Yang-Mills theory (in d spacetime dimensions)

· Specified by gauge group G : a compact Lie group

e.g. $U(N) = \{ N \times N \text{ unitary matrix} \}$ unitary group

$SU(N) = \{ N \times N \text{ unitary}, \det = 1 \}$ special unitary group

e.g. $d=4$, $G = U(1)$: Maxwell theory.

$\mathfrak{g} = \mathrm{Lie}(G)$ the Lie algebra of G "infinitesimal version of G"

e.g. $G = U(N)$: $\mathfrak{g} = \{ N \times N \text{ antihermitian matrix} \}$

$G = U(1)$: $\mathfrak{g} = i\mathbb{R} \cong \mathbb{R}$

$G = SU(N)$: $\mathfrak{g} = \{ N \times N \text{ antihermitian, traceless} \}$

field variable $A_\mu(x)$: a vector potential with values in $\mathfrak{g}$

(or $A = A_\mu(x) dx^\mu$: one form ——————)

field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$

(or $F_A = dA + \frac{1}{2}[A, A] = dA + A^2$)

the case of matrix Lie algebra

e.g. $d=4$, $G = U(1)$: $F_{0i} = \partial_0 A_i - \partial_i A_0$ electric field

$F_{ij} = \partial_i A_j - \partial_j A_i$ magnetic field

Yang-Mills action:

$$S[A] = \int -\frac{1}{4e^2} F^{\mu\nu} \cdot F_{\mu\nu} \, d^dx$$

Here "$\cdot$" is a positive definite inner product on $\mathfrak{g}$ which is invariant under the adjoint action of G, $X \mapsto gXg^{-1}$ (the infinitesimal version of conjugation $g_t \mapsto g g_t g^{-1}$):

$$gXg^{-1} \cdot gYg^{-1} = X \cdot Y.$$

e.g. for $G = SU(N)$, a standard choice is $X \cdot Y = -2 \, \mathrm{Tr} \, XY$.

$S[A]$ is invariant under a huge symmetry group:

$g(x)$: G-valued function on spacetime

$$\rightsquigarrow \quad A_\mu \mapsto A_\mu^g = g^{-1} A_\mu g + g^{-1} \partial_\mu g.$$

Under this, the field strength transforms covariantly,

$$F_{\mu\nu} \mapsto F_{\mu\nu}^g = \partial_\mu A_\nu^g - \partial_\nu A_\mu^g + [A_\mu^g, A_\nu^g] = g^{-1} F_{\mu\nu} g,$$

and thus indeed

$$S[A^g] = \int -\frac{1}{4e^2} g^{-1} F^{\mu\nu} g \cdot g^{-1} F_{\mu\nu} g \, d^dx$$

$$= \int -\frac{1}{4e^2} F^{\mu\nu} \cdot F_{\mu\nu} \, d^dx = S[A].$$

This is a generalization of invariance of Maxwell action under the gauge transformation $A_\mu \mapsto A_\mu + \partial_\mu \lambda$.

Indeed, for $G = U(1)$, $\mathfrak{g} = i\mathbb{R} \cong \mathbb{R}$ and for $g(x) = e^{i\lambda(x)}$,

$$iA_\mu^\lambda = e^{-i\lambda}\, iA_\mu\, e^{i\lambda} + e^{-\lambda}\partial_\mu e^{i\lambda} \Rightarrow A_\mu^\lambda = A_\mu + \partial_\mu \lambda.$$

As in that case, we shall call

$$A_\mu \mapsto A_\mu^g = g^{-1} A_\mu g + g^{-1}\partial_\mu g$$

the <u>gauge transformation</u> of $A_\mu(x)$ by $g(x)$, and

$$\mathcal{G} := \{\, g(x) \mid G\text{-valued function} \,\}$$

the <u>gauge transformation group</u>. We'd like to regard A and A^g as physically equivalent for any $g \in \mathcal{G}$.

I.e. we would like to physically identify them. If we put

$$\mathcal{A} := \{\, A_\mu(x) \mid \mathfrak{g}\text{-valued vector potential} \,\}$$

the space of physically inequivalent field configurations is the quotient space $\mathcal{A}/\mathcal{G}$.

A theory with such an identification of field variables is called a <u>gauge theory</u>.

Another example of a gauge theory in this sense is General Relativity, the theory of gravity. In the case of pure gravity, a metric ds^2 and its transform f^*ds^2 by a diffeomorphism f are regarded as physically equivalent and must be physically identified. The space of physically inequivalent configurations is

$$\{\text{metric}\}/\{\text{diffeomorphism}\}.$$

We would like to find a way to

quantize gauge theories

Infinitesimal gauge transformations

A $\mathfrak{g}$-valued function $\epsilon(x)$ generates a one parameter group of gauge transformations: $g_t(x) = e^{t\epsilon(x)}$:

$$A_\mu \mapsto A_\mu^{g_t} = g_t^{-1} A_\mu g + g_t^{-1} \partial_\mu g_t.$$

The infinitesimal transformation is

$$\delta_\epsilon A_\mu = \frac{d}{dt} A_\mu^{g_t} \Big|_{t=0} = -\epsilon A_\mu + A_\mu \epsilon + \partial_\mu \epsilon$$

$$= \partial_\mu \epsilon + [A_\mu, \epsilon] =: D_\mu \epsilon \quad \text{covariant derivative.}$$

The space of such $\epsilon(x)$ may be regarded as the Lie algebra of the gauge transformation group.

$$\{\epsilon(x) \mid \mathfrak{g}\text{-valued function}\} = \mathrm{Lie}(\mathcal{G}).$$

Coupling to matter fields

$\phi(x)$: a scalar field with values a representation R of G,

i.e. a vector space on which G acts linearly.

e.g. $R = \mathbb{C}^N$ for $G = U(N)$ or $SU(N)$ via matrix multiplication.

$R = \mathfrak{g}$ for a general G via adjoint action

$R =$ sum of copies of such, $\mathbb{C}^N \oplus \cdots \oplus \mathbb{C}^N \oplus \mathfrak{g} \oplus \cdots \oplus \mathfrak{g}$.

Gauge transformation by $g \in \mathcal{G}$:

$$A_\mu \mapsto A_\mu^g, \quad \phi \longmapsto \phi^g = g^{-1}\phi.$$

Infinitesimally, $\delta A_\mu = D_\mu \epsilon$, $\delta\phi = -\epsilon\phi$.

Covariant derivative $D_\mu\phi := \partial_\mu\phi + A_\mu\phi$

Its gauge transformation :

$$D_\mu\phi \mapsto \partial_\mu\phi^g + A_\mu^g\phi^g = \underbrace{\partial_\mu(g^{-1}\phi)}_{-\cancel{g^{-1}\partial_\mu g}\, g^{-1}\phi + g^{-1}\partial_\mu\phi} + (g^{-1}A_\mu g + \cancel{g^{-1}\partial_\mu g})\, g^{-1}\phi$$

$$= g^{-1}\partial_\mu\phi + g^{-1}A_\mu\phi = g^{-1}D_\mu\phi \quad \text{homogeneous.}$$

$(\phi_1, \phi_2) \longmapsto \phi_1^\dagger \phi_2$ G-invariant inner product on R

$$\mathcal{L} = -\frac{1}{4e^2} F^{\mu\nu} \cdot F_{\mu\nu} + (D^\mu \phi)^\dagger D_\mu \phi - f(\phi^\dagger \phi)$$

is gauge invariant.

The system with variable (A_μ, ϕ) and this Lagrangian is the gauge theory of gauge group G with a scalar in a representation R of G.

We may also consider a theory with a fermion ψ in a representation R of G.

$$\mathcal{L} = -\frac{1}{4e^2} F^{\mu\nu} \cdot F_{\mu\nu} + i\bar{\psi} \not{D}_A \psi - m\bar{\psi}\psi$$

where $\not{D}_A \psi = \gamma^\mu D_\mu \psi = \gamma^\mu (\partial_\mu \psi + A_\mu \psi)$.

e.g. QED with electrons of charge $Q_1, \cdots, Q_{N_f}$:

$$G = U(1), \quad e^{i\lambda} : \psi_i \longmapsto e^{iQ_i\lambda} \psi_i \quad (i = 1, \cdots, N_f)$$

e.g. QCD with color N_c and flavor N_f:

$$G = SU(N_c), \quad R = \mathbb{C}^{N_c} \oplus \cdots \oplus \mathbb{C}^{N_c} \quad (N_f \text{ copies})$$

Quantization of gauge theory (path integral)

In a gauge theory, a field configuration $(A, \phi, \psi, \cdots)$ is identified with its gauge transform $(A^g, \phi^g, \psi^g, \cdots)$.

$\mathcal{M}$ = the space of field configurations

$\mathcal{G}$ = the gauge transformation group.

The path-integral is over the quotient space $\mathcal{M}/\mathcal{G}$

$$Z = \int_{\mathcal{M}/\mathcal{G}} \underline{\text{measure}} \; e^{-S_E[A, \phi, \psi, \cdots]}$$

$$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \cdots \rangle$$

$$= \frac{1}{Z} \int_{\mathcal{M}/\mathcal{G}} \underline{\text{measure}} \; e^{-S_E[A, \phi, \psi, \cdots]} \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \cdots$$

<u>How do we do this?</u>

$\cdots$ Let us do it in a finite dimensional setting.

M: a manifold, $\dim M = n < \infty$,

G: a Lie group acting on M, $\dim G = d_G < \infty$;

$$g \in G : \phi \in M \longmapsto \phi g \in M$$

(right action : $\phi(gh) = (\phi g) h$

Assume: the action is <u>free</u>, $\phi g = \phi$ for some $\phi \Rightarrow g = 1$.

Suppose a measure $d\phi$ and a function $S_E(\phi)$ on M are G-invariant, $d(\phi g) = d\phi$, $S_E(\phi g) = S_E(\phi)$.

Want to consider the gauge theory where

$$\begin{cases} \phi \sim \phi g & \text{identified} \\ f(\phi) & \text{physically meaningful when } G\text{-invariant} \end{cases}$$

<u>Question</u> How do we define <u>measure</u> on M/G for

$$Z = \int_{M/G} \underline{\text{measure}}\; e^{-S_E(\phi)}$$

$$\langle f \rangle = \frac{1}{Z} \int_{M/G} \underline{\text{measure}}\; e^{-S_E(\phi)} f(\phi)$$

?

A naive answer:

$$Z = \frac{1}{\mathrm{Vol}\, G} \int_M d\phi \; e^{-S_E(\phi)}$$

see below for the choice of dg

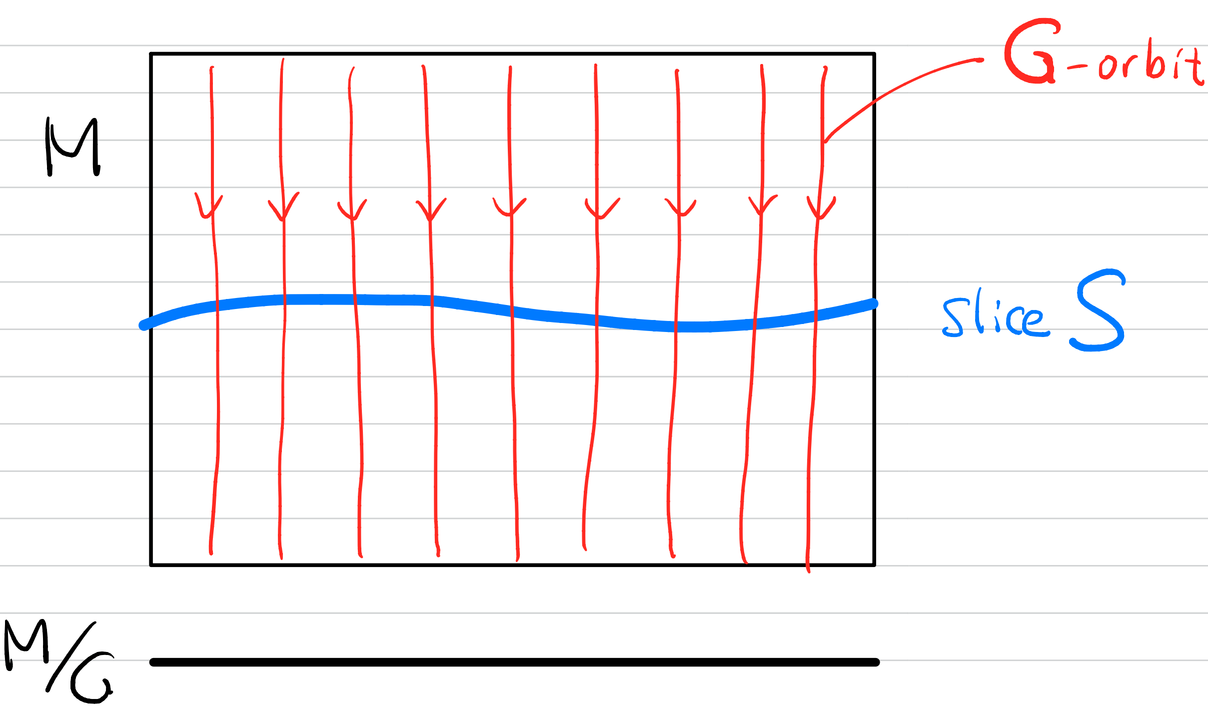
$$\langle f \rangle = \frac{1}{\mathrm{Vol}\, G} \int_M d\phi \; e^{-S_E(\phi)} f(\phi) \Big/ Z,$$

where $\mathrm{Vol}\, G = \int_G dg$ for some measure $\widetilde{dg}$.

A possible problem: $\mathrm{vol}\, G$ may be infinite

$\int_M d\phi \cdots$ may be infinite.

Suppose we can find a slice $S \subset M$, i.e. a submanifold s.t. any G-orbit has exactly one point in it.



Suppose S is the zero locus of a set of functions of M

$$\phi \in S \Leftrightarrow \chi_1(\phi) = \cdots = \chi_{d_G}(\phi) = 0.$$

$\chi(\phi) = (\chi_a(\phi))_{a=1}^{d_G}$ can be regarded as a function on M with values in $\mathfrak{g} = \mathrm{Lie}(G)$.

Note: For a monotonic function $F(y)$ of a single y,

$$\int_{-\infty}^{\infty} \delta(F(y))\, \underbrace{dF(y)}_{F'(y)\,dy} = 1.$$

- This holds for any $F(y)$ as long as monotonic.

Multivariable case:

$$y = (y_1, \cdots, y_m) \mapsto F(y) = (F_1(y), \cdots, F_m(y))$$

$$\int_{\mathbb{R}^m} \prod_{a=1}^{m} \delta(F_a(y)) \cdot \det\left(\frac{\partial F_a(y)}{\partial y_b}\right) d^m y = 1.$$

- Here $d^m y = dy_1 \cdots dy_m$ where y_i's are the ones that appear in $\partial F_a(y)/\partial y_b$.

•• This holds for any $(F_a(y))_{a=1}^{d_G}$ as long as orientation preserving.

Apply this to the function $F_a(g) = \chi_a(\phi g)$ of G for a fixed $\phi \in M$:

$$\int_G \underbrace{\prod_{a=1}^{d_G} \delta(\chi_a(\phi g))}_{\delta(\chi(\phi g))} \cdot \underbrace{\det\left(\frac{\partial \chi_a(\phi g)}{\partial g_b}\right)}_{\det(\delta^b \chi_a(\phi g))} \cdot dg = 1.$$

(i) dg is the measure $dg_1 \cdots dg_{d_G}$ where g_a's are the ones that appear in $\partial \chi_a(\phi g)/\partial g_b$.

(ii) This holds for any choice of gauge fixing function $\chi(\phi) = (\chi_a(\phi))_{a=1}^{d_G}$ as long as $g \mapsto \chi(\phi g)$ is orientation preserving.

insert $1 = \cdots$

$$\int_M d\phi\, e^{-S_E(\phi)} = \int_{M\times G} d\phi\, dg\, e^{-S_E(\phi)} \delta(\chi(\phi g))\det(\delta\chi(\phi g))$$

Change the variable $\phi g \to \phi$ and use G-invariance of $d\phi$ & $S_E(\phi)$

$$= \int_{M\times G} d\phi\, dg\, e^{-S_E(\phi)} \delta(\chi(\phi))\det(\delta\chi(\phi))$$

$$= \underbrace{\int_G dg}_{\mathrm{Vol}\,G} \int_M d\phi\, e^{-S_E(\phi)} \delta(\chi(\phi))\det(\delta\chi(\phi))$$

$$\therefore Z = \text{“}\frac{1}{\mathrm{Vol}\,G}\int_M d\phi\, e^{-S_E(\phi)}\text{”}$$

$$= \int_M d\phi\, e^{-S_E(\phi)} \delta(\chi(\phi))\det(\delta\chi(\phi)).$$

Similarly

$$\langle f \rangle = \text{"} \frac{1}{\mathrm{Vol}\,G} \int_M d\phi \, e^{-S_E(\phi)} f(\phi) \Big/ Z \text{"}$$

$$= \int_M d\phi \, e^{-S_E(\phi)} \delta(\chi(\phi)) \det(\delta\chi(\phi)) f(\phi) \Big/ Z .$$

$\chi(\phi) = (\chi_a(\phi))_{a=1}^{d_G}$ ... gauge fixing function

$\chi_1(\phi) = \cdots = \chi_{d_G}(\phi) = 0$ ... gauge fixing condition

$$\det(\delta\chi(\phi)) = \det\left(\left. \frac{\partial \chi_a(\phi g)}{\partial g_b} \right|_{g=1} \right)$$

$$= \det\left(\left. \frac{\partial \chi_a(\phi e^{\epsilon})}{\partial \epsilon_b} \right|_{\epsilon=0} \right)$$

$\{e^a\}_{a=1}^{d_G} \subset \mathfrak{g}$ basis

$\epsilon = \sum_a e^a \epsilon_a$

... <u>Faddeev-Popov determinant</u>

The above can be taken as definition of Z & $\langle f \rangle$'s.

It requires a choice of gauge fixing condition $\chi(\phi) = 0$ as well as a choice of "volume class" of the basis $\{e^a\}_{a=1}^{d_G}$ of $\mathfrak{g}$ through the appearance of the coordinates $\epsilon_1, \cdots, \epsilon_{d_G}$ in $\det \frac{\partial \chi_a(\phi e^\epsilon)}{\partial \epsilon_b}$. Does the result for Z & $\langle f \rangle$ depend on such choices?

The remarks (i) & (ii) made at the identity $1 = \int_G \delta(\chi(\phi g)) \det(\delta \chi(\phi g)) \, dg$ are relevant for this question.

By (i), the volume class of $\{e^a\}_{a=1}^{d_G}$ does affect the result for Z and the numerator of $\langle f \rangle$'s but only by a common overall multiplicative constant. In particular, it does not affect the result for $\langle f \rangle$'s.

By (ii), the result for Z & $\langle f \rangle$ does not depend on the choice of gauge fixing condition $\chi(\phi) = 0$.

Rewriting

① Use independence on the choice of $\chi(\phi)$.

② Use $\det(A_{ij}) = \int \prod_i d\bar{\eta}_i d\eta_i \, e^{-\sum_{i,j} \bar{\eta}_i A_{ij} \eta_j}$

① Replace $\chi(\phi) \to \chi(\phi) - \omega$, $\omega \in \mathfrak{g}$.

Also, $\displaystyle\int_{\mathfrak{g}} d\omega \, e^{-\frac{1}{2\xi}\omega^2} = (2\pi\xi)^{d_G/2}$

$$Z = \int_M d\phi \, e^{-S_E(\phi)} \delta(\underbrace{\chi(\phi)}_{\chi(\phi)-\omega}) \det(\underbrace{\delta\chi(\phi)}_{\delta\chi(\phi) \text{ intact}})$$

$$1 = \frac{1}{(2\pi\xi)^{d_G/2}} \int_{\mathfrak{g}} d\omega \, e^{-\frac{1}{2\xi}\omega^2}$$

$$= \frac{1}{(2\pi\xi)^{d_G/2}} \int_{M\times\mathfrak{g}} d\phi \, d\omega \, e^{-S_E(\phi) - \frac{1}{2\xi}\omega^2} \delta(\chi(\phi)-\omega) \det(\delta\chi(\phi))$$

(#)

$$= \frac{1}{(2\pi\xi)^{d_G/2}} \int_M d\phi \, e^{-S_E(\phi) - \frac{1}{2\xi}(\chi(\phi))^2} \det(\delta\chi(\phi))$$

(2) $$\det(\delta^b \chi_a(\phi)) = \int \prod_{a=1}^{d_G} d\bar{c}^a\, dc_a\, e^{-\sum_{a,b} \bar{c}^a \delta^b \chi_a(\phi) c_b}$$

$$\sum_b \delta^b \chi_a(\phi) c_b = \delta_c \chi_a(\phi)$$

infinitesimal transformation of $\chi_a(\phi)$

by $c = \sum_{a=1}^{d_G} e^a c_a$

$$= \int \prod_{a=1}^{d_G} d\bar{c}^a\, dc_a\, e^{-\sum_a \bar{c}^a \delta_c \chi_a(\phi)}$$

$$\overset{\text{or}}{=} \int_{\mathfrak{g}\times\mathfrak{g}} d\bar{c}\, dc\, e^{-\bar{c}\cdot\delta_c \chi(\phi)}$$ in a simplified form.

$c, \bar{c}$: Faddeev-Popov ghosts

Also

$$\delta(\chi(\phi)-\omega) = \int \prod_{a=1}^{d_G} \frac{dB_a}{2\pi}\, e^{iB^a(\chi_a(\phi)-\omega_a)}$$

$$\overset{\text{or}}{=} \frac{1}{(2\pi)^{d_G}} \int_{\mathfrak{g}} dB\, e^{iB\cdot(\chi(\phi)-\omega)}$$

$$\therefore \delta(\chi(\phi)-\omega) \det(\delta\chi(\phi))$$

$$= \frac{1}{(2\pi)^{d_G}} \int_{\mathfrak{g}\times\mathfrak{g}\times\mathfrak{g}} dB\, d\bar{c}\, dc\, e^{iB\cdot(\chi(\phi-\omega)) - \bar{c}\cdot\delta_c\chi(\phi)}$$

Insert this in (#) :

$$Z = \frac{1}{(2\pi\xi)^{d_a/2}} \int_{M\times\mathfrak{g}} d\phi\, d\omega\; e^{-S_E(\phi) - \frac{1}{2\xi}\omega^2}$$

$$\times \frac{1}{(2\pi)^{d_a}} \int_{\mathfrak{g}\times\mathfrak{g}\times\mathfrak{g}} dB\, d\bar{c}\, dc\; e^{iB\cdot(\chi(\phi-\omega) - \bar{c}\cdot\delta_c\chi(\phi))}$$

Perform ω- integral

$$\int d\omega\; e^{-\frac{1}{2\xi}\omega^2 - iB\cdot\omega} = (2\pi\xi)^{d_a/2}\, e^{-\frac{\xi}{2}B^2}$$

We end up with

$$Z = \frac{1}{(2\pi)^{d_a}} \int_{M\times\mathfrak{g}\times\mathfrak{g}\times\mathfrak{g}} d\phi\, dB\, d\bar{c}\, dc\; e^{-\tilde{S}_E(\phi, B, \bar{c}, c)}$$

$$\tilde{S}_E = S_E(\phi) + \frac{\xi}{2}B^2 - iB\cdot\chi(\phi) + \bar{c}\cdot\delta_c\chi(\phi)$$

... gauge fixed action

Similarly for $\langle f \rangle$.

Next Lecture :

- Remaining part of path-integral quantization of gauge theories.
  - "BRST symmetry" of the gauge fixed system.
  - A proposal for physical states/observables using BRST.
  - Back to the set-up of ∞-dim'l spaces of field configurations & gauge transformations.
- Hamiltonian formulation of classical system & Operator quantization