# Correlation functions vs vacuum expectation values

Recall $Z(t_f, q_f; \mathcal{O}_1(t_1)\,\mathcal{O}_2(t_2); t_i, q_i)$

$$:= \int_{q(t_f)=q_f,\ q(t_i)=q_i} \mathcal{D}q\, e^{i\int_{t_i}^{t_f} dt\, L(q,\dot{q})}\, \mathcal{O}_1(t_1)\,\mathcal{O}_2(t_2)$$

$$= \begin{cases} \langle q_f|\, e^{-i(t_f-t_1)\hat{H}}\,\hat{\mathcal{O}}_1\, e^{-i(t_1-t_2)\hat{H}}\,\hat{\mathcal{O}}_2\, e^{-i(t_2-t_i)\hat{H}}\,|q_i\rangle & \text{if } t_1 > t_2 \\ \langle q_f|\, e^{-i(t_f-t_2)\hat{H}}\,\hat{\mathcal{O}}_2\, e^{-i(t_2-t_1)\hat{H}}\,\hat{\mathcal{O}}_1\, e^{-i(t_1-t_i)\hat{H}}\,|q_i\rangle & \text{if } t_2 > t_1 \end{cases}$$

$$=: \langle q_f|\, e^{-it_f\hat{H}} \cdot \underbrace{T\,\widehat{\mathcal{O}_1(t_1)}\,\widehat{\mathcal{O}_2(t_2)}}_{\text{time ordered product}} \cdot e^{it_i\hat{H}}\,|q_i\rangle$$

where $\widehat{\mathcal{O}(t)} := e^{it\hat{H}}\,\hat{\mathcal{O}}\,e^{-it\hat{H}}$

Simililary

$$Z(t_f, q_f; \mathcal{O}_1(t_1)\cdots\mathcal{O}_s(t_s); t_i, q_i)$$

$$= \langle q_f|\, e^{-it_f\hat{H}} \cdot T\,\widehat{\mathcal{O}_1(t_1)}\cdots\widehat{\mathcal{O}_s(t_s)} \cdot e^{it_i\hat{H}}\,|q_i\rangle.$$

Let us take the limit $T \to e^{-i\epsilon}(+\infty)$ in

$$Z(T, q_f ; \mathcal{O}_1(t_1) \cdots \mathcal{O}_s(t_s), -T, q_i)$$

$$= \langle q_f | e^{-iT\hat{H}} \cdot T\, \widehat{\mathcal{O}_1(t_1)} \cdots \widehat{\mathcal{O}_s(t_s)} \cdot e^{-iT\hat{H}} | q_i \rangle$$

$$= \sum_{n,m} \langle q_f | e^{-iTE_n} | n \rangle \langle n | T\, \widehat{\mathcal{O}_1(t_1)} \cdots \widehat{\mathcal{O}_s(t_s)} \, e^{-iTE_m} | m \rangle \langle m | q_i \rangle$$

where $\{|n\rangle\} \subset \mathcal{H}$ is a basis consisting of Hamiltonian eigenstates $\hat{H}|n\rangle = E_n |n\rangle$.

Let us assume that $|n\rangle$ with label $n=0$ is the unique ground state, <u>the vacuum state</u> $|0\rangle$.

Then, for $n \neq 0$, $E_n > E_0$ and

$$e^{-iT(E_n - E_0)} = e^{-ie^{-i\epsilon}|T|(E_n - E_0)} \searrow 0 \quad \text{as } |T| \to \infty$$

Thus, as long as $\langle q_f | 0 \rangle \neq 0$ and $\langle 0 | q_i \rangle \neq 0$, the term $n = m = 0$ is dominant, and the other terms are exponentially small,

$$Z(T, q_f; \mathcal{O}_1(t_1) \cdots \mathcal{O}_s(t_s); -T, q_i)$$

$$= e^{-2iTE_0} \Big\{ \langle q_f | 0 \rangle \langle 0 | T \widehat{\mathcal{O}_1(t_1)} \cdots \widehat{\mathcal{O}_s(t_s)} | 0 \rangle \langle 0 | q_i \rangle + \sum_{m,n \neq 0} \underbrace{e^{-iT(E_n - E_0)} e^{-iT(E_m - E_0)}}_{\to 0 \text{ as } |T| \to \infty} *_{n,m} \Big\}$$

$$\langle \mathcal{O}_1(t_1) \cdots \mathcal{O}_s(t_s) \rangle_{T, q_f; -T, q_i}$$

$$:= \frac{Z(T, q_f; \mathcal{O}_1(t_1) \cdots \mathcal{O}_s(t_s); -T, q_i)}{Z(T, q_f; -T, q_i)}$$

$$= \frac{e^{-2iE_0T} \{ \langle q_f | 0 \rangle \langle 0 | T \widehat{\mathcal{O}_1(t_1)} \cdots \widehat{\mathcal{O}_s(t_s)} | 0 \rangle \langle 0 | q_i \rangle + \cdots \}}{e^{-2iE_0T} \{ \langle q_f | 0 \rangle \langle 0 | q_i \rangle + \cdots \}}$$

$$= \langle 0 | T \widehat{\mathcal{O}_1(t_1)} \cdots \widehat{\mathcal{O}_s(t_s)} | 0 \rangle + \cdots$$

$$\xrightarrow{T \to e^{-i\epsilon}\infty} \langle 0 | T \widehat{\mathcal{O}_1(t_1)} \cdots \widehat{\mathcal{O}_s(t_s)} | 0 \rangle$$

The vacuum expectation value of the time ordered product.

<u>Remarks</u> (i) In a potential theory, $L = \frac{m}{2}\dot{q}^2 - U(q)$, with $U(q) \to \infty$ as $|q| \to \infty$, there is a unique ground state $|0\rangle$ and $\langle q|0\rangle \neq 0$ for any $q$. Thus we may take any $q_i$ & $q_f$, say, $q_i = q_f = 0$, or $q_i = q_f =$ a minimum of $U(q)$.

(ii) In general, there can be more than one ground states $|0_I\rangle$ $(I = 1, 2, 3, \cdots)$ and/or an appropriate boundary condition needs to be specified at $t_f = T$ and $t_i = -T$. Showing the dependence on the boundary condition as a superscript, we have

$$\lim_{T \to e^{-i\epsilon}\infty} \langle \mathcal{O}_1(t_1) \cdots \mathcal{O}_s(t_s) \rangle^{I}_{T;-T} = \langle 0_I | T\, \widehat{\mathcal{O}_1(t_1)} \cdots \widehat{\mathcal{O}_s(t_s)} | 0_I \rangle.$$

(iii) We may consider, not just $t_i$ & $t_f$, but all $t_1, \cdots, t_s$ to lie on $e^{-i\epsilon}\mathbb{R}$ and then take the limit $\epsilon \searrow 0$ after the computation. Then, the LHS (before $\epsilon \searrow 0$) can be regarded as the correlation function of a slightly Wick rotated theory, or equivalently, the real time limit of the reverse Wick rotation of the Euclidean theory

$$\lim_{\epsilon \searrow 0} \langle \mathcal{O}_1(e^{-i\epsilon} t_1) \cdots \mathcal{O}_s(e^{-i\epsilon} t_s) \rangle^{I}_{e^{-i\epsilon}\mathbb{R}}$$

$$= \lim_{\epsilon \searrow 0} \lim_{T \nearrow \infty} \langle \mathcal{O}_1(e^{-i\epsilon} t_1) \cdots \mathcal{O}_s(e^{-i\epsilon} t_s) \rangle^{I}_{[-e^{-i\epsilon}T, e^{-i\epsilon}T]}$$

$$= \langle 0_I | T \widehat{\mathcal{O}_1(t_1)} \cdots \widehat{\mathcal{O}_s(t_s)} | 0_I \rangle.$$

(iv) The same holds in QFT in $d > 1$:

$$\lim_{\epsilon \searrow 0} \langle \mathcal{O}_1(x_1^\epsilon) \cdots \mathcal{O}(x_s^\epsilon) \rangle^{I}_{e^{-i\epsilon}\mathbb{R} \times \mathbb{R}^{d-1}}$$

$$= \langle 0_I | T \widehat{\mathcal{O}_1(x_1)} \cdots \widehat{\mathcal{O}_n(x_s)} | 0_I \rangle$$

where $x^\epsilon = (e^{-i\epsilon} t, \mathbf{x})$ for $x = (t, \mathbf{x})$.

Note that an appropriate boundary condition at spatial infinity $|\mathbf{x}| \to \infty$, that depends on the ground state $|0_I\rangle$, also needs to be specified.

In the Euclidean theory, where there is no time-space distinction, the boundary conditions at space & time $\infty$'s may be unified, and we may say

$$\langle \mathcal{O}_1(x_1^E) \cdots \mathcal{O}_s(x_s^E) \rangle^I_{\mathbb{R}^d_E}$$

$$\xrightarrow{\text{Minkowski limit}} \langle 0_I | T\, \mathcal{O}_1(x_1) \cdots \mathcal{O}_s(x_s) | 0_I \rangle.$$

We shall mostly consider theories with a unique ground state and suppress the label $I$ of boundary condition.

To avoid cluttering the notation, we may simply write

$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_s(x_s) \rangle := \lim_{\epsilon \searrow 0} \langle \mathcal{O}_1(x_1^\epsilon) \cdots \mathcal{O}_s(x_s^\epsilon) \rangle_{e^{-i\epsilon}\mathbb{R}\times\mathbb{R}^{d-1}}$$

for the Minkowski limit. Then, our conclusion is

$$\boxed{\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_s(x_s) \rangle = \langle 0 | T\, \mathcal{O}_1(x_1) \cdots \mathcal{O}_s(x_s) | 0 \rangle}$$

# Free field theories

A theory is said to be <u>free</u> when the action is quadratic in variables.

e.g. $n$ real variables $\phi = (\phi_1, \cdots, \phi_n)$

$$S_E(\phi) = \frac{1}{2} \sum_{i,j=1}^{n} \phi_i A_{ij} \phi_j \qquad A_{ij} = A_{ji} \text{ symmetric, positive eigenvalues}$$

$$d^n\phi = d\phi_1 \cdots d\phi_n$$

$$Z = \int d^n\phi \, e^{-S_E(\phi)} = \sqrt{\frac{(2\pi)^n}{\det A}}$$

$$\langle \phi_{i_1} \cdots \phi_{i_s} \rangle = \frac{1}{Z} \int d^n\phi \, e^{-S_E(\phi)} \phi_{i_1} \cdots \phi_{i_s} = ?$$

A trick :

$$f(A, J) := \int d^n\phi \, e^{-S_E(\phi) + \sum_{i=1}^{n} J_i \phi_i}$$

$$\frac{\partial}{\partial J_{i_1}} \cdots \frac{\partial}{\partial J_{i_s}} f(A, J) = \int d^n\phi \, e^{-S_E(\phi) + \sum_i J_i \phi_i} \phi_{i_1} \cdots \phi_{i_s}$$

$$\xrightarrow{J \to 0} \quad Z \langle \phi_{i_1} \cdots \phi_{i_s} \rangle$$

But $f(A,J)$ can be computed as

$$f(A,J) = \int d^n\phi \; e^{-\frac{1}{2}(\phi - A^{-1}J)\cdot A(\phi - A^{-1}J) + \frac{1}{2} J\cdot A^{-1} J}$$

$$= \sqrt{\frac{(2\pi)^n}{\det A}} \; e^{\frac{1}{2} J\cdot A^{-1} J} = Z \cdot e^{\frac{1}{2} J\cdot A^{-1}J}$$

$$\therefore \langle \phi_{i_1} \cdots \phi_{i_s} \rangle = \frac{1}{Z} \frac{\partial}{\partial J_{i_1}} \cdots \frac{\partial}{\partial J_{i_s}} f(A,J) \Big|_{J=0}$$

$$= \frac{\partial}{\partial J_{i_1}} \cdots \frac{\partial}{\partial J_{i_s}} e^{\frac{1}{2} J\cdot A^{-1} J} \Big|_{J=0}.$$

→ Terms where $\frac{\partial}{\partial J}$ hits only one of the two $J$'s in $\frac{1}{2} J A^{-1} J$ vanish as $J=0$. Terms that survive are those where both $J$'s in $\frac{1}{2} J A^{-1} J$ are hit by $\frac{\partial}{\partial J}$'s. Thus, the result is the sum of terms where the derivatives $\frac{\partial}{\partial J_{i_1}}, \cdots, \frac{\partial}{\partial J_{i_s}}$ form pairs, which is possible only when $s$ is even, each pair $\left\{\frac{\partial}{\partial J_{i_a}}, \frac{\partial}{\partial J_{i_b}}\right\}$ producing

$\frac{\partial}{\partial J_{i_a}} \frac{\partial}{\partial J_{i_b}} \left( \frac{1}{2} J \cdot A^{-1} J \right) = A^{-1}_{i_a i_b}$. It is the sum of pairwise contractions, called <u>Wick contractions</u>:

$$\langle \phi_i \rangle = 0,$$

$$\langle \phi_i \phi_j \rangle = \wick{\c1\phi_i \c1\phi_j} = A^{-1}_{ij},$$

$$\langle \phi_i \phi_j \phi_k \rangle = 0,$$

$$\langle \phi_i \phi_j \phi_k \phi_l \rangle = \wick{\c1\phi_i \c1\phi_j \c2\phi_k \c2\phi_l} + \wick{\c1\phi_i \c2\phi_j \c1\phi_k \c2\phi_l} + \wick{\c1\phi_i \c2\phi_j \c2\phi_k \c1\phi_l}$$

$$= A^{-1}_{ij} A^{-1}_{kl} + A^{-1}_{ik} A^{-1}_{jl} + A^{-1}_{il} A^{-1}_{jk},$$

$$\vdots$$

- We see that everything is determined by the two point function

$$\langle \phi_i \phi_j \rangle = \wick{\c1\phi_i \c1\phi_j} = A^{-1}_{ij}.$$

- The logic holds also when $n = \infty$, i.e. in QFT in dimension $d \geq 1$. We now apply this to important examples. The focus will be two point functions.

# e.g. real scalar field in d dimensions

variable: $\phi(x)$ a function of $x \in \mathbb{R}^d_E$

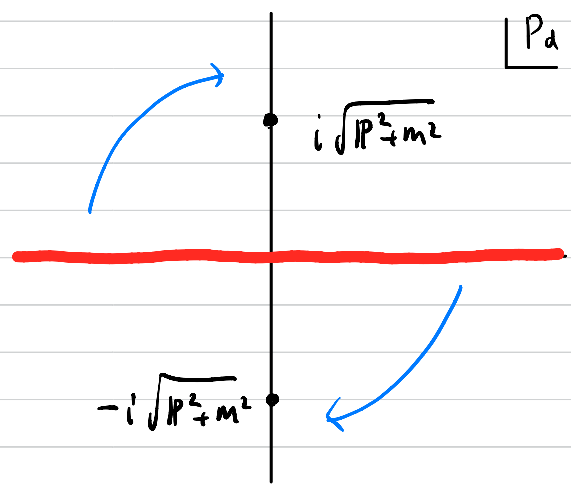
$$S_E[\phi] = \int d^d x_E \left( \frac{1}{2} \partial\phi \cdot \partial\phi + \frac{m^2}{2}\phi^2 \right)$$

The B.C. at $|x_E| \to \infty$ must allow this partial int'n.
e.g. $\phi \partial_n \phi \to 0$ at $\infty$ is OK

$$= \frac{1}{2} \int d^d x_E \, \phi(x_E) \left( -\partial^2 + m^2 \right) \phi(x_E)$$

$$\langle \phi(x_E)\, \phi(y_E) \rangle_{\mathbb{R}^d_E} = \left(-\partial^2 + m^2\right)^{-1}_{x_E y_E} = \int \frac{d^d p_E}{(2\pi)^d} \frac{e^{-i p_E \cdot (x_E - y_E)}}{p_E^2 + m^2}$$

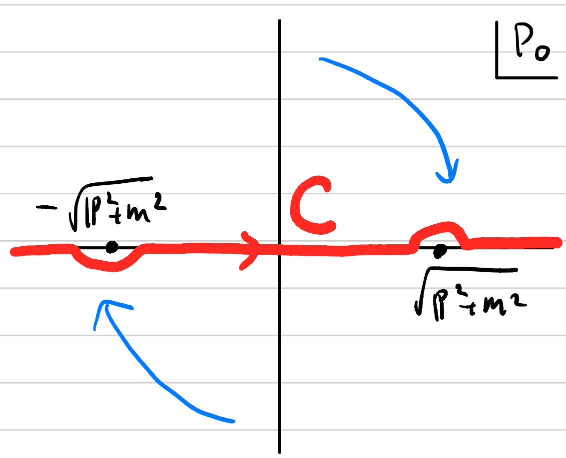
$$= \int \frac{d^{d-1}\mathbb{P}\, dp_d}{(2\pi)^d} \frac{e^{-i \mathbb{P} \cdot (\mathbb{x} - \mathbb{y}) - i p_d (x^d - y^d)}}{\mathbb{P}^2 + p_d^2 + m^2}$$



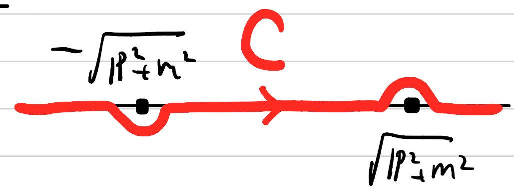

**reverse Wick rotation**

$x^d \to i x^0, \quad y^d \to i y^0; \quad p_d \to -i p_0$

$$\to \int_{\mathbb{R}^{d-1} \times C} \frac{d^{d-1}\mathbb{P}\, (-i\, dp_0)}{(2\pi)^d} \frac{e^{-i \mathbb{P} \cdot (\mathbb{x} - \mathbb{y}) - i p_0 (x^0 - y^0)}}{\mathbb{P}^2 - p_0^2 + m^2}$$



$$\langle \phi(x)\phi(y)\rangle = \int_{\mathbb{R}^{d-1}\times C} \frac{d^{d-1}\mathbb{p}\, dp_0}{(2\pi)^d} \frac{i\, e^{-ip\cdot(x-y)}}{p_0^2-\mathbb{p}^2-m^2}$$




Equivalently, slightly moving the poles,

$$\langle \phi(x)\phi(y)\rangle = \int_{\mathbb{R}^d} \frac{d^d p}{(2\pi)^d} \frac{i\, e^{-ip\cdot(x-y)}}{p^2-m^2+i\cdot 0}$$

$$p^2 := \eta^{\mu\nu}p_\mu p_\nu = p_0^2 - \mathbb{p}^2 \quad \left( \begin{array}{c} \text{Our convention:} \\ \eta_{00}=1, \quad \eta_{ij} = -\delta_{ij} \end{array} \right)$$

This is the inverse of the Kinetic operator in the Minkowski action ($\times\, i$ from $e^{iS[\phi]}$)

$$S[\phi] = \int d^d x \left( \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - \frac{m^2}{2}\phi^2 \right)$$

$$= \frac{1}{2}\int d^d x\, \phi(x)\left(-\eta^{\mu\nu}\partial_\mu\partial_\nu - m^2\right)\phi(x)$$

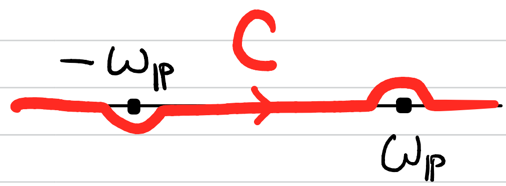
with the prescription $\dfrac{1}{p^2-m^2} \to \dfrac{1}{p^2-m^2+i\cdot 0}$ to avoid poles.

Let us continue the computation. Defining

$$\omega_{\mathbb{p}} := \sqrt{\mathbb{p}^2 + m^2},$$

$$\langle \phi(x)\phi(y) \rangle$$

$$= \int_{\mathbb{R}^{d-1}} \frac{d^{d-1}\mathbb{p}}{(2\pi)^{d-1}} \int_{C} \frac{dp_0}{2\pi} \frac{i\, e^{-ip_0(x^0-y^0) - i\mathbb{p}\cdot(\mathbb{x}-\mathbb{y})}}{(p_0 - \omega_{\mathbb{p}})(p_0 + \omega_{\mathbb{p}})}$$

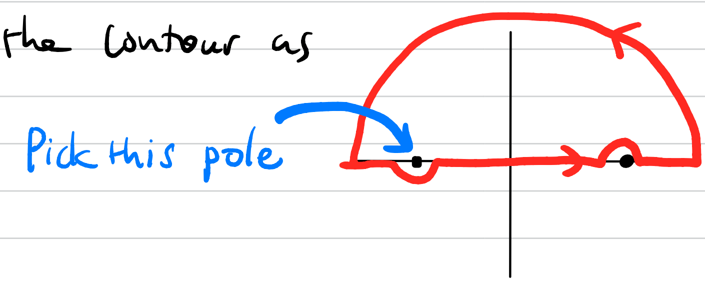



We may perform the $p_0$-integration

... Look at $e^{-ip_0(x^0-y^0)}$

$x^0 - y^0 > 0$ : close the contour as


$x^0 - y^0 < 0$ : close the contour as




The result

$$\langle \phi(x)\phi(y) \rangle = \int \frac{d^{d-1}\mathbb{p}}{(2\pi)^{d-1}\, 2\omega_{\mathbb{p}}} e^{-i\omega_{\mathbb{p}}|x^0-y^0| - i\mathbb{p}\cdot(\mathbb{x}-\mathbb{y})}$$

## Comparison with operator result

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\,\partial_\mu\phi - \frac{m^2}{2}\phi^2$$

$$\phi(\mathbb{x}) = \int\frac{d^{d-1}\mathbb{p}}{(2\pi)^{d-1}}\, e^{i\mathbb{p}\cdot\mathbb{x}}\,\phi(\mathbb{p}) \,, \qquad \phi(\mathbb{p})^* = \phi(-\mathbb{p})$$

$$L = \int d^{d-1}\mathbb{x}\,\mathcal{L}$$

$$= \int\frac{d^{d-1}\mathbb{p}}{(2\pi)^{d-1}}\left\{\frac{1}{2}\dot\phi(-\mathbb{p})\dot\phi(\mathbb{p}) - \frac{1}{2}\underbrace{(\mathbb{p}^2+m^2)}_{\omega_{\mathbb{p}}^2}\phi(-\mathbb{p})\phi(\mathbb{p})\right\}$$

$$\pi(\mathbb{p}) = \frac{\delta L}{\delta\dot\phi(\mathbb{p})} = \frac{1}{(2\pi)^{d-1}}\dot\phi(-\mathbb{p}) \,, \qquad \pi(\mathbb{p})^* = \pi(-\mathbb{p})$$

$$H = \int d^{d-1}\mathbb{p}\ \pi(\mathbb{p})\dot\phi(\mathbb{p}) - L$$

$$= \int d^{d-1}\mathbb{p}\left\{\frac{(2\pi)^{d-1}}{2}\pi(\mathbb{p})\pi(-\mathbb{p}) + \frac{\omega_{\mathbb{p}}^2}{2(2\pi)^{d-1}}\phi(-\mathbb{p})\phi(\mathbb{p})\right\}$$

Let us quantize the system.

(We omit hat ^ for operators.)

By the reality of variables, $\phi(\mathbb{p})^\dagger = \phi(-\mathbb{p})$, $\pi(\mathbb{p})^\dagger = \pi(-\mathbb{p})$.

Canonical commutation relation is

$$[\phi(\mathbb{P}_1), \pi(\mathbb{P}_2)] = i\delta^{d-1}(\mathbb{P}_1 - \mathbb{P}_2)$$

$$[\phi(\mathbb{P}_1), \phi(\mathbb{P}_2)] = [\pi(\mathbb{P}_1), \pi(\mathbb{P}_2)] = 0$$

Looking at H, we see that the system is just the sum of copies of harmonic oscillators. This motivates us to take

$$a(\mathbb{P}) := \sqrt{\frac{\omega_{\mathbb{P}}}{2(2\pi)^{d-1}}}\,\phi(\mathbb{P}) + i\sqrt{\frac{(2\pi)^{d-1}}{2\omega_{\mathbb{P}}}}\,\pi(-\mathbb{P})$$

$$a(\mathbb{P})^{\dagger} = \sqrt{\frac{\omega_{\mathbb{P}}}{2(2\pi)^{d-1}}}\,\phi(-\mathbb{P}) - i\sqrt{\frac{(2\pi)^{d-1}}{2\omega_{\mathbb{P}}}}\,\pi(\mathbb{P})$$

Then,

$$[a(\mathbb{P}_1), a(\mathbb{P}_2)^{\dagger}] = \delta^{d-1}(\mathbb{P}_1 - \mathbb{P}_2)$$

$$[a(\mathbb{P}_1), a(\mathbb{P}_2)] = [a(\mathbb{P}_1)^{\dagger}, a(\mathbb{P}_2)^{\dagger}] = 0$$

$$H = \int d^{d-1}\mathbb{P}\ \omega_{\mathbb{P}} \left( \frac{1}{2} a(\mathbb{P})^{\dagger} a(\mathbb{P}) + \frac{1}{2} a(-\mathbb{P}) a(-\mathbb{P})^{\dagger} \right)$$

$$= \int d^{d-1}\mathbb{P}\ \omega_{\mathbb{P}} \left( a(\mathbb{P})^{\dagger} a(\mathbb{P}) + \frac{1}{2}\delta^{d-1}(0) \right)$$

$$[H, a(\mathbb{p})] = -\omega_{\mathbb{p}}\, a(\mathbb{p}), \quad [H, a(\mathbb{p})^\dagger] = \omega_{\mathbb{p}}\, a(\mathbb{p})^\dagger.$$

Thus, $a(\mathbb{p})^\dagger / a(\mathbb{p})$ are indeed creation/annihilation operators.

The state $|0\rangle$ annihilated by all $a(\mathbb{p})$ is the unique ground state with energy $E_0 = \int d^{d-1}\mathbb{p}\, \frac{1}{2}\omega_{\mathbb{p}}\, \underbrace{\delta^{d-1}(0)}$.

best understood by putting the system in a finite volume

Other states are obtained from $|0\rangle$ by operating $a(\mathbb{p})^\dagger$'s.

Each operation increases the energy by $\omega_{\mathbb{p}}$.

$$\phi(\mathbb{x}) = \int \frac{d^{d-1}\mathbb{p}}{(2\pi)^{d-1}}\, e^{i\mathbb{p}\cdot\mathbb{x}} \sqrt{\frac{(2\pi)^{d-1}}{2\omega_{\mathbb{p}}}} \left( a(\mathbb{p}) + a(-\mathbb{p})^\dagger \right)$$

$$= \int \frac{d^{d-1}\mathbb{p}}{\sqrt{(2\pi)^{d-1} 2\omega_{\mathbb{p}}}} \left( e^{i\mathbb{p}\cdot\mathbb{x}} a(\mathbb{p}) + e^{-i\mathbb{p}\cdot\mathbb{x}} a(\mathbb{p})^\dagger \right)$$

$$\phi(t, \mathbb{x}) = e^{itH} \phi(\mathbb{x})\, e^{-itH}$$

$$= \int \frac{d^{d-1}\mathbb{p}}{\sqrt{(2\pi)^{d-1} 2\omega_{\mathbb{p}}}} \left( e^{i\mathbb{p}\cdot\mathbb{x}} e^{-i\omega_{\mathbb{p}} t} a(\mathbb{p}) + e^{-i\mathbb{p}\cdot\mathbb{x} + i\omega_{\mathbb{p}} t} a(\mathbb{p})^\dagger \right)$$

$\langle 0 | T \phi(x) \phi(y) | 0 \rangle$

$$\overset{x^0 > y^0}{=} \langle 0 | \phi(x) \phi(y) | 0 \rangle$$

$$= \int \frac{d^{d-1}\mathbb{p}_1 \, d^{d-1}\mathbb{p}_2}{(2\pi)^{d-1} 2\sqrt{\omega_{\mathbb{p}_1}\omega_{\mathbb{p}_2}}} \underbrace{\langle 0 | e^{i\mathbb{p}_1\cdot\mathbb{x} - i\omega_{\mathbb{p}_1}x^0} a(\mathbb{p}_1) \, e^{-i\mathbb{p}_2\cdot\mathbb{y} + i\omega_{\mathbb{p}_2}y^0} a(\mathbb{p}_2)^\dagger | 0 \rangle}_{e^{-i\omega_{\mathbb{p}_1}x^0 + i\omega_{\mathbb{p}_2}y^0 + i\mathbb{p}_1\cdot\mathbb{x} - i\mathbb{p}_2\cdot\mathbb{y}} \, \delta^{d-1}(\mathbb{p}_1 - \mathbb{p}_2)}$$

$$= \int \frac{d^{d-1}\mathbb{p}}{(2\pi)^{d-1} 2\omega_{\mathbb{p}}} \, e^{-i\omega_{\mathbb{p}}(x^0 - y^0) + i\mathbb{p}\cdot(\mathbb{x} - \mathbb{y})}$$

$$\overset{y^0 > x^0}{=} \int \frac{d^{d-1}\mathbb{p}}{(2\pi)^{d-1} 2\omega_{\mathbb{p}}} \, e^{-i\omega_{\mathbb{p}}(y^0 - x^0) + i\mathbb{p}\cdot(\mathbb{y} - \mathbb{x})}$$

In eigther case

$\langle 0 | T \phi(x) \phi(y) | 0 \rangle$

$$= \int \frac{d^{d-1}\mathbb{p}}{(2\pi)^{d-1} 2\omega_{\mathbb{p}}} \, e^{-i\omega_{\mathbb{p}}|x^0 - y^0| - i\mathbb{p}\cdot(\mathbb{x} - \mathbb{y})}$$

It matches with $\langle \phi(x) \phi(y) \rangle$.

# e.g. A part of gauge fixed Maxwell theory

$$\tilde{\mathcal{L}}_E = \frac{1}{4e^2}\sum_{\mu,\nu} F_{\mu\nu}^2 + \frac{e^2\xi}{2}B^2 - iB\cdot\partial^\mu A_\mu + \bar{c}\cdot\partial^2 c$$

Eliminate B

$$\tilde{\mathcal{L}}_E = \underbrace{\frac{1}{4e^2}\sum_{\mu,\nu} F_{\mu\nu}^2 + \frac{1}{2e^2\xi}(\partial\cdot A)^2}_{\text{Consider this part}} + \bar{c}\cdot\partial^2 c$$

$$S_E[A] = \int d^d x_E \left( \frac{1}{4e^2}\sum_{\mu,\nu} F_{\mu\nu}^2 + \frac{1}{2e^2}(\partial\cdot A)^2 \right)$$

$$= \int d^d x_E \frac{1}{2e^2}\sum_{\mu,\nu} A_\mu(x_E)\left(-\delta_{\mu\nu}\partial^2 + \partial_\mu\partial_\nu - \frac{1}{\xi}\partial_\mu\partial_\nu\right)A_\nu(x_E)$$

$$=: \frac{1}{2} A\cdot\Delta A$$

The boundary condition at $|x_E| \to \infty$ must allow this partial integration. e.g. $F_{\mu\nu} \to 0$ at $\infty$ does.

$$\langle A_\mu(x_E)A_\nu(y_E)\rangle_{\mathbb{R}^d_E} = \Delta^{-1}_{(\mu,x_E),(\nu,y_E)}$$

$$= e^2\int \frac{d^d p_E}{(2\pi)^d}\frac{e^{-ip_E(x-y)}}{p_E^2}\left(\delta_{\mu\nu} + (\xi-1)\frac{p^E_\mu p^E_\nu}{p_E^2}\right)$$

reverse Wick rotation: $x^d \to i x^0$ ; $p_d \to -i p_0$

$$A_d(x_E) \to -i A_0(x)$$

$$\left.\begin{array}{lcl} \delta_{00} & \to & -1 \\ \delta_{ij} & \to & \delta_{ij} \end{array}\right\} \quad \delta_{\mu\nu} \to -\eta_{\mu\nu}$$

$$\langle A_\mu(x) A_\nu(y) \rangle$$

$$= e^2 \int \frac{d^d p}{(2\pi)^d} \frac{i\, e^{-ip(x-y)}}{p^2+i0} \left( -\eta_{\mu\nu} - (\xi-1)\frac{p_\mu p_\nu}{p^2+i0} \right)$$

The result depends on $\xi$ $\longleftrightarrow$ $A_\mu$ is not physical,

$$\delta_B A_\mu = \partial_\mu C \neq 0.$$

$$\langle F_{\mu\nu}(x) F_{\rho\lambda}(y) \rangle = ?$$

$$\langle \partial_\mu A_\nu(x)\, \partial_\rho A_\lambda(y) \rangle$$

$$= e^2 \int \frac{d^d p}{(2\pi)^d} \frac{i\, e^{-ip(x-y)}}{p^2+i0} \Big( -p_\mu p_\rho \eta_{\nu\lambda} - (\xi-1) \underbrace{\frac{p_\mu p_\nu p_\rho p_\lambda}{p^2+i0}} \Big)$$

Symmetric in $(\mu\nu)$, $(\rho\lambda)$

$\to$ vanishes in $[\mu\nu]$, $[\rho\lambda]$ alternating sum

$$\langle F_{\mu\nu}(x) F_{\rho\lambda}(y) \rangle$$

$$= e^2 \int \frac{d^d p}{(2\pi)^d} \frac{i\, e^{-ip(x-y)}}{p^2 + i0}$$

$$\left( - p_\mu p_\rho \eta_{\nu\lambda} + p_\nu p_\rho \eta_{\mu\lambda} + p_\mu p_\lambda \eta_{\nu\rho} - p_\nu p_\lambda \eta_{\mu\rho} \right)$$

This is the full correlation function of the gauge fixed theory :

$$\langle F_{\mu\nu}(x) F_{\rho\lambda}(y) \rangle_{\text{full}} = \text{the same,}$$

Since the ghost path-integral simply produces

$$\frac{\int \mathcal{D}\bar{c}\, \mathcal{D}c\, e^{\int d^d x\, \bar{c}\, \partial^2 c}}{\int \mathcal{D}\bar{c}\, \mathcal{D}c\, e^{\int d^d x\, \bar{c}\, \partial^2 c}} = 1.$$

Since $F_{\mu\nu}$ is physical, $\delta_B F_{\mu\nu} = 0$, this is a physically meaningful result. Indeed, there is no $\xi$ - dependence.

**Exercise** Compute the same in the canonical quantization of Maxwell theory, and compare.

<u>Remark</u> The expression for $\langle A_\mu(x) A_\nu(y) \rangle$ simplifies at

$\xi = 1$ called <u>Feynman gauge</u>:

$$\langle A_\mu(x) A_\nu(y) \rangle = e^2 \int \frac{d^d p}{(2\pi)^d} \frac{i e^{-ip(x-y)}}{p^2 + i0} (-\eta_{\mu\nu})$$

(Simplification at $\xi = 1$ is obvious in $S[A]$.)

Of course, the physics (of the full gauge fixed system, with "physical = BRST cohomology" taken into account) should not depend on $\xi$. In other words, $\xi = 1$ is a convenient choice.

Let us continue with the computation of $\langle A_\mu(x) A_\nu(y) \rangle$. As it simplifies at $\xi = 1$, we just use it. Then, we can borrow the result for real scalar and find

$$\langle A_\mu(x) A_\nu(y) \rangle$$

$$= -e^2 \eta_{\mu\nu} \int \frac{d^{d-1} \mathbb{P}}{(2\pi)^{d-1} 2|\mathbb{P}|} e^{-i|\mathbb{P}| \cdot |x^0 - y^0| - i \mathbb{P} \cdot (x-y)}.$$

Let us compare this with operator result (continuing with $\xi = 1$).

$$S[A] = \int d^d x \left( -\frac{1}{4e^2} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2e^2} (\partial^\mu A_\mu)^2 \right)$$

$$= \int d^d x \, \frac{1}{2e^2} A_\mu(x) \eta^{\mu\nu} \partial^2 A_\nu(x)$$

$$L = \int d^{d-1} \mathbb{x} \left[ -\frac{1}{2e^2} \dot{A}_0^2 + \frac{1}{2e^2} (\nabla A_0)^2 + \frac{1}{2e^2} \sum_i \left( \dot{A}_i^2 - (\nabla A_i)^2 \right) \right]$$

$$A_\mu(\mathbb{x}) = \int \frac{d^{d-1} \mathbb{p}}{(2\pi)^{d-1}} e^{i \mathbb{p} \cdot \mathbb{x}} A_\mu(\mathbb{p}) , \qquad A_\mu(\mathbb{p})^* = A_\mu(-\mathbb{p})$$

$$L = \int \frac{d^{d-1} \mathbb{p}}{(2\pi)^{d-1}} \Big[ -\frac{1}{2e^2} \dot{A}_0(-\mathbb{p}) A_0(\mathbb{p}) + \frac{\mathbb{p}^2}{2e^2} A_0(-\mathbb{p}) A_0(\mathbb{p})$$
$$+ \frac{1}{2e^2} \sum_i \left( \dot{A}_i(-\mathbb{p}) A_i(\mathbb{p}) - \mathbb{p}^2 A_i(-\mathbb{p}) A_i(\mathbb{p}) \right) \Big]$$

$$\Pi^0(\mathbb{p}) = -\frac{1}{(2\pi)^{d-1} e^2} \dot{A}_0(-\mathbb{p}) , \quad \Pi^i(\mathbb{p}) = \frac{1}{(2\pi)^{d-1} e^2} \dot{A}_i(-\mathbb{p}) ,$$

$$\Pi^\mu(\mathbb{p})^* = \Pi^\mu(-\mathbb{p}) ,$$

$$H = \int d^{d-1} \mathbb{p} \Big[ -\frac{(2\pi)^{d-1} e^2}{2} \Pi^0(\mathbb{p}) \Pi^0(-\mathbb{p}) - \frac{\mathbb{p}^2}{2(2\pi)^{d-1} e^2} A_0(-\mathbb{p}) A_0(\mathbb{p})$$
$$+ \sum_i \left( \frac{(2\pi)^{d-1} e^2}{2} \Pi^i(\mathbb{p}) \Pi^i(-\mathbb{p}) + \frac{\mathbb{p}^2}{2(2\pi)^{d-1} e^2} A_i(-\mathbb{p}) A_i(\mathbb{p}) \right) \Big]$$

Quantization

$$A_\mu(\mathbb{P})^\dagger = A_\mu(-\mathbb{P}), \quad \Pi^\mu(\mathbb{P})^\dagger = \Pi^\mu(-\mathbb{P})$$

$$[A_\mu(\mathbb{P}_1), \Pi^\nu(\mathbb{P}_2)] = i\delta_\mu^\nu \delta^{d-1}(\mathbb{P}_1 - \mathbb{P}_2)$$

$$[A_\mu(\mathbb{P}_1), A_\nu(\mathbb{P}_2)] = [\Pi^\mu(\mathbb{P}_1), \Pi^\nu(\mathbb{P}_2)] = 0$$

If we put

$$a_\mu(\mathbb{P}) := \sqrt{\frac{|\mathbb{P}|}{2(2\pi)^{d-1}e^2}}\, A_\mu(\mathbb{P}) + i\sqrt{\frac{(2\pi)^{d-1}e^2}{2|\mathbb{P}|}}\, \Pi^\mu(-\mathbb{P})$$

$$a_\mu(\mathbb{P})^\dagger = \sqrt{\frac{|\mathbb{P}|}{2(2\pi)^{d-1}e^2}}\, A_\mu(-\mathbb{P}) - i\sqrt{\frac{(2\pi)^{d-1}e^2}{2|\mathbb{P}|}}\, \Pi^\mu(\mathbb{P})$$

Then, $[a_\mu(\mathbb{P}_1), a_\nu(\mathbb{P}_2)^\dagger] = \delta_{\mu,\nu}\delta^{d-1}(\mathbb{P}_1 - \mathbb{P}_2)$

$$[a_\mu(\mathbb{P}_1), a_\nu(\mathbb{P}_2)] = [a_\mu(\mathbb{P}_1)^\dagger, a_\nu(\mathbb{P}_2)^\dagger] = 0$$

$$H = \int d^{d-1}\mathbb{P} \Big[ -|\mathbb{P}| \left( a_0(\mathbb{P})^\dagger a_0(\mathbb{P}) + \frac{1}{2}\delta^{d-1}(0) \right)$$
$$+ \sum_i |\mathbb{P}| \left( a_i(\mathbb{P})^\dagger a_i(\mathbb{P}) + \frac{1}{2}\delta^{d-1}(0) \right) \Big]$$

$$[H, a_0(\mathbb{P})] = |\mathbb{P}|\, a_0(\mathbb{P}), \quad [H, a_0(\mathbb{P})^\dagger] = -|\mathbb{P}|\, a_0(\mathbb{P})^\dagger$$

$$[H, a_i(\mathbb{P})] = -|\mathbb{P}|\, a_i(\mathbb{P}), \quad [H, a_i(\mathbb{P})^\dagger] = |\mathbb{P}|\, a_i(\mathbb{P})^\dagger$$

$a_0(\mathbb{P})$, $a_i(\mathbb{P})^\dagger$ : creation operators

$a_0(\mathbb{P})^\dagger$, $a_i(\mathbb{P})$ : annihilation operators

The state $|0\rangle$ annihilated by $a_0(\mathbb{P})^\dagger$ and $a_i(\mathbb{P})$ is the unique ground state, with energy

$$E_0 = \int d^{d-1}\mathbb{P}\, \frac{d}{2}|\mathbb{P}|\, \delta^{d-1}(0).$$

Other states are obtained from $|0\rangle$ by operating $a_0(\mathbb{P})$ & $a_i(\mathbb{P})^\dagger$ each operation increasing energy by $|\mathbb{P}|$.

e.g. The 1st excitate states:

$$|\mathbb{P};0\rangle = a_0(\mathbb{P})|0\rangle, \quad |\mathbb{P};i\rangle = a_i(\mathbb{P})^\dagger|0\rangle.$$

Note: assuming $\langle 0|0\rangle = 1$,

$$\langle \mathbb{P}_1; i|\mathbb{P}_2; j\rangle = \langle 0|a_i(\mathbb{P}_1)\, a_j(\mathbb{P}_2)^\dagger|0\rangle$$

$$= \delta_{ij}\,\delta^{d-1}(\mathbb{P}_1-\mathbb{P}_2),$$ this is normal.

$$\langle \mathbb{P}_1; 0|\mathbb{P}_2; 0\rangle = \langle 0|a_0(\mathbb{P}_1)^\dagger a_0(\mathbb{P}_2)|0\rangle$$

$$= -\delta_{ij}\,\delta^{d-1}(\mathbb{P}_1-\mathbb{P}_2),$$ negative norm states!

$$A_\mu(\mathbb{x}) = e\int \frac{d^{d-1}\mathbb{P}}{\sqrt{(2\pi)^{d-1}2|\mathbb{P}|}} e^{i\mathbb{P}\cdot\mathbb{x}} \left( a_\mu(\mathbb{P}) + a_\mu(-\mathbb{P})^\dagger \right)$$

$$= e\int \frac{d^{d-1}\mathbb{P}}{\sqrt{(2\pi)^{d-1}2|\mathbb{P}|}} \left( e^{i\mathbb{P}\cdot\mathbb{x}} a_\mu(\mathbb{P}) + e^{-i\mathbb{P}\cdot\mathbb{x}} a_\mu(\mathbb{P})^\dagger \right)$$

$$A_\mu(t,\mathbb{x}) = e^{itH} A_\mu(\mathbb{x}) e^{-itH}$$

$$= \begin{cases} e\int \frac{d^{d-1}\mathbb{P}}{\sqrt{(2\pi)^{d-1}2|\mathbb{P}|}} \left( e^{-i|\mathbb{P}|t + i\mathbb{P}\cdot\mathbb{x}} a_i(\mathbb{P}) + e^{i|\mathbb{P}|t - i\mathbb{P}\cdot\mathbb{x}} a_i(\mathbb{P})^\dagger \right) & \mu = i \\ e\int \frac{d^{d-1}\mathbb{P}}{\sqrt{(2\pi)^{d-1}2|\mathbb{P}|}} \left( e^{i|\mathbb{P}|t + i\mathbb{P}\cdot\mathbb{x}} a_0(\mathbb{P}) + e^{-i|\mathbb{P}|t - i\mathbb{P}\cdot\mathbb{x}} a_0(\mathbb{P})^\dagger \right) & \mu = 0 \end{cases}$$

$$\langle 0 | T A_i(x) A_j(y) | 0 \rangle$$

$$= e^2 \delta_{ij} \int \frac{d^{d-1}\mathbb{P}}{(2\pi)^{d-1}2|\mathbb{P}|} e^{-i|\mathbb{P}||x^0-y^0| - i\mathbb{P}\cdot(\mathbb{x}-\mathbb{y})}$$

$$\langle 0 | T A_0(x) A_0(y) | 0 \rangle$$

$$= -e^2 \int \frac{d^{d-1}\mathbb{P}}{(2\pi)^{d-1}2|\mathbb{P}|} e^{-i|\mathbb{P}||x^0-y^0| - i\mathbb{P}\cdot(\mathbb{x}-\mathbb{y})}$$

... match with

$$\langle A_\mu(x) A_\nu(y) \rangle = -e^2 \eta_{\mu\nu} \int \frac{d^{d-1}\mathbb{P}}{(2\pi)^{d-1}2|\mathbb{P}|} e^{-i|\mathbb{P}||x^0-y^0| - i\mathbb{P}\cdot(\mathbb{x}-\mathbb{y})}$$