Symmetry in classical mechanics (in Lagrangian)

Suppose $\exists$ a symmetry $q \mapsto q + \delta q$ $\left(\delta q = \epsilon\, u(q, \dot{q}) \right)$

$$\delta L(q, \dot{q}) = \epsilon \frac{d}{dt}\left(\cdots \right) \quad \text{total derivative}$$

Allow variational parameter ϵ to depend on time, $\epsilon(t)$,

s.t. $\epsilon(t_f) = \epsilon(t_i) = 0$:

$$\delta S = \delta \int_{t_i}^{t_f} dt\, L(q, \dot{q}) = \int_{t_i}^{t_f} dt\, \dot{\epsilon}(t)\, Q(q, \dot{q})$$

This $Q = Q(q, \dot{q})$ is called the <u>Noether charge</u>.

<u>Noether's theorem</u> Q is conserved. I.e. it is time-independent for a solution to equation of motion.

<u>proof</u> A solution is s.t. $\delta S = 0$ for $\forall \delta q$ s.t. $\delta q|_{t_f, t_i} = 0$.

For $\forall \epsilon(t)$ s.t. $\epsilon(t_f) = \epsilon(t_i) = 0$, under $q \to q + \epsilon(t)\, u(q, \dot{q})$,

$$0 = \delta S = \int_{t_i}^{t_f} dt\, \dot{\epsilon}(t)\, Q = -\int_{t_i}^{t_f} dt\, \epsilon(t) \frac{dQ}{dt}.$$

$$\therefore \quad \frac{dQ}{dt} = 0 \qquad \underline{\text{Q.E.D.}}$$

Example $L = \frac{m}{2}\dot{q}^2$: a free particle without potential

$\delta q = \epsilon$: translation in q

$$\delta S = \int_{t_i}^{t_f} dt \; \frac{m}{2} 2\dot{q}\dot{\epsilon} = \int_{t_i}^{t_f} dt \, \dot{\epsilon} \, m\dot{q}$$

$\therefore \; Q = m\dot{q}$: momentum.

Example $L = \frac{m}{2}(\dot{q}_1^2 + \dot{q}_2^2) - V(q_1^2 + q_2^2)$

$$g_\alpha : \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \longmapsto \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$$ rotational symmetry

Infinitesimal version :

$$\delta q_1 = -\epsilon q_2, \quad \delta q_2 = \epsilon q_1$$

$$\delta S = \int_{t_i}^{t_f} dt \; \frac{m}{2}\left(2\dot{q}_1(-\dot{\epsilon} q_2) + 2\dot{q}_2(\dot{\epsilon} q_1)\right)$$

$$= \int_{t_i}^{t_f} dt \; \dot{\epsilon} \; m(q_1\dot{q}_2 - q_2\dot{q}_1)$$

$\therefore \; Q = mq_1\dot{q}_2 - mq_2\dot{q}_1$: angular momentum.

Example $L(q,\dot q)$ general (no explicit t-dependence).

$$\delta q = \epsilon \dot q \quad : \text{ time translation.}$$

$$\delta S = \int_{t_i}^{t_f} dt \left(\epsilon \dot q \frac{\partial L}{\partial q} + \underbrace{\frac{d}{dt}(\epsilon \dot q)}_{\epsilon \ddot q + \dot\epsilon \dot q} \frac{\partial L}{\partial \dot q} \right)$$

$$\epsilon \frac{d}{dt} L + \dot\epsilon \dot q \frac{\partial L}{\partial \dot q}$$

$$= \int_{t_i}^{t_f} dt\, \dot\epsilon \left(\dot q \frac{\partial L}{\partial \dot q} - L \right) dt$$

$$\therefore Q = \dot q \frac{\partial L}{\partial \dot q}(q,\dot q) - L(q,\dot q) =: E(q,\dot q) \quad \text{energy}$$

c.f. If we solve $\frac{\partial L}{\partial \dot q}(q,\dot q) \overset{!}{=} p$ for $\dot q$

and plug the solution $\dot q = \dot q(p,q)$, then

$$E(q,\dot q(p,q)) = \dot q(p,q)\, p - L(q,\dot q(p,q))$$

$$= H(p,q) \quad \text{Hamiltonian}$$

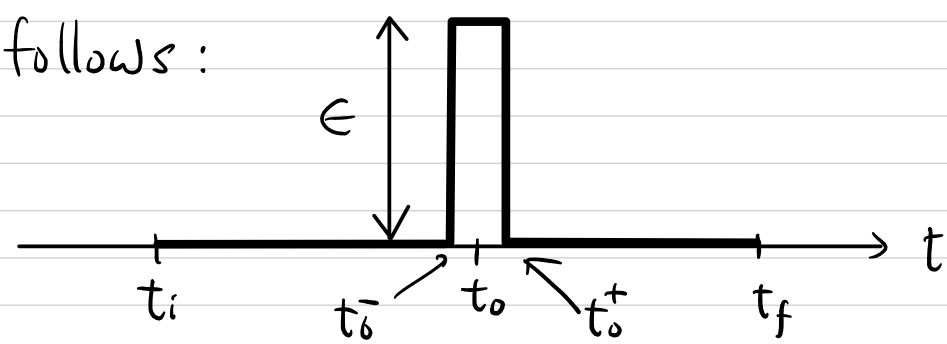
Symmetry in quantum mechanics

Suppose $\exists$ a symmetry $\delta q = \epsilon\, u(q,\dot q)$ of the classical system & it is also a symmetry of the path-integral measure $\mathcal{D}q$.

Apply $\delta q = \epsilon(t)\, u(q,\dot q)$ in the integrand of

$$Z(t_f, q_f; \mathcal{O}(t_0); t_i, q_i) = \int_{q(t_f)=q_f,\ q(t_i)=q_i} \mathcal{D}q\ e^{\frac{i}{\hbar}\int_{t_i}^{t_f} dt\, L(q,\dot q)}\ \mathcal{O}(t_0)$$

for $\epsilon(t)$ as follows:



Note: $\dot\epsilon(t) = \epsilon\,\delta(t - t_0^-) - \epsilon\,\delta(t - t_0^+)$

$$0 \overset{\text{Ward id}}{=} \int \delta\left(\mathcal{D}q\ e^{\frac{i}{\hbar}S[q]}\ \mathcal{O}(t_0)\right)$$

$$= \int \mathcal{D}q\ e^{\frac{i}{\hbar}S[q]}\left(\frac{i}{\hbar}\delta S[q]\,\mathcal{O}(t_0) + \delta\mathcal{O}(t_0)\right)$$

$$0 = \int \mathcal{D}q \, e^{\frac{i}{\hbar}S[q]} \left(\frac{i}{\hbar} \underbrace{\delta S[q]}_{\int_{t_i}^{t_f} dt\, \dot{\epsilon} Q \,=\, \epsilon Q(t_0^-) - \epsilon Q(t_0^+)} \mathcal{O}(t_0) + \delta \mathcal{O}(t_0) \right)$$

i.e.

$$Z(t_f, q_f ; \delta\mathcal{O}(t_0) ; t_i, q_i)$$

$$= Z\left(t_f, q_f ; \left(\frac{i\epsilon}{\hbar} Q(t_0^+) - \frac{i\epsilon}{\hbar} Q(t_0^-) \right) \mathcal{O}(t_0) ; q_i, t_i\right)$$

$$\hat{Z}_{t_f, t_i}(\delta\mathcal{O}(t_0)) = \hat{Z}_{t_f, t_i}\left(\left(\frac{i\epsilon}{\hbar} Q(t_0^+) - \frac{i\epsilon}{\hbar} Q(t_0^-) \right) \mathcal{O}(t_0)\right)$$

Take the limit $t_0^+ \searrow t_0$ and $t_0^- \nearrow t_0$:

$$\widehat{\delta\mathcal{O}} = \frac{i\epsilon}{\hbar} \hat{Q} \circ \hat{\mathcal{O}} - \hat{\mathcal{O}} \circ \frac{i\epsilon}{\hbar} \hat{Q}$$

Put $\epsilon \to 1$:

$$\boxed{\widehat{\delta\mathcal{O}} = \frac{i}{\hbar} [\hat{Q}, \hat{\mathcal{O}}]}$$

Ward identity in quantum mechanics

(in operator formalism)

In classical mechanics, a continuous symmetry yields
a conserved charge (Noether charge).

After quantization, the Noether charge generates
the symmetry transformation.

The case of time translation symmetry:

$$\widehat{\frac{d}{dt}\mathcal{O}} = \frac{i}{\hbar}\left[\widehat{H(p,q)}, \hat{\mathcal{O}}\right].$$

On the other hand, using $\hat{H}$ defined by $\hat{Z}_{t_f,t_i} = e^{-\frac{i}{\hbar}(t_f-t_i)\hat{H}}$,

we also know

$$e^{-\frac{i}{\hbar}(t_f-t)\hat{H}}\,\widehat{\frac{d}{dt}\mathcal{O}}\,e^{-\frac{i}{\hbar}(t-t_i)\hat{H}} = \hat{Z}_{t_f,t_i}\left(\frac{d}{dt}\mathcal{O}(t)\right)$$

$$= \frac{d}{dt}\hat{Z}_{t_f,t_i}(\mathcal{O}(t)) = \frac{d}{dt}\left(e^{-\frac{i}{\hbar}(t_f-t)\hat{H}}\,\hat{\mathcal{O}}\,e^{-\frac{i}{\hbar}(t-t_i)\hat{H}}\right)$$

$$= e^{-\frac{i}{\hbar}(t_f-t)\hat{H}}\,\frac{i}{\hbar}[\hat{H},\hat{\mathcal{O}}]\,e^{-\frac{i}{\hbar}(t-t_i)\hat{H}}$$

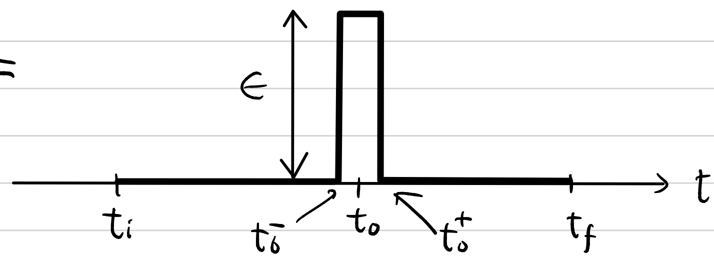
$$\therefore\ \widehat{\frac{d}{dt}\mathcal{O}} = \frac{i}{\hbar}[\hat{H},\hat{\mathcal{O}}].$$

Comparison $\Rightarrow$ $\hat{H} = \widehat{H(p,q)}$ + c-number.

$\hat{H}$ is the operator corresponding to Hamiltonian (modulo a c-number shift).

The case of q-translation (not a symmetry in general).

Apply $\delta q(t) = \epsilon(t) =$



in the integrand of $Z(t_f, q_f; q(t_0); t_i, q_i)$:

$$0 = \int \delta\left(\mathcal{D}q\, e^{\frac{i}{\hbar}S[q]}\, q(t_0)\right)$$

$$= \int \mathcal{D}q\, e^{\frac{i}{\hbar}S[q]}\left(\frac{i}{\hbar}\delta S[q]\, q(t_0) + \epsilon\right)$$

$$\delta S[q] = \int_{t_i}^{t_f} dt\left(\epsilon(t)\frac{\partial L}{\partial q} + \dot{\epsilon}(t)\frac{\partial L}{\partial \dot{q}}\right)$$

Conjugate momentum p

$$= \int_{t_0^-}^{t_0^+} dt\, \epsilon\frac{\partial L}{\partial q} + \epsilon\, p(t_0^-) - \epsilon\, p(t_0^+)$$

$$= \epsilon\int \mathcal{D}q\, e^{\frac{i}{\hbar}S[q]}\left\{\frac{i}{\hbar}\left(\int_{t_0^-}^{t_0^+} dt\,\frac{\partial L}{\partial q} + p(t_0^-) - p(t_0^+)\right) q(t_0) + 1\right\}$$

Take the limit $t_0^+ \searrow t_0$, $t_0^- \nearrow t_0$:

The part $\int_{t_0^-}^{t_0^+} dt \frac{\partial L}{\partial q} q(t_0)$ vanishes in this limit.

$$\therefore \quad 0 = \frac{i}{\hbar}\left(\hat{q} \circ \hat{p} - \hat{p} \circ \hat{q}\right) + 1$$

$$\boxed{\therefore \quad [\hat{q}, \hat{p}] = i\hbar}$$

The canonical commutation relation!

- Remark on terminology

  We used "local observable" for $\mathcal{O}(t)$, but it is common to call it "local operator" even inside path-integral.

  We have chosen "observable" to emphasize the distinction between path-integral & operator formalisms.

- It is instructive to do path-integrals in explicit examples. Please do it yourself. For your convenience, a note on it is uploaded.

- In a classical field theory in a general dimension, a continuous symmetry yields a conserved current (Noether current). Just like in quantum mechanics one can derive Ward identity involving Noether current. A note on it will be uploaded.

Fermion path integrals

To describe fermions, we consider path integrals

$$\int d\psi_1 \cdots d\psi_n \, e^{-S_E(\psi_1, \cdots, \psi_n)} f(\psi_1, \cdots, \psi_n)$$

of anticommuting variables $\psi_i \psi_j = -\psi_j \psi_i$.

Algebra & Calculus

Grassmann algebra (graded commutative algebra)

... algebra/$\mathbb{C}$ with bose vs fermi statistics of elements.

If a & b have definite statistics,

$$a \cdot b = (-1)^{|a||b|} \, b \cdot a$$

where $|a| \underset{\text{mod } 2}{\equiv} \begin{cases} 0 & a \text{ even (bosonic)} \\ 1 & a \text{ odd (fermionic)} \end{cases}$

- even elements commute with everything.
- odd elements anticommute with each other.

- Rule of complex conjugation : $(a \cdot b)^* = b^* \cdot a^*$.

- A function $f(\psi) = f(\psi_1, \dots, \psi_n)$ of odd variables $\psi_1, \dots, \psi_n$:

$$\psi_i \psi_j = -\psi_j \psi_i \qquad 1 \leq \forall i, j \leq n.$$

$$i=j : \quad \psi_i \psi_i = -\psi_i \psi_i \quad \Rightarrow \quad \psi_i^2 = \psi_i \psi_i = 0$$

For finite n, $f(\psi)$ has only finitely many terms

$$f(\psi) = f_0 + \sum_i f_i \psi_i + \sum_{i<j} f_{ij} \psi_i \psi_j + \cdots + f_{1 \dots n} \psi_1 \cdots \psi_n$$

$$\left(1 + n + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n \text{ terms at most.} \right)$$

- Integration of functons of odd variables

$$\int d^n\psi \, f(\psi) = \int d\psi_1 \cdots d\psi_n \, f(\psi_1, \dots, \psi_n)$$

Want : linearity

$$\int d^n\psi \left(f(\psi)\, a + g(\psi)\, b \right) \overset{!}{=} \left(\int d^n\psi \, f(\psi) \right) a + \left(\int d^n\psi \, g(\psi) \right) b,$$

translation invariance

$$\int d^n\psi \, f(\psi + \eta) \overset{!}{=} \int d^n\psi \, f(\psi), \quad \eta = (\eta_1, \dots, \eta_n)$$

independent of $\psi_1, \dots, \psi_n$.

- One variable case $f(\psi) = a + \psi b$

$$\int d\psi\,(a+\psi b) \overset{\text{linearity}}{=\!=} \left(\int d\psi \cdot 1\right)a + \left(\int d\psi \cdot \psi\right)b$$

$$\text{transl.inv} \quad \Vert$$

$$\int d\psi\,(a+(\psi+\eta)b) \overset{\text{linearity}}{=\!=} \left(\int d\psi \cdot 1\right)(a+\eta b) + \left(\int d\psi \cdot \psi\right)b$$

$$\therefore \left(\int d\psi \cdot 1\right)\eta b = 0 \qquad \forall_{\text{odd}}\,\eta,\ \forall b.$$

$$\Rightarrow \quad \boxed{\int d\psi \cdot 1 = 0}$$

- To have non-zero result $\int d\psi \cdot \psi \neq 0$. We set

$$\boxed{\int d\psi \cdot \psi = 1}$$

- $\int d\psi\,(a + b\psi) = \int d\psi\,(a + (-1)^{|b|}\psi b)$ (if b has a definite stat.)

$$= (-1)^{b}\, b$$

i.e. $\boxed{d\psi\, b = (-1)^{b}\, b\, d\psi}$

· Multivariable case : determined by iteration.

$$\int d\psi_1 d\psi_2 \left(f_0 + \psi_1 f_1 + \psi_2 f_2 + \psi_1 \psi_2 f_{12} \right)$$

$$= \int d\psi_1 \int d\psi_2 \left(f_0 + \psi_1 f_1 + \psi_2 f_2 + \psi_1 \psi_2 f_{12} \right)$$

$$= \int d\psi_1 \left(\underbrace{\int d\psi_2 f_0}_{0} + \underbrace{\int d\psi_2 \psi_1 f_1}_{0} + \underbrace{\int d\psi_2 \psi_2 f_2}_{1} + \underbrace{\int d\psi_2 \psi_1 \psi_2 f_{12}}_{-\psi_1} \right)$$

$$= \underbrace{\int d\psi_1 f_2}_{0} - \underbrace{\int d\psi_1 \psi_1}_{1} f_{12}$$

$$= -f_{12} \qquad \left(\Rightarrow d\psi_1 d\psi_2 = -d\psi_2 d\psi_1 \right)$$

$\vdots$

$$\int d\psi_1 \cdots d\psi_n \left(f_0 + \sum_i \psi_i f_i + \sum_{i<j} \psi_i \psi_j f_{ij} + \cdots + \underbrace{\psi_1 \cdots \psi_n}_{(-1)^{\frac{n(n-1)}{2}} \psi_n \cdots \psi_1} f_{1\cdots n} \right)$$

$$= (-1)^{\frac{n(n-1)}{2}} f_{1\cdots n}$$

$$\int d\psi_1 \cdots d\psi_n \left(f_0 + \sum_i f_i \psi_i + \cdots + f_{1\cdots n} \psi_1 \cdots \psi_n \right)$$

$$= (-1)^{|f_{1\cdots n}| + \frac{n(n-1)}{2}} f_{1\cdots n}$$

Change of variables

$A = (A_{ij})$ $n \times n$ invertible matrix (even)

Commuting case : $x_i = \sum_{j=1}^{n} A_{ij} x_j' \quad (i = 1, \cdots, n)$

$$\Rightarrow \quad dx_1 \cdots dx_n = \det A \cdot dx_1' \cdots dx_n'$$

anticommuting case : $\psi_i = \sum_{j=1}^{n} A_{ij} \psi_j' \quad (i = 1, \cdots, n)$

$$\Rightarrow \quad d\psi_1 \cdots d\psi_n = (\det A)^{-1} d\psi_1' \cdots d\psi_n'$$

For $f(\psi) = f_0 + \sum_i \underbrace{\psi_i}_{\sum_j A_{ij}\psi_j'} f_i + \cdots + \underbrace{\psi_1 \cdots \psi_n}_{= \det A\, \psi_1' \cdots \psi_n'} f_{1 \cdots n}$,

$$\int d\psi_1 \cdots d\psi_n f(\psi) = (-1)^{\frac{n(n-1)}{2}} f_{1 \cdots n}$$

$$\int d\psi_1' \cdots d\psi_n' f(\psi) = (-1)^{\frac{n(n-1)}{2}} \det A\, f_{1 \cdots n}$$

$$\therefore \int d\psi_1 \cdots d\psi_n f(\psi) = (\det A)^{-1} \int d\psi_1' \cdots d\psi_n' f(\psi)$$

$\square$

"Gaussian" integral

$$\int d\bar{\psi}\, d\psi\, e^{-\bar{\psi}\psi} = \int d\bar{\psi}\, d\psi \,(1-\bar{\psi}\psi) = 1.$$

$$\int d\bar{\psi}\, d\psi\, e^{-\bar{\psi}a\psi} = \int d\bar{\psi}\, d\psi \,(1-\bar{\psi}a\psi) = a.$$

Alternatively, we may use the rule of change of variables

$$\psi = a^{-1}\psi' \quad \Rightarrow \quad d\psi = a\, d\psi',$$

$$\int d\bar{\psi}\, d\psi\, e^{-\bar{\psi}a\psi} = \int d\bar{\psi}\, a\, d\psi'\, e^{-\bar{\psi}\psi'} = a.$$

The latter is useful for generalization to n-pairs:

$$\boxed{\int d\bar{\psi}_1 d\psi_1 \cdots d\bar{\psi}_n d\psi_n\, e^{-\sum_{i,j} \bar{\psi}_i A_{ij} \psi_j} = \det A}$$

c.f. $$\int dx_1 \cdots dx_n = e^{-\frac{1}{2}\sum_{i,j} x_i A_{ij} x_j} = \sqrt{\frac{(2\pi)^n}{\det A}}$$

$$\int d\bar{z}_1 dz_1 \cdots d\bar{z}_n dz_n\, e^{-\sum_{i,j} \bar{z}_i A_{ij} z_j} \overset{*}{=} \frac{(2\pi i)^n}{\det A}$$

* There was an error in the class. This is the correct version

Other useful relations

- $$\int d\psi\,(\psi-\eta)\,f(\psi) \overset{\text{transl.inv}}{=} \int d\psi\,\psi\,f(\psi+\eta)$$
$$= f(\eta)$$

$\therefore$ $\psi-\eta$ is like the δ-function "$\delta(\psi-\eta)$".

- $$\int d\psi_1 d\psi_2\, \psi_2\psi_1\, a = a \qquad \text{for } a\in\mathbb{C}$$
$$\Rightarrow \int \underbrace{(a\psi_2\psi_1)^*}_{\psi_1^*\psi_2^*a^*}(d\psi_1 d\psi_2)^* = a^*$$
$$\|$$
$$\int (d\psi_1 d\psi_2)^*\,\psi_1^*\psi_2^*\,a^*$$

$\therefore$ $(d\psi_1 d\psi_2)^* = d\psi_2^*\,d\psi_1^*$.

Fermionic quantum mechanics

Consider the classical mechanics with a pair of anticommuting variables $\psi(t)$, $\bar{\psi}(t)$ and Lagrangian $L = i\bar{\psi}\dot{\psi} - \omega\bar{\psi}\psi$

where $\omega \in \mathbb{R}$.

Preview of the next :

We shall quantize the system.

We take a mixed way :

Start with path-integral & find canonical commutation reln :

$$\{\hat{\psi}, \hat{\bar{\psi}}\} = \hbar, \quad \{\hat{\psi}, \hat{\psi}\} = \{\hat{\bar{\psi}}, \hat{\bar{\psi}}\} = 0 \quad \text{(Clifford algebra)}$$

Follow the operator quantization (find the representation of this algebra and determine the spectrum etc).

Then go back to path-integral to find expression for transition amplitudes & partition function.