Fermionic quantum mechanics

Consider the classical mechanics with

a pair of anticommuting variables $\psi(t)$, $\bar{\psi}(t)$ and

Lagrangian $\quad L = i\bar{\psi}\dot{\psi} - \omega\bar{\psi}\psi$

where $\omega \in \mathbb{R}$.

Symmetries

① time translation $\quad \psi(t) \to \psi(t+\delta t)$, $\bar{\psi}(t) \to \bar{\psi}(t+\delta t)$

$$\rightsquigarrow \quad \delta\psi = \epsilon\dot{\psi}\,, \quad \delta\bar{\psi} = \epsilon\dot{\bar{\psi}}$$

The Noether charge is $E = \omega\bar{\psi}\psi$ (energy)

② phase rotation $\quad \psi(t) \to e^{-i\alpha}\psi(t)$, $\bar{\psi}(t) \to e^{i\alpha}\bar{\psi}(t)$.

$$\rightsquigarrow \quad \delta\psi = -i\epsilon\psi\,, \quad \delta\bar{\psi} = i\epsilon\bar{\psi}$$

The Noether charge is $Q = \bar{\psi}\psi$ (fermion number)

Exercise: Show that the Noether charges are as given above.

• L is real (modulo total derivative) under

$$\psi^* = \bar\psi, \quad \bar\psi^* = \psi :$$

$$L^* = -i\dot\psi^*\bar\psi^* - \omega\psi^*\bar\psi^* = -i\dot{\bar\psi}\psi - \omega\bar\psi\psi$$

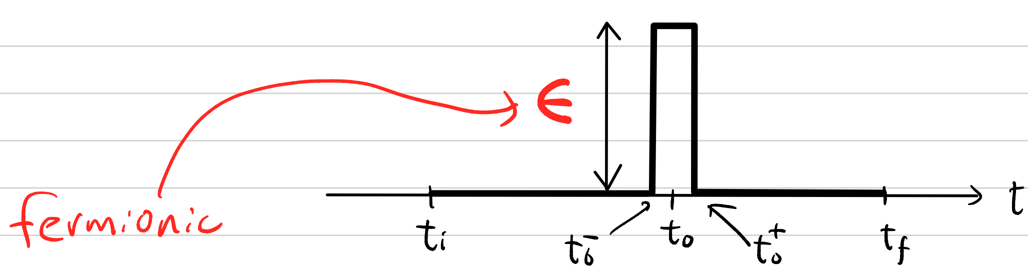
$$= \underbrace{i\bar\psi\dot\psi - \omega\bar\psi\psi}_{L} + \frac{d}{dt}\left(-i\bar\psi\psi\right).$$

$$\underline{\text{Let us quantize the system}} \left\{ \begin{array}{ll} \text{Path-integral} & ? \\ \text{Operator} & ? \end{array} \right.$$

We shall take the "mixed" way: Use the path-integral to find the commutation relation of $\hat\psi$ & $\hat{\bar\psi}$ and then find the representation of the algebra.

Consider the variation

$$\delta\psi(t) = \epsilon(t), \quad \delta\bar\psi(t) = 0$$



$$\delta S[\psi,\bar\psi] = \delta \int_{t_i}^{t_f} dt \left(i\bar\psi\dot\psi - \omega\bar\psi\psi \right)$$

$$= \int_{t_i}^{t_f} dt \left(i\bar\psi(t)\underbrace{\dot\epsilon(t)}_{\epsilon\delta(t-t_0^-)-\epsilon\delta(t-t_0^+)} - \omega\bar\psi(t)\epsilon(t) \right)$$

$$= i\bar\psi(t_0^-)\epsilon - i\bar\psi(t_0^+) - \int_{t_0^-}^{t_0^+} dt\, \omega\bar\psi\epsilon$$

Ward identity :

$$0 = \int \delta\left(\mathcal{D}\bar\psi\mathcal{D}\psi\, e^{\frac{i}{\hbar}S[\psi,\bar\psi]}\, \psi(t_0) \right)$$

$$= \int \mathcal{D}\bar\psi\mathcal{D}\psi \left(\frac{i}{\hbar}\delta S[\psi,\bar\psi]\psi(t_0) + \epsilon \right)$$

$$= \int \mathcal{D}\bar\psi\mathcal{D}\psi \Big(-\frac{1}{\hbar}\bar\psi(t_0^-)\epsilon\cdot\psi(t_0) + \frac{1}{\hbar}\bar\psi(t_0^+)\epsilon\cdot\psi(t_0)$$

$$- \frac{i}{\hbar}\int_{t_0^-}^{t_0^+} dt\, \omega\bar\psi(t)\epsilon\cdot\psi(t_0) + \epsilon \Big)$$

This implies an identity among the operators $\hat\psi$, $\hat{\bar\psi}$ corresponding to ψ, $\bar\psi$.

Take the limit $t_0^+ \searrow t_0$, $t_0^- \nearrow t_0$:

Recalling the time ordered product and noting

$\int_{t_0^-}^{t_0^+} dt\, \omega \bar{\psi} \epsilon \cdot \psi(t_0) \longrightarrow 0$ in the limit, we find

$$0 = -\frac{1}{\hbar} \hat{\psi} \circ \hat{\bar{\psi}} \epsilon + \frac{1}{\hbar} \hat{\bar{\psi}} \epsilon \circ \hat{\psi} + \epsilon$$

$\epsilon, \hat{\psi}, \hat{\bar{\psi}}$ fermionic

$$= \left[-\frac{1}{\hbar} \hat{\psi} \circ \hat{\bar{\psi}} - \frac{1}{\hbar} \hat{\bar{\psi}} \circ \hat{\psi} + 1 \right] \epsilon.$$

$$\therefore \quad \hat{\psi} \circ \hat{\bar{\psi}} + \hat{\psi} \circ \hat{\bar{\psi}} = \hbar.$$

If we use the notation

$$\{A, B\} := AB + BA \quad \text{(anticommutator)}$$

this can be written as

$$\{\hat{\psi}, \hat{\bar{\psi}}\} = \hbar.$$

Similarly,

- $$0 = \int \delta\left(\mathcal{D}\bar{\psi}\,\mathcal{D}\psi\, e^{\frac{i}{\hbar}S[\psi,\bar{\psi}]}\,\bar{\psi}(t_0)\right) \quad \text{for the same } \delta$$

$$\Rightarrow \quad \hat{\bar{\psi}} \circ \hat{\bar{\psi}} = 0$$

- $$0 = \int \delta\left(\mathcal{D}\bar{\psi}\,\mathcal{D}\psi\, e^{\frac{i}{\hbar}S[\psi,\bar{\psi}]}\,\psi(t_0)\right)$$

for $\delta\psi(t) = 0$ and $\delta\bar{\psi}(t) =$ the same $\epsilon(t)$

$$\Rightarrow \quad \hat{\psi} \circ \hat{\psi} = 0$$

The canonical commutation relation of $\hat{\psi}$ and $\hat{\bar{\psi}}$ is

$$\boxed{\begin{gathered} \{\hat{\psi}, \hat{\bar{\psi}}\} = \hbar, \\ \hat{\psi}^2 = \hat{\bar{\psi}}^2 = 0. \end{gathered}}$$

... "Clifford algebra".

Representation of the algebra.

Reality of the variables $\psi^* = \bar{\psi}$, $\bar{\psi}^* = \psi$

$\rightsquigarrow$ hermiticity of the operators $\hat{\psi}^\dagger = \hat{\bar{\psi}}$, $\hat{\bar{\psi}}^\dagger = \hat{\psi}$.

The commutation relation reads

$$\boxed{\begin{aligned} &\{\hat{\psi}, \hat{\psi}^\dagger\} = \hbar \\ &\{\hat{\psi}, \hat{\psi}\} = \{\hat{\psi}^\dagger, \hat{\psi}^\dagger\} = 0. \end{aligned}}$$

This is like the algebra of annihilation/creation operators in harmonic oscillators. As in that case, we may prepare a state $|0\rangle$ annihilated by $\hat{\psi}$,

$$\hat{\psi}|0\rangle = 0$$

and build other states by multiplying powers of $\hat{\psi}^\dagger$,

$$\hat{\psi}^\dagger|0\rangle, \ \hat{\psi}^{\dagger 2}|0\rangle, \ \hat{\psi}^{\dagger 3}|0\rangle, \ \cdots.$$

But, by the relation $\hat{\psi}^{\dagger 2} = 0$, $\hat{\psi}^{\dagger 2}|0\rangle = \hat{\psi}^{\dagger 3}|0\rangle = \cdots = 0$.

Only $\hat{\psi}^\dagger|0\rangle$ among them can be non-zero.

Also,

$$\hat{\varphi}(\hat{\varphi}^+|0\rangle) = (\underbrace{\hat{\varphi}\hat{\varphi}^+ + \hat{\varphi}^+\hat{\varphi}}_{\hbar})|0\rangle - \hat{\varphi}^+\underbrace{\hat{\varphi}|0\rangle}_{0} = \hbar|0\rangle.$$

Thus $\hat{\varphi}^+|0\rangle$ is indeed non-zero. <u>We have a 2-dimensional representation</u> $\mathcal{H}$, with basis $|0\rangle$, $\hat{\varphi}^+|0\rangle$.

With respect to this basis, $\hat{\varphi}$ and $\hat{\varphi}^+$ are represented by matrices

$$\hat{\varphi} \doteq \begin{pmatrix} 0 & \hbar \\ 0 & 0 \end{pmatrix}, \quad \hat{\varphi}^+ \doteq \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

If $|0\rangle$ is normalized as $\| |0\rangle \|^2 = \langle 0|0\rangle = 1$,

$$\left.\begin{aligned} &\| |0\rangle \|^2 = \langle 0|0\rangle = 1, \\ &\| \hat{\varphi}^+|0\rangle \|^2 = \langle 0|\hat{\varphi}\hat{\varphi}^+|0\rangle = \langle 0|\hbar|0\rangle = \hbar. \end{aligned}\right\} \text{positive definite!}$$

<u>Energy spectrum?</u>

Recall $E = \omega\bar{\varphi}\varphi$.

$$\rightsquigarrow \hat{H} = \omega\hat{\varphi}^+\hat{\varphi} \doteq \begin{cases} 0 & \text{on } |0\rangle \\ \hbar\omega & \text{on } \hat{\varphi}^+|0\rangle \end{cases}$$

※ There is an <u>operator ordering ambiguity</u>.

Classically, $\bar{\psi}\psi = (1-s)\bar{\psi}\psi - s\psi\bar{\psi}$ for any s, but they correspond to different operators in the quantum theory:

$$\hat{H} = (1-s)\omega\hat{\psi}^{+}\hat{\psi} - s\omega\hat{\psi}\hat{\psi}^{+} = \omega\hat{\psi}^{+}\hat{\psi} - s\hbar\omega.$$

$$= \begin{cases} -s\hbar\omega & \text{on } |0\rangle \\ (1-s)\hbar\omega & \text{on } \hat{\psi}^{+}|0\rangle. \end{cases}$$

e.g. the "symmetric" ordering $s=\frac{1}{2}$:

$$\hat{H} = \frac{1}{2}\omega\hat{\psi}^{+}\hat{\psi} - \frac{1}{2}\omega\hat{\psi}\hat{\psi}^{+} = \frac{\omega}{2}[\hat{\psi}^{+}, \hat{\psi}]$$

$$= \begin{cases} -\frac{1}{2}\hbar\omega & \text{on } |0\rangle \\ \frac{1}{2}\hbar\omega & \text{on } \hat{\psi}^{+}|0\rangle. \end{cases}$$

<u>Also</u> $Q = \bar{\psi}\psi \rightsquigarrow \hat{Q} = \hat{\psi}^{+}\hat{\psi}$

or $\hat{Q} = (1-s)\hat{\psi}^{+}\hat{\psi} - s\hat{\psi}\hat{\psi}^{+} = \hat{\psi}^{+}\hat{\psi} - s\hbar$

$$= \begin{cases} -s\hbar & \text{on } |0\rangle \\ (1-s)\hbar & \text{on } \hat{\psi}^{+}|0\rangle. \end{cases}$$

$\hat{\psi}$ & $\hat{\psi}^\dagger$ eigenstates

As a passage to (go back to) the path-integral, we find eigenstates of $\hat{\psi}$ and $\hat{\psi}^\dagger$ (just like eigenstates $|q\rangle$ & $|p\rangle$ for $\hat{q}$ & $\hat{p}$).

Suppose
$$|\psi\rangle = a|0\rangle + b\hat{\psi}^\dagger|0\rangle \qquad \qquad \hat{\psi}|\psi\rangle = \psi|\psi\rangle$$
satisfy
$$|\bar{\psi}\rangle = c|0\rangle + d\hat{\psi}^\dagger|0\rangle \qquad \qquad \hat{\psi}^\dagger|\bar{\psi}\rangle = \bar{\psi}|\bar{\psi}\rangle.$$

As
$$\hat{\psi}|\psi\rangle = (-1)^{|a|} a \underbrace{\hat{\psi}|0\rangle}_{0} + (-1)^{|b|} b \underbrace{\hat{\psi}\hat{\psi}^\dagger|0\rangle}_{\hbar|0\rangle}$$
$$\overset{!}{\|}$$
$$\psi|\psi\rangle = \psi a|0\rangle + \psi b\hat{\psi}^\dagger|0\rangle$$

$$\hat{\psi}^\dagger|\bar{\psi}\rangle = (-1)^{|c|} c\hat{\psi}^\dagger|0\rangle + (-1)^{|d|} d \underbrace{\hat{\psi}^\dagger\hat{\psi}^\dagger}_{0}|0\rangle$$
$$\overset{!}{\|}$$
$$\bar{\psi}|\bar{\psi}\rangle = \bar{\psi}c|0\rangle + \bar{\psi}d\hat{\psi}^\dagger|0\rangle$$

We find
$$\psi a = (-1)^{|b|} b\hbar, \qquad \psi b = 0$$
$$\bar{\psi}c = 0, \qquad \bar{\psi}d = (-1)^{|c|} c$$

A solution
$$(a, b) = (1, -\psi/\hbar)$$
$$(c, d) = (\bar{\psi}, -1)$$

$$|\psi\rangle = \left(1 - \frac{\psi}{\hbar}\hat{\psi}^\dagger\right)|0\rangle = \left(1 + \frac{1}{\hbar}\hat{\psi}^\dagger\psi\right)|0\rangle,$$

$$|\bar{\psi}) = (\bar{\psi} - \hat{\psi}^\dagger)|0\rangle.$$

$\Bigg\}$ ✪

We have chosen a random normalization, but this turns out to be a useful one.

(A more symmetric normalization would be

$$|\psi\rangle = \hbar^{\frac{1}{4}}\left(1 + \frac{1}{\hbar}\hat{\psi}^\dagger\psi\right)|0\rangle, \quad |\bar{\psi}) = \hbar^{-\frac{1}{4}}(\bar{\psi} - \hat{\psi}^\dagger)|0\rangle.$$

This is also good, but we shall use ✪)

We also prepare "bra" eigenstates for $\hat{\psi}$ & $\hat{\psi}^\dagger$:

$$\langle\psi|\hat{\psi} = \langle\psi|\psi, \quad \langle\bar{\psi}|\hat{\psi}^\dagger = \langle\bar{\psi}|\bar{\psi}.$$

For these we may take

$$\langle\psi| = |\bar{\psi}\rangle^\dagger = \langle 0|(\psi - \hat{\psi})$$

$$\langle\bar{\psi}| = |\psi\rangle^\dagger = \langle 0|\left(1 + \frac{1}{\hbar}\bar{\psi}\hat{\psi}\right).$$

[(∵) Take the hermitian conjugate of

$\hat{\psi}^\dagger|\bar{\psi}\rangle = \bar{\psi}|\bar{\psi}\rangle$ and $\hat{\psi}|\psi\rangle = \psi|\psi\rangle$. □]

Just like $\int dq\, |q\rangle\langle q| = \int dp\, |p\rangle\langle p| = \mathrm{id}_{\mathcal{H}}$, we have

$$\boxed{\int d\psi\, |\psi\rangle\langle\psi| = \int d\bar\psi\, |\bar\psi\rangle\langle\bar\psi| = \mathrm{id}_{\mathcal{H}}.}$$

<u>proof</u>

$$\int d\psi\, |\psi\rangle\langle\psi|0\rangle = \int d\psi\, \left(1+\frac{1}{\hbar}\hat\psi^{+}\psi\right)|0\rangle\langle 0|(\psi-\underbrace{\hat\psi)|0\rangle}_{0}$$

$|0\rangle\langle 0|$ is even

$$= \int d\psi\, \left(1+\frac{1}{\hbar}\hat\psi^{+}\underbrace{\psi\right)\psi}_{0}|0\rangle\langle 0|0\rangle = |0\rangle.$$

$$\int d\psi\, |\psi\rangle\langle\psi|\hat\psi^{+}|0\rangle = \int d\psi\, \left(1+\frac{1}{\hbar}\hat\psi^{+}\psi\right)|0\rangle\langle 0|(\psi-\underbrace{\hat\psi)\hat\psi^{+}}_{-\hbar}|0\rangle$$

$$= \int d\psi\, \left(1-\psi\frac{1}{\hbar}\hat\psi^{+}\right)|0\rangle\langle 0|(-\hbar)|0\rangle = \hat\psi^{+}|0\rangle.$$

$$\int d\bar\psi\, |\bar\psi\rangle\langle\bar\psi|0\rangle = \int d\bar\psi\, (\bar\psi-\hat\psi^{+})|0\rangle\langle 0|\left(1+\frac{1}{\hbar}\bar\psi\underbrace{\hat\psi\right)|0\rangle}_{0}$$

$$= \int d\bar\psi\, (\bar\psi-\hat\psi^{+})|0\rangle\langle 0|0\rangle = |0\rangle.$$

$$\int d\bar\psi\, |\bar\psi\rangle\langle\bar\psi|\hat\psi^{+}|0\rangle = \int d\bar\psi\, (\bar\psi-\hat\psi^{+})|0\rangle\langle 0|\left(1+\frac{1}{\hbar}\bar\psi\underbrace{\hat\psi\right)\hat\psi^{+}}_{\hbar}|0\rangle$$

$$= \int d\bar\psi\, (\bar\psi-\hat\psi^{+})|0\rangle\langle 0|\bar\psi|0\rangle$$

$|0\rangle\langle 0|$ is even

$$= \int d\bar\psi\, (\overbrace{\bar\psi-\hat\psi^{+}}^{0})\bar\psi|0\rangle\langle 0|0\rangle = \hat\psi^{+}|0\rangle.$$

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<u>Q.E.D.</u>

Note also

$$\langle \psi_1 | \bar{\psi}_2 \rangle = \langle 0 | (\psi_1 - \hat{\psi})(\bar{\psi}_2 - \hat{\psi}^{\dagger}) | 0 \rangle = \langle 0 | (\overbrace{\psi_1 \bar{\psi}_2}^{\text{even}} + \hbar) | 0 \rangle$$

$$= \hbar + \psi_1 \bar{\psi}_2 = \hbar \left(1 + \frac{1}{\hbar} \psi_1 \bar{\psi}_2 \right) = \hbar \, e^{\frac{1}{\hbar} \psi_1 \bar{\psi}_2},$$

$$\langle \bar{\psi}_1 | \psi_2 \rangle = \langle 0 | \left(1 + \frac{1}{\hbar} \bar{\psi}_1 \hat{\psi}\right)\left(1 + \frac{1}{\hbar} \hat{\psi}^{\dagger} \psi_2\right) | 0 \rangle$$

$$= \langle 0 | \left(1 + \frac{1}{\hbar} \bar{\psi}_1 \psi_2\right) | 0 \rangle = 1 + \frac{1}{\hbar} \bar{\psi}_1 \psi_2 = e^{\frac{1}{\hbar} \bar{\psi}_1 \psi_2},$$

$$\langle \psi_1 | \psi_2 \rangle = \langle 0 | (\psi_1 - \hat{\psi}) \left(1 + \frac{1}{\hbar} \hat{\psi}^{\dagger} \psi_2\right) | 0 \rangle$$

$$= \langle 0 | (\psi_1 - \psi_2) | 0 \rangle = (-1)^{| |0\rangle |} (\psi_1 - \psi_2),$$

$$\langle \bar{\psi}_1 | \bar{\psi}_2 \rangle = \langle 0 | \left(1 + \frac{1}{\hbar} \bar{\psi}_1 \hat{\psi}\right)(\bar{\psi}_2 - \hat{\psi}^{\dagger}) | 0 \rangle$$

$$= \langle 0 | (\bar{\psi}_2 - \bar{\psi}_1) | 0 \rangle = -(-1)^{| |0\rangle |} (\bar{\psi}_1 - \bar{\psi}_2).$$

(The latter two can also be used for derivation of

$$\int d\psi \, |\psi\rangle\langle\psi| = \int d\bar{\psi} \, |\bar{\psi}\rangle\langle\bar{\psi}| = \mathrm{id}_{\mathcal{H}}$$

[$$\int d\psi \, |\psi\rangle\langle\psi|\psi_1\rangle = \int d\psi \, |\psi\rangle (-1)^{| |0\rangle |} (\psi - \psi_1)$$

$| |\psi\rangle | = | |0\rangle |$

$$= \int d\psi \, (\psi - \psi_1) |\psi\rangle = |\psi_1\rangle \quad \text{etc.}$$])

We just use $(\psi - \psi_1) = \text{"}\delta(\psi - \psi_1)\text{"}$.

For $A \in \mathrm{End}_{\mathbb{C}} \mathcal{H}$

$$\mathrm{Tr}_{\mathcal{H}} A = \int d\bar\psi \, d\psi \, \langle -\bar\psi | A | \psi \rangle \langle \psi | \bar\psi \rangle$$

It is enough to show this for $A = \mathrm{id}, \hat\psi, \hat\psi^\dagger, \hat\psi^\dagger\hat\psi$ as they span $\mathrm{End}_{\mathbb{C}} \mathcal{H}$.

$A = \mathrm{id}$: $\displaystyle \int d\bar\psi \, d\psi \, \underbrace{\langle -\bar\psi | \psi \rangle}_{e^{-\frac{1}{\hbar}\bar\psi\psi}} \underbrace{\langle \psi | \bar\psi \rangle}_{\hbar e^{\frac{1}{\hbar}\psi\bar\psi}} = \int \hbar \, d\bar\psi \, d\psi \, e^{-\frac{2}{\hbar}\bar\psi\psi} = 2$

$= \mathrm{tr}\, \mathrm{id}$ ✓

$A = \hat\psi$: $\displaystyle \int d\bar\psi \, d\psi \, \langle -\bar\psi | \underbrace{\hat\psi | \psi \rangle}_{\psi | \psi \rangle} \langle \psi | \bar\psi \rangle = \int \hbar \, d\bar\psi \, d\psi (\pm\psi) e^{-\frac{2}{\hbar}\bar\psi\psi} = 0$

$= \mathrm{tr}\, \hat\psi$ ✓

$A = \hat\psi^\dagger$: $\displaystyle \int d\bar\psi \, d\psi \, \underbrace{\langle -\bar\psi | \hat\psi^\dagger}_{\langle -\bar\psi | (-\bar\psi)} | \psi \rangle \langle \psi | \bar\psi \rangle = \int \hbar \, d\bar\psi \, d\psi (\mp\bar\psi) e^{-\frac{2}{\hbar}\bar\psi\psi} = 0$

$= \mathrm{tr}\, \hat\psi^\dagger$ ✓

$A = \hat\psi^\dagger\hat\psi$: $\displaystyle \int d\bar\psi \, d\psi \, \langle -\bar\psi | \underbrace{\hat\psi^\dagger \hat\psi}_{-\bar\psi\psi} | \psi \rangle \langle \psi | \bar\psi \rangle = \int \hbar \, d\bar\psi \, d\psi (-\bar\psi\psi) e^{-\frac{2}{\hbar}\bar\psi\psi}$

$= \hbar = \mathrm{tr}(\hat\psi^\dagger\hat\psi)$ ✓

□

We also have

$$\mathrm{tr}_{\mathcal{H}} A = \int d\bar\psi\, d\psi \, \langle \psi | A | \bar\psi\rangle\langle\bar\psi|-\psi\rangle$$

$$= (-1)^{|\,|0\rangle|} \int d\psi \, \langle\psi|A|-\psi\rangle$$

$$= (-1)^{|\,|0\rangle|}\int d\bar\psi \, \langle -\bar\psi|A|\bar\psi\rangle$$

bar $\bar\psi$ was missing in the note used in Lecture

If we define $(-1)^F$ by $(-1)^F = \begin{cases} +1 & \text{on } |0\rangle \\ -1 & \text{on } \hat\psi^\dagger|0\rangle \end{cases}$,

then $(-1)^F|\psi\rangle = |-\psi\rangle$ $\qquad |\psi\rangle = (1 + \frac{1}{\hbar}\hat\psi^\dagger\psi)|0\rangle$,

& $\langle\bar\psi|(-1)^F = \langle-\bar\psi|$.

$$\boxed{\therefore \quad \mathrm{tr}_{\mathcal{H}} (-1)^F A = \int d\bar\psi\, d\psi \, \langle\bar\psi|A|\psi\rangle\langle\psi|\bar\psi\rangle}$$

States as wave functions

We can represent states as functions of $\bar{\psi}$:

$$|\Psi\rangle \longleftrightarrow \Psi(\bar{\psi}) = \langle \bar{\psi} | \Psi \rangle .$$

Note: $\Psi(\bar{\psi})^* = \langle \bar{\psi} | \Psi \rangle^* = \langle \Psi | \psi \rangle .$

- The inner product of states is represented as

$$\langle \Psi_1 | \Psi_2 \rangle = \langle \Psi_1 | \underbrace{\int d\psi \, |\psi\rangle\langle\psi|}_{\text{even}} \int d\bar{\psi} \, |\bar{\psi}\rangle\langle\bar{\psi}|\Psi_2\rangle$$

$$= \langle \Psi_1 | \int \underbrace{d\bar{\psi}\, d\psi}_{\text{even}} \, |\psi\rangle\langle\psi|\bar{\psi}\rangle\langle\bar{\psi}|\Psi_2\rangle$$

$$= \int d\bar{\psi}\, d\psi \, \underbrace{\langle \Psi_1 | \psi \rangle}_{\Psi_1(\bar{\psi})^*} \underbrace{\langle \psi | \bar{\psi} \rangle}_{\hbar e^{\frac{1}{\hbar}\psi\bar{\psi}}} \underbrace{\langle \bar{\psi} | \Psi_2 \rangle}_{\Psi_2(\bar{\psi})}$$

$$= \int \hbar \, d\bar{\psi}\, d\psi \, \Psi_1(\psi)^* \, e^{\frac{1}{\hbar}\psi\bar{\psi}} \, \Psi_2(\bar{\psi})$$

• Time evolution of states:

$$\left(e^{-i\frac{t_f-t_i}{\hbar}\hat{H}}\Psi\right)(\bar{\psi}) = \langle\bar{\psi}|e^{-i\frac{t_f-t_i}{\hbar}\hat{H}}|\Psi\rangle$$

$$= \langle\bar{\psi}|e^{-i\frac{t_f-t_i}{\hbar}\hat{H}}\int d\psi'|\psi'\rangle\langle\psi'|\int d\bar{\psi}'|\bar{\psi}'\rangle\langle\bar{\psi}'|\Psi\rangle$$

$$= \int d\bar{\psi}'d\psi'\,\langle\bar{\psi}|e^{-i\frac{t_f-t_i}{\hbar}\hat{H}}|\psi'\rangle\underbrace{\langle\psi'|\bar{\psi}'\rangle}_{\hbar e^{\frac{1}{\hbar}\psi'\bar{\psi}'}}\underbrace{\langle\bar{\psi}'|\Psi\rangle}_{\Psi(\bar{\psi}')}$$

As in the bosonic case, let us define the transition amplitude by

$$Z(t_f,\bar{\psi}_f\,;\,t_i,\psi_i) := \langle\bar{\psi}_f|e^{-i\frac{(t_f-t_i)}{\hbar}\hat{H}}|\psi_i\rangle.$$

Then the time evolution of states is given by

$$\left(e^{-i\frac{t_f-t_i}{\hbar}\hat{H}}\Psi\right)(\bar{\psi}_f)$$

$$= \int\hbar\,d\bar{\psi}_i\,d\psi_i\;Z(t_f,\bar{\psi}_f\,;\,t_i,\psi_i)\,e^{\frac{1}{\hbar}\psi_i\bar{\psi}_i}\,\Psi(\bar{\psi}_i).$$

Path-integral expression for $Z(t_f, \overline{\psi}_f ; t_i, \psi_i)$

Let us divide $t_f - t_i$ into N pieces, $t_f - t_i = N\epsilon$,

$$e^{-i\frac{t_f - t_i}{\hbar}\hat{H}} = \underbrace{e^{-\frac{i\epsilon}{\hbar}\hat{H}} \cdots\cdots e^{-\frac{i\epsilon}{\hbar}\hat{H}}}_{N}$$

and insert

$$\mathrm{id} = \int d\psi_{j+1} |\psi_{j+1}\rangle\langle\psi_{j+1}| \int d\overline{\psi}_j |\overline{\psi}_j\rangle\langle\overline{\psi}_j|$$

$$= \int d\overline{\psi}_j \, d\psi_{j+1} |\psi_{j+1}\rangle\langle\psi_{j+1}|\overline{\psi}_j\rangle\langle\overline{\psi}_j|$$

$$= \int \hbar \, d\overline{\psi}_j \, d\psi_{j+1} |\psi_{j+1}\rangle e^{-\frac{1}{\hbar}\overline{\psi}_j \psi_{j+1}} \langle\overline{\psi}_j|$$

into the j-th slot $(j = 1, 2, \cdots, N-1)$ and use

$$\langle\overline{\psi}_j| e^{-i\frac{\epsilon}{\hbar}\hat{H}} |\psi_j\rangle = \langle\overline{\psi}_j| \left(1 - i\frac{\epsilon}{\hbar} H(\overline{\psi}_j, \psi_j) + O(\epsilon^2)\right) |\psi_j\rangle$$

$$= \left(1 - \frac{i\epsilon}{\hbar} H(\overline{\psi}_j, \psi_j) + O(\epsilon^2)\right) \langle\overline{\psi}_j|\psi_j\rangle$$

$$= e^{\frac{1}{\hbar}\overline{\psi}_j\psi_j - \frac{i\epsilon}{\hbar} H(\overline{\psi}_j, \psi_j)} + O(\epsilon^2)$$

Then, $Z(t_f, \bar{\psi}_f ; t_i, \psi_i)$

$$= \int \prod_{j=1}^{N-1} \hbar\, d\bar{\psi}_j\, d\psi_{j+1}\; e^{\frac{1}{\hbar}\bar{\psi}_f \psi_N - \frac{i\epsilon}{\hbar} H(\bar{\psi}_f, \psi_N)}$$

$$\cdot e^{-\frac{1}{\hbar}\bar{\psi}_{N-1}\psi_N} \cdot e^{\frac{1}{\hbar}\bar{\psi}_{N-1}\psi_{N-1} - \frac{i\epsilon}{\hbar} H(\bar{\psi}_{N-1}, \psi_{N-1})}$$

$$\cdots\cdots \cdot e^{-\frac{1}{\hbar}\bar{\psi}_1 \psi_2} \cdot e^{\frac{1}{\hbar}\bar{\psi}_1 \psi_i - \frac{i\epsilon}{\hbar} H(\bar{\psi}_1, \psi_i)} + O(N\epsilon^2)$$

$$= \int \prod_{j=1}^{N-1} \hbar\, d\bar{\psi}_j\, d\psi_{j+1}\; e^{\frac{1}{\hbar}\bar{\psi}_N \psi_N - \frac{i\epsilon}{\hbar} H(\bar{\psi}_N, \psi_N)}$$

$$\cdot \underbrace{\prod_{j=1}^{N-1} e^{-\frac{1}{\hbar}\bar{\psi}_j (\psi_{j+1} - \psi_j) - \frac{i\epsilon}{\hbar} H(\bar{\psi}_j, \psi_j)}}_{= e^{\frac{i}{\hbar}\sum_{j=1}^{N-1} \epsilon \left(i\bar{\psi}_j \frac{\psi_{j+1}-\psi_j}{\epsilon} - H(\bar{\psi}_j, \psi_j)\right)}} + O(N\epsilon^2)$$

with $\bar{\psi}_N := \bar{\psi}_f$, $\psi_1 := \psi_i$

$N \to \infty$

$N\epsilon = t_f - t_i$: fixed

$$\longrightarrow \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\; e^{\frac{1}{\hbar}\bar{\psi}\psi\big|_{t_f} + \frac{i}{\hbar}\int_{t_i}^{t_f} dt \left(i\bar{\psi}\dot{\psi} - H(\bar{\psi}, \psi)\right)}$$

$\bar{\psi}(t_f) = \bar{\psi}_f$, $\psi(t_i) = \psi_i$

$Z(t_f, \bar\psi_f ; t_i, \psi_i)$

$$= \int \mathcal{D}\bar\psi \mathcal{D}\psi \; e^{\underbrace{\frac{1}{\hbar}\bar\psi\psi|_{t_f} + \frac{i}{\hbar}\int_{t_i}^{t_f} dt \underbrace{\left(i\bar\psi\dot\psi - H(\bar\psi,\psi)\right)}_{\text{Lagrangian}}}_{\text{What is this ?}}}$$

$$\bar\psi(t_f) = \bar\psi_f, \quad \psi(t_i) = \psi_i$$

Rmk $\bar\psi(t_f)$, $\psi(t_i)$ fixed

$\Rightarrow$ E.O.M. comes from E.L. eqn for

$$-i\bar\psi\psi|_{t_f} + \int_{t_i}^{t_f} dt \left(i\bar\psi\dot\psi - H(\bar\psi,\psi)\right).$$

∵ $\delta\left(-i\bar\psi\psi|_{t_f} + \int_{t_i}^{t_f} dt \left(i\bar\psi\dot\psi - H(\bar\psi,\psi)\right)\right)$

$$= -i\,\cancel{\delta\bar\psi}\,\psi|_{t_f} - i\bar\psi\,\delta\psi|_{t_f}$$

$$+ \int_{t_i}^{t_f} dt \left(i\delta\bar\psi\,\dot\psi + \underbrace{i\bar\psi\,\delta\dot\psi}_{\frac{d}{dt}(\bar\psi\delta\psi) - \dot{\bar\psi}\delta\psi} - \delta H(\bar\psi,\psi) \right)$$

$$= -i\,\cancel{\bar\psi\delta\psi}|_{t_f} + i\,\cancel{\bar\psi\delta\psi}|_{t_f} - i\bar\psi\,\cancel{\delta\psi}|_{t_i}$$

$$+ \int_{t_i}^{t_f} dt \left(\underbrace{i\delta\bar\psi\,\dot\psi - i\dot{\bar\psi}\,\delta\psi - \delta H(\bar\psi,\psi)}_{\Rightarrow \text{E.O.M}} \right)$$

//

For other boundary conditions, EOM is obtained by EL eqn for

$\psi(t_f), \bar{\psi}(t_i)$ fixed: $\int_{t_i}^{t_f} dt\,(i\bar{\psi}\dot{\psi} - H(\bar{\psi},\psi)) + i\bar{\psi}\psi|_{t_i}$,

$\psi(t_f), \psi(t_i)$ fixed: $\int_{t_i}^{t_f} dt\,(i\bar{\psi}\dot{\psi} - H(\bar{\psi},\psi))$,

$\bar{\psi}(t_f), \bar{\psi}(t_i)$ fixed: $\int_{t_i}^{t_f} dt\,(-i\dot{\bar{\psi}}\psi - H(\bar{\psi},\psi))$.

Correspondingly,

$$Z(t_f,\psi_f;t_i,\bar{\psi}_i) = \langle\psi_f|e^{-i\frac{t_f-t_i}{\hbar}\hat{H}}|\bar{\psi}_i\rangle$$

$$= \int_{\psi(t_f)=\psi_f,\ \bar{\psi}(t_i)=\bar{\psi}_i} \mathcal{D}\bar{\psi}\mathcal{D}\psi\ e^{\frac{i}{\hbar}\int_{t_i}^{t_f} dt\,(i\bar{\psi}\dot{\psi} - H(\bar{\psi},\psi)) - \frac{1}{\hbar}\bar{\psi}\psi|_{t_i}}$$

$$Z(t_f,\psi_f;t_i,\psi_i) = \langle\psi_f|e^{-i\frac{t_f-t_i}{\hbar}\hat{H}}|\psi_i\rangle$$

$$= \int_{\psi(t_f)=\psi_f,\ \psi(t_i)=\psi_i} \mathcal{D}\bar{\psi}\mathcal{D}\psi\ e^{\frac{i}{\hbar}\int_{t_i}^{t_f} dt\,(i\bar{\psi}\dot{\psi} - H(\bar{\psi},\psi))}$$

$$Z(t_f,\bar{\psi}_f;t_i,\bar{\psi}_i) = \langle\bar{\psi}_f|e^{-i\frac{t_f-t_i}{\hbar}\hat{H}}|\bar{\psi}_i\rangle$$

$$= \int_{\bar{\psi}(t_f)=\bar{\psi}_f,\ \bar{\psi}(t_i)=\bar{\psi}_i} \mathcal{D}\bar{\psi}\mathcal{D}\psi\ e^{\frac{i}{\hbar}\int_{t_i}^{t_f} dt\,(-i\dot{\bar{\psi}}\psi - H(\bar{\psi},\psi))}$$

Path-integral expression for partition functions

$$\mathrm{Tr}_{\mathcal{H}}\, e^{-\frac{T}{\hbar}\hat{H}} = \int d\bar{\psi}\, d\psi\, \langle -\bar{\psi}| e^{-\frac{T}{\hbar}\hat{H}} |\psi\rangle\langle\psi|\bar{\psi}\rangle$$

- $\langle\psi|\bar{\psi}\rangle = \hbar\, e^{-\frac{1}{\hbar}\bar{\psi}\psi}$
- $\langle -\bar{\psi}| e^{-\frac{T}{\hbar}\hat{H}} |\psi\rangle$

$$= \int \prod_{j=1}^{N-1} \hbar\, d\bar{\psi}_j\, d\psi_{j+1} \cdot e^{\frac{1}{\hbar}\bar{\psi}_N\psi_N - \frac{\epsilon}{\hbar}H(\bar{\psi}_N,\psi_N)} \cdot e^{-\frac{1}{\hbar}\sum_{j=1}^{N-1}\epsilon\left(\bar{\psi}_j \frac{\psi_{j+1}-\psi_j}{\epsilon} + H(\bar{\psi}_j,\psi_j)\right)} + O(N\epsilon^2)$$

with $\bar{\psi}_N := -\bar{\psi}$, $\psi_1 := \psi$

$$= \int \hbar\, d\bar{\psi}\, d\psi \prod_{j=1}^{N-1} \hbar\, d\bar{\psi}_j\, d\psi_{j+1}\, e^{-\frac{1}{\hbar}\bar{\psi}\psi + \frac{1}{\hbar}\bar{\psi}_N\psi_N - \frac{\epsilon}{\hbar}H(\bar{\psi}_N,\psi_N)} \cdot e^{-\frac{1}{\hbar}\sum_{j=1}^{N-1}\epsilon\left(\bar{\psi}_j \frac{\psi_{j+1}-\psi_j}{\epsilon} + H(\bar{\psi}_j,\psi_j)\right)} + O(N\epsilon^2)$$

with $\bar{\psi}_N := -\bar{\psi}$, $\psi_1 := \psi$

Write $\bar{\psi} = \bar{\psi}_0 = -\bar{\psi}_N$, $\psi = \psi_1 = -\psi_{N+1}$

$$= \int \prod_{j=0}^{N-1} \hbar\, d\bar{\psi}_j\, d\psi_{j+1}\, e^{-\frac{1}{\hbar}\sum_{j=1}^{N}\epsilon\left(\bar{\psi}_j \frac{\psi_{j+1}-\psi_j}{\epsilon} + H(\bar{\psi}_j,\psi_j)\right)} + O(N\epsilon^2)$$

with $\bar{\psi}_N := -\bar{\psi}_0$, $\psi_{N+1} := -\psi_1$

Let us define $\psi(\epsilon j), \overline{\psi}(\epsilon j)$ for $j \in \mathbb{Z}$ by

$$\psi(\epsilon(j+1)) = \psi_{j+1}, \quad \overline{\psi}(\epsilon j) = \overline{\psi}_j \quad \text{for } j = 0, 1, \cdots, N-1$$

& $$\psi(\epsilon j + T) = \psi(\epsilon(j+N)) = -\psi(\epsilon j)$$

$$\overline{\psi}(\epsilon j + T) = \overline{\psi}(\epsilon(j+N)) = -\overline{\psi}(\epsilon j)$$

(Consistent by $\overline{\psi}_N = -\overline{\psi}_0$ & $\psi_{N+1} = -\psi_1$).

Then, the exponent is

$$-\frac{1}{\hbar} \sum_{j \in \mathbb{Z}/N\mathbb{Z}} \epsilon \left(\overline{\psi}(\epsilon j) \frac{\psi(\epsilon j + \epsilon) - \psi(\epsilon j)}{\epsilon} + H(\overline{\psi}(\epsilon j), \psi(\epsilon j)) \right)$$

$$\xrightarrow[N\epsilon = T]{N \to \infty} -\frac{1}{\hbar} \int_{\mathbb{R}/T\mathbb{Z}} d\tau \underbrace{\left(\overline{\psi}(\tau) \frac{d\psi(\tau)}{d\tau} + H(\overline{\psi}(\tau), \psi(\tau)) \right)}_{L_E(\overline{\psi}, \psi, \frac{d\psi}{d\tau})}$$

$$\therefore \operatorname{Tr}_{\mathcal{H}} e^{-\frac{T}{\hbar}\hat{H}}$$

$$= \int \mathcal{D}\overline{\psi} \mathcal{D}\psi \, e^{-\frac{1}{\hbar}\int_{\mathbb{R}/T\mathbb{Z}} d\tau \, L_E(\overline{\psi}, \psi, \frac{d\psi}{d\tau})}$$

$$\overline{\psi}(\tau + T) = -\overline{\psi}(\tau), \quad \psi(\tau + T) = -\psi(\tau)$$

antiperiodic

$$\mathrm{Tr}_{\mathcal{H}}(-1)^F e^{-\frac{T}{\hbar}\hat{H}} = \int d\bar{\psi}\, d\psi\, \langle \bar{\psi} | e^{-\frac{T}{\hbar}\hat{H}} | \psi \rangle \langle \psi | \bar{\psi} \rangle$$

$$= \int \hbar\, d\bar{\psi}\, d\psi \prod_{j=1}^{N-1} \hbar\, d\bar{\psi}_j\, d\psi_{j+1}\; e^{-\frac{1}{\hbar}\bar{\psi}\psi + \frac{1}{\hbar}\bar{\psi}_N\psi_N - \frac{\epsilon}{\hbar}H(\bar{\psi}_N, \psi_N)} \cdot e^{-\frac{1}{\hbar}\sum_{j=1}^{N-1}\epsilon\left(\bar{\psi}_j \frac{\psi_{j+1}-\psi_j}{\epsilon} + H(\bar{\psi}_j, \psi_j)\right)} + O(N\epsilon^2)$$

with $\bar{\psi}_N := \bar{\psi}$, $\psi_1 := \psi$

$\Big[$ Write $\bar{\psi} = \bar{\psi}_0 = \bar{\psi}_N$, $\psi = \psi_1 = \psi_{N+1}$

$$= \int \prod_{j=0}^{N-1} \hbar\, d\bar{\psi}_j\, d\psi_{j+1}\; e^{-\frac{1}{\hbar}\sum_{j=1}^{N}\epsilon\left(\bar{\psi}_j \frac{\psi_{j+1}-\psi_j}{\epsilon} + H(\bar{\psi}_j, \psi_j)\right)} + O(N\epsilon^2)$$

with $\bar{\psi}_N := \bar{\psi}_0$, $\psi_{N+1} := \psi_1$

$$= \int \prod_{j\in\mathbb{Z}/N\mathbb{Z}} \hbar\, d\bar{\psi}(\epsilon j)\, d\psi(\epsilon j+\epsilon)\; e^{-\frac{1}{\hbar}\sum_{j\in\mathbb{Z}/N\mathbb{Z}}\epsilon\left(\bar{\psi}(\epsilon j)\frac{\psi(\epsilon j+\epsilon)-\psi(\epsilon j)}{\epsilon} + H(\bar{\psi}(\epsilon j), \psi(\epsilon j))\right)} + O(N\epsilon^2)$$

$$\bar{\psi}(\epsilon j + T) = \bar{\psi}(\epsilon j), \quad \psi(\epsilon j + T) = \psi(\epsilon j)$$

$$\xrightarrow[N\epsilon = T]{N\to\infty} \int D\bar{\psi}\, D\psi\; e^{-\frac{1}{\hbar}\int_{\mathbb{R}/T\mathbb{Z}} d\tau\, L_E\left(\bar{\psi}, \psi, \frac{d\psi}{d\tau}\right)}$$

$$\bar{\psi}(\tau+T) = \bar{\psi}(\tau), \quad \psi(\tau+T) = \psi(\tau)$$

periodic

To summarize,

$$Z(t_f,\bar{\psi}_f;t_i,\psi_i)$$

$$= \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\ e^{\frac{1}{\hbar}\bar{\psi}\psi|_{t_f} + \frac{i}{\hbar}\int_{t_i}^{t_f} dt\left(i\bar{\psi}\dot{\psi} - H(\bar{\psi},\psi)\right)},$$

$$\bar{\psi}(t_f)=\bar{\psi}_f,\quad \psi(t_i)=\psi_i$$

$$\mathrm{Tr}_{\mathcal{H}}\, e^{-\frac{T}{\hbar}\hat{H}} = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\ e^{-\frac{1}{\hbar}\int_{\mathbb{R}/T\mathbb{Z}} d\tau\, L_E}$$

$$\bar{\psi}(\tau+T) = -\bar{\psi}(\tau),\quad \psi(\tau+T) = -\psi(\tau),$$

antiperiodic

$$\mathrm{Tr}_{\mathcal{H}}(-1)^F e^{-\frac{T}{\hbar}\hat{H}} = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\ e^{-\frac{1}{\hbar}\int_{\mathbb{R}/T\mathbb{Z}} d\tau\, L_E}$$

$$\bar{\psi}(\tau+T) = \bar{\psi}(\tau),\quad \psi(\tau+T) = \psi(\tau).$$

periodic

- Once again, it is instructive to do path-integral in explicit examples. Please do it yourself.

  For your convenience, a note on it is uploaded.

Yang-Mills theory (in d spacetime dimensions)

... Specified by gauge group G : a compact Lie group

e.g. $U(N) = \{ N\times N \text{ unitary matrix} \}$ unitary group

$SU(N) = \{ N\times N \text{ unitary}, \det = 1 \}$ special unitary group

e.g. $d=4$, $G=U(1)$: Maxwell theory

$\mathfrak{g} = \mathrm{Lie}(G)$ the Lie algebra of G

e.g. $G=U(N)$: $\mathfrak{g} = \{ N\times N \text{ antihermitian matrix} \}$

$G=SU(N)$: $\mathfrak{g} = \{ N\times N \text{ antihermitian, traceless} \}$

field variable $A_\mu(x)$: a vector potential with values in $\mathfrak{g}$

(or $A = A_\mu(x)dx^\mu$: one form ———————)

field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$

(or $F_A = dA + \frac{1}{2}[A,A] = dA + A^2$)

matrix Lie algebra

e.g. $d=4$, $G=U(1)$: $F_{0i} = \partial_0 A_i - \partial_i A_0$ electric field

$F_{ij} = \partial_i A_j - \partial_j A_i$ magnetic field

Yang-Mills action :

$$S[A] = \int -\frac{1}{4e^2} F^{\mu\nu}\cdot F_{\mu\nu}\, d^d x$$

Here "$\cdot$" is a positive definite inner product on $\mathfrak{g}$ which is invariant under the adjoint action of G :

$$gXg^{-1}\cdot gYg^{-1} = X\cdot Y$$

e.g. for $G = U(N)$ or $SU(N)$, $X\cdot Y = -2\,\mathrm{Tr}\, XY$

It is invariant under a huge symmetry group :

$g(x)$: G-valued function on spacetime

$$\rightsquigarrow \quad A_\mu \longmapsto A_\mu^g = g^{-1} A_\mu g + g^{-1}\partial_\mu g .$$

e.g. $G = U(1)$, $A_\mu = i\,a_\mu$, $g(x) = e^{i\alpha(x)}$

$$\rightsquigarrow \quad a_\mu^g = a_\mu + \partial_\mu \alpha \quad \text{(the well-known transformation).}$$

Under this, the field strength transform covariantly,

$$F_{\mu\nu} \longmapsto F_{\mu\nu}^g = \partial_\mu A_\nu^g - \partial_\nu A_\mu^g + [A_\mu^g, A_\nu^g] = g^{-1} F_{\mu\nu}\, g .$$

e.g. $G = U(1)$, $F_{\mu\nu}^g = F_{\mu\nu}$ invariant.

In particular,

$$S[A^g] = \int -\frac{1}{4e^2}\, g^{-1} F^{\mu\nu} g \cdot g^{-1} F_{\mu\nu} g \; d^d x$$

$$= \int -\frac{1}{4e^2} F^{\mu\nu} \cdot F_{\mu\nu} \, d^d x = S[A].$$

$\therefore$ $S[A]$ is invariant under a huge group:

$$\mathcal{G} = \{ G\text{-valued function } g(x) \}$$

We would like to regard A_μ and A_μ^g for any $g \in \mathcal{G}$ as <u>physically equivalent</u>. I.e. we would like to <u>physically identify</u> them.

Such a symmetry group is called a <u>local symmetry group</u> or <u>gauge symmetry group</u>. And a theory with such an identification of field variables is called a <u>gauge theory</u>.

We would like to find a way to quantize gauge theories.