(\frac{m^2}{\Lambda^2}, \frac{p^2}{\Lambda^2}, \frac{q^2}{\Lambda^2}, \frac{pq}{\Lambda^2}\right)\right]$$

where

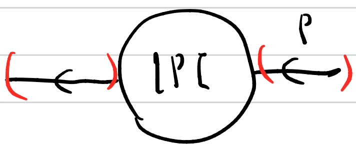
$$\Delta_0 = y(1-y)q^2 + z(1-z)p^2 + 2yz\,pq + (1-z)m^2$$

$$X^{\mu} = (-y\not{q} + z\not{p})\gamma^{\mu}((1-y)\not{q} + z\not{p}) + 2m((1-2y)q + 2zp)^{\mu} + m^2\gamma^{\mu}$$

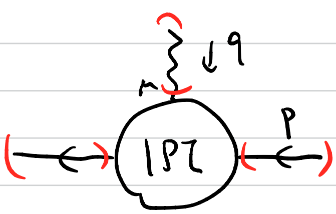
Recall $\Gamma[\Phi] = S_{\text{free}}[\Phi]$ – generating function of 1PI diagrams.



$= \Pi^{\mu\nu}(p)$



$= -\Sigma(p)$



$= -e\Gamma^{\mu}(p,q)$

divergent 1PI vertices.

$\Gamma[A,\psi,\bar\psi]$

$$= \int \frac{d^4p}{(2\pi)^4}\left\{ \frac{1}{2} A_\mu(-p)\left(\delta^{\mu\nu}p^2 - p^\mu p^\nu + \frac{1}{\xi}p^\mu p^\nu - \Pi^{\mu\nu}(p)\right)A_\nu(p)\right.$$

$$+ \bar\psi(-p)\left(-\not{p} + m + \Sigma(p)\right)\psi(p)$$

$$+ \int \frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\, \bar\psi(-p-q)\, e\Gamma^\mu(p,q)\, A_\mu(q)\, \psi(p)$$

+ higher powers of fields

↖ Here we recovered the general gauge parameter ξ

Digression: Ward identity for 1PI effective action

Let us consider a general QFT with variable $\phi = (\phi_1, \cdots, \phi_N)$ measure $d\phi$ and action $S(\phi)$.

Suppose $\phi \to \phi + \delta\phi$ is a symmetry, $\delta\left(d\phi\, e^{-S(\phi)}\right) = 0$.

Then, we have Ward identity

$$0 = \int \delta\left(d\phi\, e^{-S(\phi) + J\cdot\phi}\right)$$

$$= \int d\phi\, e^{-S(\phi)+J\cdot\phi}\, J\cdot\delta\phi = e^{-W(J)}\, J\cdot\langle\delta\phi\rangle_J$$

Set $J = J(\phi)$ and use $\frac{\partial \Gamma}{\partial \phi_i}(\phi) = J_i(\phi)$. We obtain

$$\boxed{\sum_i \langle \delta\phi_i\rangle_{J(\phi)} \frac{\partial\Gamma}{\partial\phi_i}(\phi) = 0. \qquad \text{Slavnov-Taylor identity} \\ \text{i.e. } \Gamma(\phi) \text{ is invariant under } \phi \to \phi + \langle\delta\phi\rangle_{J(\phi)}.}$$

For an at most linear symmetry: $\delta\phi_i = M_{ij}\phi_j + C_i$,

$$\langle\delta\phi_i\rangle_{J(\phi)} = M_{ij}\langle\phi_j\rangle_{J(\phi)} + C_i = M_{ij}\phi_j + C_i = \delta\phi_i.$$

So $\Gamma(\phi)$ is invariant under the original symmetry.

A variant : non symmetry

e.g. $$\delta\left(d\phi\, e^{-S(\phi)}\right) = d\phi\, e^{-S(\phi)}\left(-\delta S(\phi)\right).$$

Then $$0 = \int d\phi\, e^{-S(\phi)+J\cdot\phi}\left(-\delta S(\phi) + J\cdot\delta\phi\right)$$

$$= e^{-W(J)}\left(-\langle \delta S(\phi)\rangle_J + J\cdot\langle\delta\phi\rangle_J\right)$$

and setting $J = J(\phi)$ we have

$$\boxed{\sum_i \langle\delta\phi_i\rangle_{J(\phi)} \frac{\partial\Gamma}{\partial\phi_i}(\phi) = \langle\delta S(\phi)\rangle_{J(\phi)}.}$$

If both $\delta\phi$ & $\delta S(\phi)$ are at most linear,

$$\delta\Gamma(\phi) = \delta S(\phi).$$

End of Digression

Let us apply this to the gauge fixed QED and the gauge transformation

$$\delta A_\mu = -\frac{1}{e}\partial_\mu \alpha, \quad \delta\psi = i\alpha\psi, \quad \delta\bar\psi = \bar\psi(-i\alpha).$$

This is at most linear, and also

$$\delta S = \int d^4x \frac{1}{\xi}\left(-\frac{1}{e}\partial^2\alpha\right)\partial\cdot A \quad \text{is linear.}$$

Therefore (the variant of) Slavnov-Taylor identity reads

$$\delta\Gamma[A,\psi,\bar\psi] = \int d^4x \frac{1}{\xi}\left(-\frac{1}{e}\partial^2\alpha\right)\partial\cdot A.$$

That is, if we set

$$\Gamma[A,\psi,\bar\psi] = \int d^4x \frac{1}{2\xi}(\partial\cdot A)^2 + \Gamma^{inv}[A,\psi,\bar\psi],$$

then, $\Gamma^{inv}[A,\psi,\bar\psi]$ is gauge invariant,

$$\delta\Gamma^{inv}[A,\psi,\bar\psi] = 0.$$

In $\Gamma[A,\psi,\bar\psi]$, the gauge fixing term of the classical Lagrangian is the only term that breaks the gauge symmetry.

I.e. "Gauge fixing term is not renormalized."

Note: $\Gamma^{inv}[A,\psi,\bar\psi]$

$$= \int \frac{d^4p}{(2\pi)^4}\left\{ \frac{1}{2} A_\mu(-p)\left(\delta^{\mu\nu}p^2 - p^\mu p^\nu - \Pi^{\mu\nu}(p)\right) A_\nu(p) \right.$$

$$\left. + \bar\psi(-p)\left(-\not{p} + m + \Sigma(p)\right)\psi(p)\right.$$

$$+ \int \frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\, \bar\psi(-p-q)\, e\Gamma^\mu(p,q)\, A_\mu(q)\,\psi(p)$$

+ higher power

$$\delta A_\mu(p) = \frac{i}{e} p_\mu \alpha(p)$$

$$\delta\psi(p) = \int \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}(2\pi)^4\delta(p_1+p_2-p)\, i\alpha(p_1)\psi(p_2)$$

$$\delta\bar\psi(-p) = \int \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}(2\pi)^4\delta(p+p_1-p_2)\,\bar\psi(-p_2)\left(-i\alpha(p_1)\right)$$

$$\delta\Gamma^{inv}[A,\psi,\bar\psi] = 0$$

$$\Rightarrow \begin{cases} p_\mu \Pi^{\mu\nu}(p) = 0 \\ q_\mu \Gamma^\mu(p,q) - \not{q} + \Sigma(p+q) - \Sigma(p) = 0 \end{cases}$$

The identities are derived in another way (though the origin is the same).

Another application

The (gauge fixed) Lagrangian of massless QED

$$\mathcal{L} = \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + i\bar{\Psi}(-\gamma^\mu D_\mu)\Psi + \frac{1}{2\xi}(\partial\cdot A)^2$$

has axial symmetry ($\gamma_5 = \gamma^1\gamma^2\gamma^3\gamma^4$, β : constant)

$$A_\mu \to A_\mu, \quad \Psi \to e^{i\beta\gamma_5}\Psi, \quad \bar{\Psi} \to \bar{\Psi}e^{i\beta\gamma_5}.$$

Indeed, $\gamma^\mu\gamma_5 = -\gamma_5\gamma^\mu$ and hence

$$e^{i\beta\gamma_5}\gamma^\mu e^{i\beta\gamma_5} = e^{i\beta\gamma_5}e^{-i\beta\gamma_5}\gamma^\mu = \gamma^\mu.$$

(A mass term $\propto \bar{\Psi}\Psi$ would not be invariant.)

Suppose the path-integral measure is also invariant.

As the transformation is at most linear, by Slavnov-Taylor identity, $\Gamma[A, \Psi, \bar{\Psi}]$ is also invariant. In particular,

$$e^{i\beta\gamma_5}\Sigma(p)e^{i\beta\gamma_5} \overset{!}{=} \Sigma(p).$$

This requires $\Sigma(p) \propto \not{p}$ in the massless theory.

If we recover $m \neq 0$, $\Sigma(p)$ must be of the form

$$\Sigma(p) = A(p^2)\not{p} + B(p^2)m.$$

Structure of divergence

On dimensional ground, the divergence as the UV cut off Λ is removed $(\Lambda \to \infty)$ must be of the form

$$\Pi^{\mu\nu}(p) = (\delta^{\mu\nu} p^2 - p^\mu p^\nu)\, a_1 \log\Lambda + \text{finite},$$

$$\Sigma(p) = a_2 \log\Lambda\, \not{p} + a_3 \Lambda + a_4 \log\Lambda\, m + \text{finite},$$

$$\Gamma^\mu(p,q) = a_5 \log\Lambda\, \gamma^\mu + \text{finite}$$

with some constants a_1, a_2, a_3, a_4, a_5

- By $q_\mu \Gamma^\mu(p,q) = \not{q} + \Sigma(p) - \Sigma(p+q)$,

  we find $a_5 = -a_2$.

- By the axial symmetry of the $m=0$ theory

  we also find $a_3 = 0$.

Indeed, at the one loop level (with $M_1 \sim M_2 \sim \Lambda$),

$$a_1 = -\frac{8e^2}{3(4\pi)^2}, \quad a_2 = -\frac{2e^2}{(4\pi)^2}, \quad a_3 = 0, \quad a_4 = \frac{8e^2}{(4\pi)^2},$$

$$a_5 = \frac{2e^2}{(4\pi)^2}.$$

$a_2 + a_5 = a_3 = 0$ is satisfied.

Renormalization

In view of the structure of divergence, after regularization, we can renormalize the theory as

$$\mathcal{L} = \frac{1}{4e_0^2} F_0^{\mu\nu} F_{0\mu\nu} + \bar{\psi}_0(-i\not{D}_{A_0} + m_0)\psi_0 + \frac{1}{2e_0^2\xi_0}(\partial\cdot A_0)^2$$

$$\boxed{\begin{array}{c} \dfrac{1}{e_0^2} = \dfrac{Z_3}{e^2}, \quad \psi_0 = \sqrt{Z_2}\,\psi, \quad \bar{\psi}_0 = \sqrt{Z_2}\,\bar{\psi}, \quad Z_2 m_0 = Z_m m \\ A_0 = eA, \qquad e_0^2\xi_0 = e^2\xi \end{array}}$$

$$= \frac{Z_3}{4} F^{\mu\nu}F_{\mu\nu} + \bar{\psi}(-iZ_2\not{\partial} + Z_m m)\psi + eZ_2\bar{\psi}\not{A}\psi + \frac{1}{2\xi}(\partial\cdot A)^2$$

- Z_3 takes care of the divergence of $\Pi^{\mu\nu}(p) \leftrightarrow F^{\mu\nu}F_{\mu\nu}$.
- Z_2 takes care of the common divergence of

  $\not{p}$ part of $\Sigma(p) \leftrightarrow i\bar{\psi}\not{\partial}\psi$ and $\Gamma^{\mu}(p,q) \leftrightarrow e\bar{\psi}\not{A}\psi$.
- Z_m takes care of the divergence of

  $m\cdot id$ part of $\Sigma(p) \leftrightarrow \bar{\psi}m\psi$.

$$Z_3 = 1 + \delta_3^{(1)} + \delta_3^{(2)} + \cdots$$

$$Z_2 = 1 + \delta_2^{(1)} + \delta_2^{(2)} + \cdots$$

$$Z_m = 1 + \delta_m^{(1)} + \delta_m^{(2)} + \cdots$$

We determine $\delta_3^{(a)}, \delta_2^{(a)}, \delta_m^{(a)}$ order by order in perturbation theory so that

$$\Gamma_0[A_0, \psi_0, \bar{\psi}_0, e_0(\Lambda), m_0(\Lambda), \xi_0 ; \Lambda]$$

$$= \Gamma[A, \psi, \bar{\psi}, e, m, \xi ; \Lambda]$$

is finite as a function of $A, \psi, \bar{\psi}, e, m, \xi$ as the UV cut-off Λ is removed.

There is an ambiguity in the choice of renormalized fields and couplings, but that is fixed by renormalization condition.

Renormalization condition

$$\Pi^{\mu\nu}(q) = (\delta^{\mu\nu} q^2 - q^\mu q^\nu)\, \Pi(q^2)$$

$$\Sigma(p) = A(p^2)\, \not{p} + B(p^2)\, m$$

$$\begin{cases} \Pi(0) = 0 \\ A(-m^2) = 0 \\ B(-m^2) = 0 \end{cases}$$

On shell renormalization

$$\begin{cases} \Pi(\mu^2) = 0 \\ A(\mu^2) = 0 \\ B(\mu^2) = 0 \end{cases}$$

Another R.C.

μ: a mass scale.

Solution at 1-loop

$$\Pi^{(1)}(q^2) = \Pi_2(q^2) - \delta_3^{(1)},$$

$$A^{(1)}(p^2) = A_2(p^2) - \delta_2^{(1)},$$

$$B^{(1)}(p^2) = B_2(p^2) + \delta_m^{(1)},$$

where (for $\xi = 1$)

$$\Pi_2(q^2) = -\frac{8e^2}{(4\pi)^2}\int_0^1 dx\, x(1-x)\log\left(\frac{M^2}{m^2+x(1-x)q^2}\right)$$

$$A_2(p^2) = -\frac{2e^2}{(4\pi)^2}\int_0^1 dx\, x\log\left(\frac{x\Lambda^2}{(1-x)m^2+x(1-x)p^2}\right)$$

$$B_2(p^2) = \frac{4e^2}{(4\pi)^2}\int_0^1 dx\, \log\left(\frac{x\Lambda^2}{(1-x)m^2+x(1-x)p^2}\right)$$

On shell $\delta_3^{(1)} = \Pi_2(0)$, $\delta_2^{(1)} = A_2(-m^2)$, $\delta_m^{(1)} = -B_2(-m^2)$

Another $\delta_3^{(1)} = \Pi_2(\mu^2)$, $\delta_2^{(1)} = A_2(\mu^2)$, $\delta_m^{(1)} = -B_2(\mu^2)$

The result (on shell R.C.):

$$\Pi^{(1)}(q^2) = \frac{8e^2}{(4\pi)^2}\int_0^1 dx\, x(1-x)\log\left(\frac{m^2+x(1-x)q^2}{m^2}\right)$$

$$\Sigma^{(1)}(p) = \frac{2e^2}{(4\pi)^2}\int_0^1 dx\,(x\not{p}-2m)\log\left(\frac{m^2+xp^2}{(1-x)m^2}\right)$$

$$\Gamma^{\mu(1)}(p,q) = \gamma^\mu - \frac{2e^2}{(4\pi)^2}\int_{\triangle} dy\,dz\left[\log\left(\frac{(1-z)\Delta_0}{z^3m^2}\right)\gamma^\mu - \frac{X^\mu}{\Delta_0}\right]$$

$$\Delta_0 = y(1-y)q^2 + z(1-z)p^2 + 2yz\,pq + (1-z)m^2$$

$$X^\mu = (-y\not{q}+z\not{p})\gamma^\mu((1-y)\not{q}+z\not{p}) + 2m((1-2y)q+2zp)^\mu + m^2\gamma^\mu$$

meaning of $\Pi(q^2)$: It enters into $\Gamma[A,\psi,\bar\psi]$ as

$$\Gamma^{inv}[A,\psi,\bar\psi] = \int \frac{d^4q}{(2\pi)^4}\,\frac{1}{2}A_\mu(-q)\left(\delta^{\mu\nu}q^2 - q^\mu q^\nu\right)\left(1-\Pi(q^2)\right)A_\nu(q)$$

$$+ \cdots$$

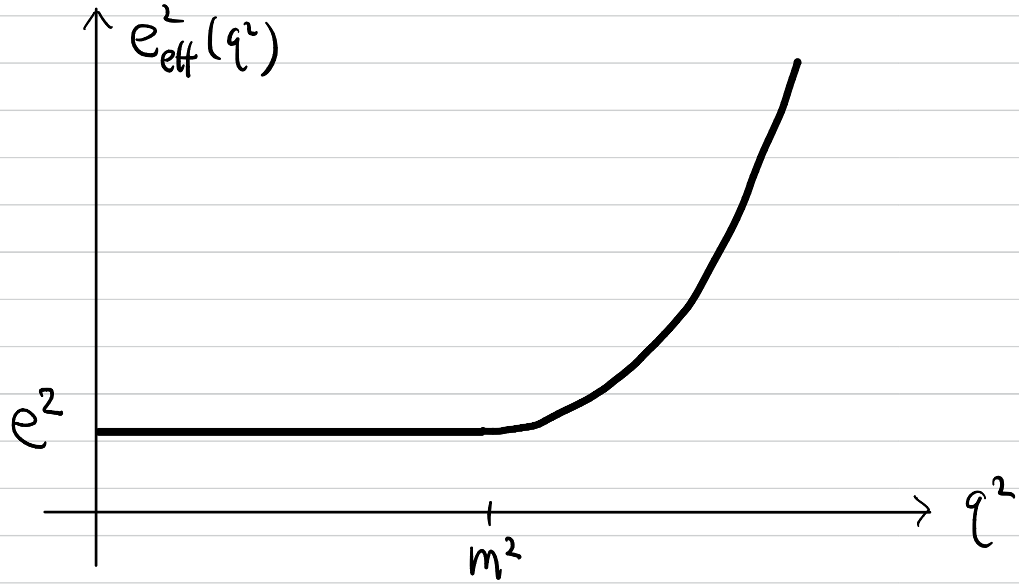
$$\rightsquigarrow \quad e_{eff}^2(q^2) = \frac{e^2}{1-\Pi(q^2)}$$

effective gauge coupling constant

at energy scale $q^2 \approx$ length scale $1/q^2$.

$$\Pi(q^2) \longrightarrow \begin{cases} O(q^2/m^2) & \text{as} \quad q^2/m^2 \to 0 \\[2ex] \dfrac{4e^2}{3(4\pi)^2}\left[\log\left(\dfrac{q^2}{m^2}\right) - \dfrac{5}{3} + O\left(\dfrac{m^2}{q^2}\right)\right] & \text{as} \quad q^2 \gg m^2 \end{cases}$$

$$e_{eff}^2(q^2) \to \begin{cases} e^2 & \text{as} \quad q^2/m^2 \to 0 \\[2ex] \dfrac{e^2}{1 - \dfrac{4e^2}{3(4\pi)^2}\left(\log(q^2/m^2) - 5/3\right)} & \text{as} \quad q^2/m^2 \gg 1 \end{cases}$$

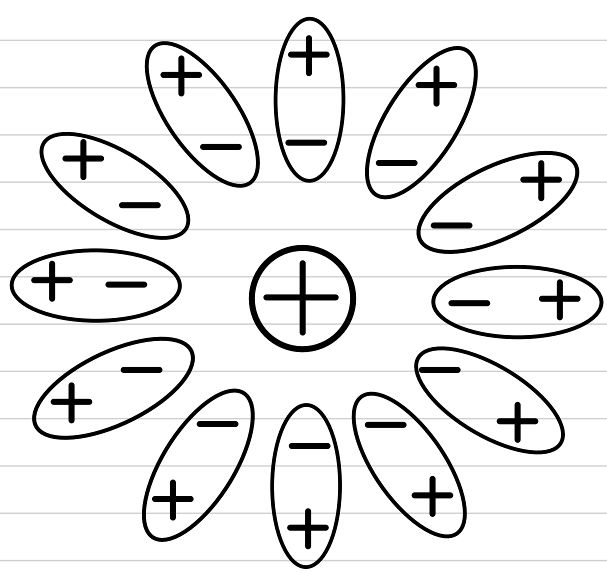


$e_{eff}^2(q^2) \sim e^2$ at long distances $|q|^{-1} \gg \frac{1}{m}$

$e_{eff}^2(q^2)$ grows at short distances $|q|^{-1} \ll \frac{1}{m}$

Interpretation:

"charge screening by vacuum polarization"



Renormalization group

Take the "Another renormalization condition", and denote the 1PI effective action as $\Gamma[A, \psi, \bar{\psi}, e, m, \xi; \mu, \Lambda]$.

Change the renormalization point μ while fixing bare fields/couplings $A_0, \psi_0, \bar{\psi}_0, e_0, m_0, \xi_0$. Then, we have RGE

$$0 = \mu \frac{d}{d\mu} \Gamma[A_0, \psi_0, \bar{\psi}_0, e_0, m_0, \xi_0; \Lambda]$$

$$= \mu \frac{d}{d\mu} \Gamma[A, \psi, \bar{\psi}, e, m, \xi; \mu, \Lambda]$$

We denote

$$\mu \frac{d}{d\mu} \Phi_I = -\gamma_I \Phi_I \quad \text{for } \Phi_I = A, \psi, \bar{\psi}, m \text{ and}$$

$$\mu \frac{d}{d\mu} e = \beta.$$

As $A_0 = eA$ and $e_0^2 \xi_0 = e^2 \xi$, we find

$$0 = \beta A + e(-\gamma_A A), \quad 0 = 2e\beta\xi + e^2 \mu \frac{d\xi}{d\mu}.$$

i.e. $\gamma_A = \frac{1}{e}\beta$ & $\mu \frac{d\xi}{d\mu} = -2\frac{1}{e}\beta\xi = -2\gamma_A \xi$

Thus, the RGE (in the limit $\Lambda \to \infty$) reads

$$\left(\mu \frac{\partial}{\partial \mu} - \frac{1}{e}\beta A \cdot \frac{\delta}{\delta A} - \gamma_\psi \psi \frac{\delta}{\delta \psi} - \gamma_\psi \bar{\psi} \frac{\delta}{\delta \bar{\psi}} - \gamma_m m \frac{\partial}{\partial m} \right.$$

$$\left. + \beta \frac{\partial}{\partial e} - \frac{2}{e}\beta \xi \frac{\partial}{\partial \xi} \right) \Gamma[A, \psi, \bar{\psi}, e, m, \xi ; \mu] = 0.$$

As $\frac{1}{e_0^2} = \frac{Z_3}{e^2}$, $\psi_0 = \sqrt{Z_2}\,\psi$, $\bar{\psi}_0 = \sqrt{Z_2}\,\bar{\psi}$, $Z_2 m_0 = Z_m m$,

$$0 = -2e^{-3}\beta Z_3 + e^{-2}\mu \frac{d}{d\mu} Z_3 ,$$

$$0 = \mu \frac{d}{d\mu}\sqrt{Z_2}\,\psi + \sqrt{Z_2}(-\gamma_\psi \psi),$$

$$0 = \mu \frac{d}{d\mu} Z_2^{-1} Z_m m + Z_2^{-1} \mu \frac{d}{d\mu} Z_m m + Z_2^{-1} Z_m (-\gamma_m m).$$

$$\beta = \frac{1}{2} e \mu \frac{d}{d\mu} \log Z_3$$

$$\gamma_\psi = \frac{1}{2} \mu \frac{d}{d\mu} \log Z_2$$

$$\gamma_m = \mu \frac{d}{d\mu} \log Z_m - \mu \frac{d}{d\mu} \log Z_2$$

Computation at 1-loop (at $\xi = 1$) :

$$\beta^{(1)} = \frac{1}{2} e \mu \frac{d}{d\mu} \delta_3^{(1)} = \frac{4e^3}{(4\pi)^2} \int_0^1 dx \; x(1-x) \frac{x(1-x)\, 2\mu^2}{m^2 + x(1-x)\mu^2}$$

$$\gamma_\psi^{(1)} = \frac{1}{2} \mu \frac{d}{d\mu} \delta_2^{(1)} = \frac{e^2}{(4\pi)^2} \int_0^1 dx \; x \; \frac{x\, 2\mu^2}{m^2 + x\mu^2}$$

$$\gamma_m^{(1)} = \mu \frac{d}{d\mu} \left(\delta_m^{(1)} - \delta_2^{(1)} \right) = \frac{2e^2}{(4\pi)^2} \int_0^1 dx\, (2-x) \frac{x\, 2\mu^2}{m^2 + x\mu^2}$$

The result for $\beta^{(1)}$ is valid for any ξ

since $\psi(x)\psi(y)$ is independent of ξ.

The result fo $\gamma_\psi^{(1)}$ & $\gamma_m^{(1)}$ depends on ξ.

(See the additional note for the expressions.)

Note that

$$\beta^{(1)} \simeq \begin{cases} \dfrac{4e^3}{3(4\pi)^2} & \mu \gg m \\[2em] \dfrac{4e^2}{15(4\pi)^2} \dfrac{\mu^2}{m^2} \sim 0 & \mu \ll m \end{cases}$$

Just like the ϕ^4 theory.

$\underline{\mu \gg m}$: $\beta \sim \frac{4e^3}{3(4\pi)^2} \quad \leftrightarrow \quad \mu \frac{d}{d\mu} \frac{1}{e^2} \sim -\frac{8}{3(4\pi)^2}$

$$\frac{1}{e^2(\mu)} - \frac{1}{e^2(\mu_0)} \sim -\frac{8}{3(4\pi)^2} \log(\mu/\mu_0)$$

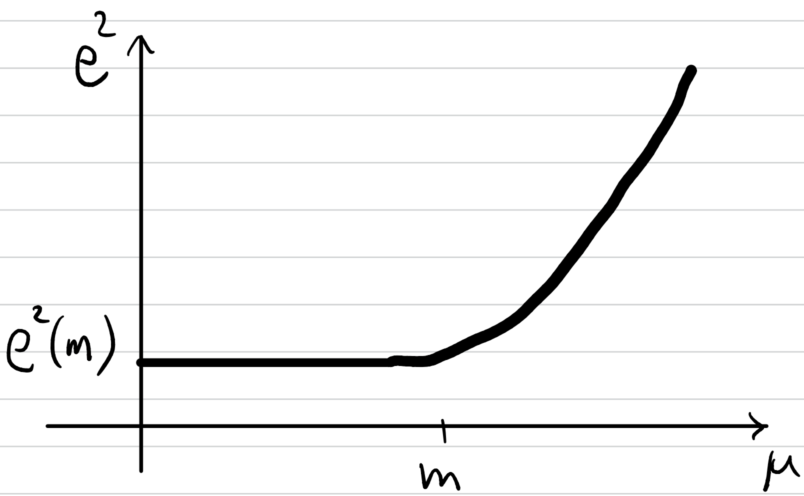
$$e^2(\mu) \sim \frac{e^2(\mu_0)}{1 - \frac{8e^2(\mu_0)}{3(4\pi)^2} \log(\mu/\mu_0)}$$

The gauge coupling is weaker at lower energies

or stronger at higher energies,

$\underline{\mu \ll m}$: $\beta \sim 0$

The gauge coupling stops running.



As we've already seen.

The massless theory $m=0$:

$$e^2(\mu) \sim \frac{e^2(\mu_0)}{1 - \frac{8e^2(\mu_0)}{3(4\pi)^2}\log(\mu/\mu_0)}$$ valid at low μ

$$\longrightarrow 0 \quad \text{as} \quad \mu \to 0$$

infra-red free!

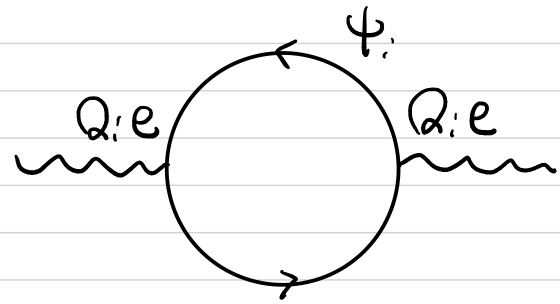
A simple generalization

QED with electrons with charges $Q_1, \cdots, Q_N$

$$\mathcal{L} = \frac{1}{4e^2} F^{\mu\nu} F_{\mu\nu} + \sum_{i=1}^{N} \overline{\Psi}_i (-i \not{D}_A + m_i) \Psi_i$$

$$\not{D}_A \Psi_i = \gamma^\mu (\partial_\mu + i Q_i A_\mu) \Psi_i$$

$$\Pi_2(q) = \sum_i$$



$$= \sum_i - \frac{8(Q_i e)^2}{(4\pi)^2} \int_0^1 dx\, x(1-x) \log\left(\frac{M^2}{m_i^2 + x(1-x) q^2} \right)$$

With "another R.C." $\Pi(\mu^2) = 0, \cdots,$

$$\beta^{(1)} = \frac{1}{2} e \mu \frac{d}{d\mu} \delta_3^{(1)} = \frac{1}{2} e \mu \frac{d}{d\mu} \Pi_2(\mu^2)$$

$$= \sum_i e \frac{8(Q_i e)^2}{(4\pi)^2} \int_0^1 dx\, x(1-x) \frac{x(1-x)\mu^2}{m_i^2 + x(1-x)\mu^2}$$

Suppose the masses are well-separated

$$m_1 \ll m_2 \ll \cdots \ll m_N.$$

At the energy scale $m_i \ll \mu \ll m_{i+1}$,

$$\beta \sim \frac{4e^3}{3(4\pi)^2} \sum_{j=1}^{i} Q_j^2$$

The slope depends on the energy scale