One thing I forgot to say in the last lecture.

-- <u>particle interpretation</u>.

For a real scalar field, we have seen that the state $|0\rangle$ annihilated by all $a(\mathbb{p})$'s is the unique ground state and other states are obtained from $|0\rangle$ by operating $a(\mathbb{p})^\dagger$'s, each increasing the energy by $\omega_{\mathbb{p}} = \sqrt{\mathbb{p}^2 + m^2}$.

<u>Interpretation</u>

$|0\rangle$ -- the vacuum

$a(\mathbb{p})^\dagger$ -- creation of a <u>particle</u> of mass m and momentum $\mathbb{p}$ ($\Rightarrow$ energy $\sqrt{\mathbb{p}^2+m^2}$)

e.g.

$a(\mathbb{p})^\dagger |0\rangle$ -- a one particle state

$a(\mathbb{p}_1)^\dagger a(\mathbb{p}_2)^\dagger |0\rangle$ --- a two particle state

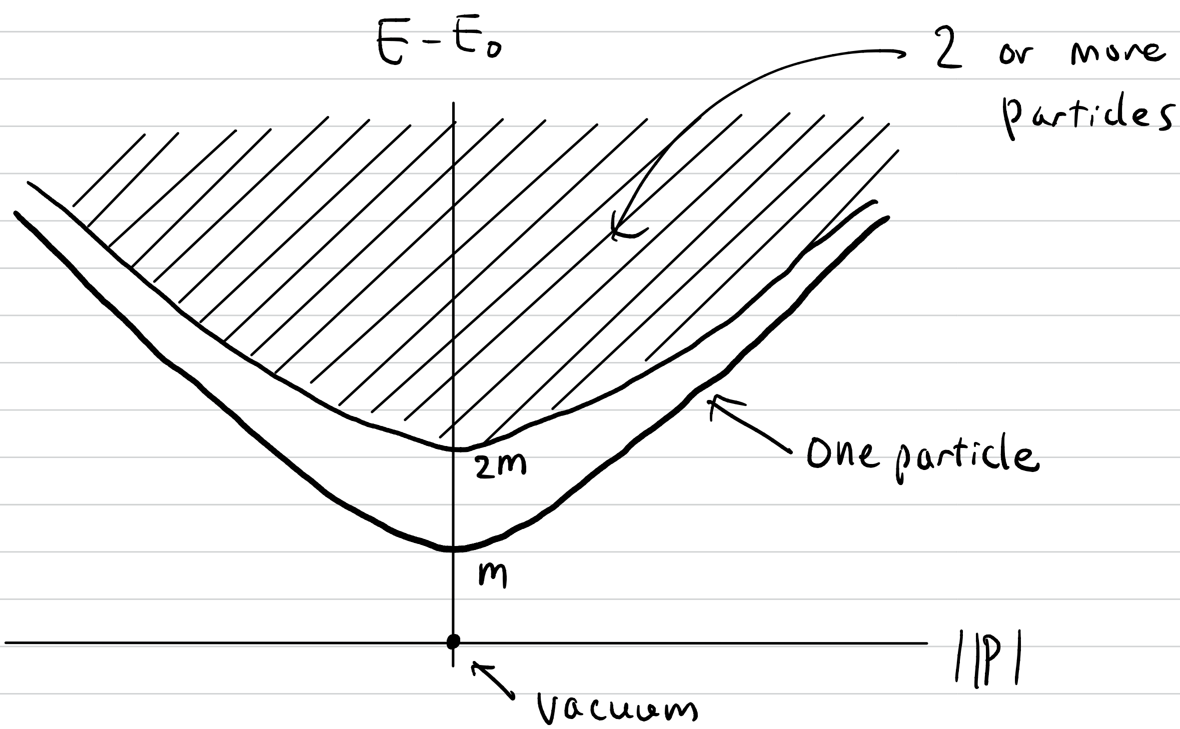
$a(\mathbb{p}_1)^\dagger a(\mathbb{p}_2)^\dagger a(\mathbb{p}_3)^\dagger |0\rangle$ --- a three particle state

$\vdots$

These respectively have

momentum	energy $- E_0$
$\mathbb{P}$	$\sqrt{\mathbb{P}^2+m^2}$
$\mathbb{P}_1+\mathbb{P}_2$	$\sqrt{\mathbb{P}_1^2+m^2}+\sqrt{\mathbb{P}_2^2+m^2}$
$\mathbb{P}_1+\mathbb{P}_2+\mathbb{P}_3$	$\sqrt{\mathbb{P}_1^2+m^2}+\sqrt{\mathbb{P}_2^2+m^2}+\sqrt{\mathbb{P}_3^2+m^2}$
$\vdots$	$\vdots$

Plot of momentum-energy spectrum



Free fermions

A finite system: n pairs of anticommuting variables

$$\psi_1, \bar{\psi}^1, \psi_2, \bar{\psi}^2, \dots, \psi_n, \bar{\psi}^n$$

$$S_E = \sum_{i,j} \bar{\psi}^i A_i{}^j \psi_j$$

$$d\bar{\psi}\, d\psi = d\bar{\psi}^n \cdots d\bar{\psi}^1 d\psi_1 \cdots d\psi_n = d\bar{\psi}^1 d\psi_1 \cdots d\bar{\psi}^n d\psi_n$$

Partition function is

$$Z = \int d\bar{\psi}\, d\psi\, e^{-S_E} = \det A$$

To compute correlation functions, let us introduce

$$f(A, \bar{\eta}, \eta) := \int d\bar{\psi}\, d\psi\, e^{-S_E + \sum_i (\bar{\eta}^i \psi_i + \bar{\psi}^i \eta_i)}.$$

Note:

$$\frac{\partial}{\partial \bar{\eta}^{i_1}} \frac{\partial}{\partial \bar{\eta}^{i_2}} \cdots \frac{\partial}{\partial \bar{\eta}^{i_s}} e^{\bar{\eta}\psi + \bar{\psi}\eta} \frac{\overleftarrow{\partial}}{\partial \eta_{j_1}} \frac{\overleftarrow{\partial}}{\partial \eta_{j_2}} \cdots \frac{\overleftarrow{\partial}}{\partial \eta_{j_t}}$$

$$= e^{\bar{\eta}\psi} \psi_{i_1} \cdots \psi_{i_s} \bar{\psi}^{j_1} \cdots \bar{\psi}^{j_t} e^{\bar{\psi}\eta}$$

Thus,

$$\frac{\partial}{\partial \bar{\eta}^{i_1}} \frac{\partial}{\partial \bar{\eta}^{i_2}} \cdots \frac{\partial}{\partial \bar{\eta}^{i_s}} f(A, \bar{\eta}, \eta) \frac{\overleftarrow{\partial}}{\partial \eta_{j_1}} \frac{\overleftarrow{\partial}}{\partial \eta_{j_2}} \cdots \frac{\overleftarrow{\partial}}{\partial \eta_{j_t}} \Bigg|_{\bar{\eta} = \eta = 0}$$

$$= Z \langle \psi_{i_1} \cdots \psi_{i_s} \bar{\psi}^{j_1} \cdots \bar{\psi}^{j_t} \rangle$$

$$f(A,\bar\eta,\eta) = \int d\bar\psi\, d\psi\, e^{-(\bar\psi - \bar\eta A^{-1})A(\psi - A^{-1}\eta) + \bar\eta A^{-1}\eta}$$

$$= Z\, e^{\bar\eta A^{-1}\eta}$$

$$\therefore \langle \psi_{i_1} \cdots \psi_{i_s} \bar\psi^{j_1} \cdots \bar\psi^{j_t} \rangle$$

$$= \frac{\partial}{\partial \bar\eta^{i_1}} \frac{\partial}{\partial \bar\eta^{i_2}} \cdots \frac{\partial}{\partial \bar\eta^{i_s}} e^{\bar\eta A^{-1}\eta} \frac{\overleftarrow{\partial}}{\partial \eta_{j_1}} \frac{\overleftarrow{\partial}}{\partial \eta_{j_2}} \cdots \frac{\overleftarrow{\partial}}{\partial \eta_{j_t}} \Bigg|_{\bar\eta = \eta = 0}$$

$$= (A^{-1}\eta)_{i_1} \cdots (A^{-1}\eta)_{i_s} \frac{\overleftarrow{\partial}}{\partial \eta_{j_1}} \frac{\overleftarrow{\partial}}{\partial \eta_{j_2}} \cdots \frac{\overleftarrow{\partial}}{\partial \eta_{j_t}} \Bigg|_{\eta = 0}$$

This is non-zero only if $s = t$.

e.g. $\langle \psi_i \bar\psi^j \rangle = \underbrace{(A^{-1}\eta)_i}_{A^{-1}{}_i{}^k \eta_k} \frac{\overleftarrow{\partial}}{\partial \eta_j} = A^{-1}{}_i{}^j$

$$\langle \psi_i \psi_j \bar\psi^k \bar\psi^l \rangle = (A^{-1}\eta)_i (A^{-1}\eta)_j \frac{\overleftarrow{\partial}}{\partial \eta_k} \frac{\overleftarrow{\partial}}{\partial \eta_l}$$

$$= A^{-1}{}_i{}^l A^{-1}{}_j{}^k - A^{-1}{}_i{}^k A^{-1}{}_j{}^l$$

$\frac{\overleftarrow{\partial}}{\partial \eta_k}$ passes through η in $(A^{-1}\eta)_j$

The result can also be presented as the sum of Wick contractions, with the understanding that a (-1) is produced each time two fermionic objects are swapped:

$$\langle \psi_i \bar{\psi}^j \rangle = \psi_i \bar{\psi}^j = A^{-1}{}_i{}^j$$

$$\langle \psi_i \psi_j \bar{\psi}^k \bar{\psi}^l \rangle = \psi_i \psi_j \bar{\psi}^k \bar{\psi}^l + \psi_i \psi_j \bar{\psi}^k \bar{\psi}^l$$

$$= \psi_i \bar{\psi}^l \psi_j \bar{\psi}^k - \psi_i \bar{\psi}^k \psi_j \bar{\psi}^l$$

$$= A^{-1}{}_i{}^l A^{-1}{}_j{}^k - A^{-1}{}_i{}^k A^{-1}{}_j{}^l$$

$$\vdots$$

- We see that everything is determined by the two point functions
$$\langle \psi_i \bar{\psi}^j \rangle = \psi_i \bar{\psi}^j = A^{-1}{}_i{}^j .$$

- The logic holds also when $n = \infty$, e.g. in QFT in dimension $d \geq 1$. We now apply this to important examples. The focus will thus be two point functions.

Dirac fermions in d dimensions

Gamma matrices

The algebra over $\mathbb{C}$ generated by $\gamma^0, \cdots, \gamma^{d-1}$ with relation

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$$

$$\text{i.e.} \begin{cases} (\gamma^0)^2 = 1, \quad (\gamma^i)^2 = -1 \quad \text{for } i = 1, \cdots, d-1, \\ \gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu \quad \text{for } 0 \leq \mu < \nu \leq d-1 \end{cases}$$

.... called the Clifford algebra in dimension $(d-1, 1)$

has an irreducible representation on a vector space S of dimension

$$d_S = 2^{[d/2]} = 2^n \begin{cases} d = 2n & \text{even} \\ d = 2n+1 & \text{odd.} \end{cases}$$

- S has a positive definite hermitian inner product s.t.

$$\gamma^{0\dagger} = \gamma^0 \ \text{hermitian}, \quad \gamma^{i\dagger} = -\gamma^i \ \text{antihermitian.}$$

- $\mathrm{tr}_S(\gamma^{\mu_1} \cdots \gamma^{\mu_\ell}) = 0$ for distinct $\mu_1, \cdots, \mu_\ell$'s

except $\ell = d = 2n+1$.

In particular, $\mathrm{tr}_S \gamma^\mu = 0$ if $d > 1$.

- S is a representation of Lorentz group $SO(d-1,1)$ $\left(\begin{matrix}\text{to be precise} \\ Spin(d-1,1)\end{matrix}\right)$

$$e^{\omega} \in SO(d-1,1) : \varphi \in S \mapsto e^{\frac{1}{2}\gamma(\omega)} \varphi \in S$$

where $\gamma(\omega) := \frac{1}{2}\omega_{\mu\nu}[\gamma^\mu, \gamma^\nu] =: \omega_{\mu\nu}\gamma^{\mu\nu}$.

$\begin{cases} d \text{ odd} : & \text{irreducible} \\ d \text{ even} : & \text{splits to two irreducibles, distinguished by the action of } \gamma^0\gamma^1 \cdots \gamma^{d-1}. \end{cases}$

- For $\varphi \in S$, define $\overline{\varphi} \in S^*$ by

$$\overline{\varphi}\varphi' := \varphi^\dagger \gamma^0 \varphi' \quad \text{for } \varphi' \in S.$$

Then, $\overline{\varphi}\varphi'$ is a Lorentz scalar and

$\overline{\varphi}\gamma^\mu\varphi'$ is a Lorentz vector.

$\because$ $(\gamma^0\gamma^i)^\dagger = -\gamma^i\gamma^0 = \gamma^0\gamma^i$, $(\gamma^i\gamma^j)^\dagger = \gamma^j\gamma^i = -\gamma^i\gamma^j$ $\quad i \neq j$

$\therefore (\gamma^{\mu\nu})^\dagger\gamma^0 = -\gamma^0\gamma^{\mu\nu}$ $\quad \therefore \left(e^{\frac{1}{2}\gamma(\omega)}\right)^\dagger \gamma^0 = \gamma^0 e^{-\frac{1}{2}\gamma(\omega)}$.

$$\left[\tfrac{1}{2}\gamma(\omega), \gamma^\mu\right] = \tfrac{1}{4}\omega_{\rho\lambda}[\gamma^\rho\gamma^\lambda, \gamma^\mu] = \gamma^\rho \underbrace{\{\gamma^\lambda, \gamma^\mu\}}_{2\eta^{\lambda\mu}} - \underbrace{\{\gamma^\rho, \gamma^\mu\}}_{2\eta^{\rho\mu}}\gamma^\lambda$$

$$= -\omega^\mu{}_\nu \gamma^\nu.$$

$$\therefore e^{-\frac{1}{2}\gamma(\omega)}\gamma^\mu e^{\frac{1}{2}\gamma(\omega)} = (e^\omega)^\mu{}_\nu \gamma^\nu. \qquad //$$

The system

Variable : an S-valued anticommuting function $\psi(x)$ of d dimensional spacetime.

Lagrangian : $$\mathcal{L} = i\bar\psi \not\partial \psi - m\bar\psi\psi$$

where
- $\not\partial := \gamma^\mu \partial_\mu$
- m is a real parameter $m^* = m$.

- $\mathcal{L}$ is real modulo total derivative

$$\mathcal{L}^* = \underbrace{-i\,\partial_\mu\psi^\dagger \underbrace{(\gamma^0\gamma^\mu)^\dagger}_{\gamma^0\gamma^\mu}\psi}_{-i\partial_\mu\bar\psi\gamma^\mu\psi} - m\underbrace{\psi^\dagger\underbrace{\gamma^{0\dagger}}_{\gamma^0}\psi}_{\bar\psi\psi}$$

$$= \partial_\mu(-i\bar\psi\gamma^\mu\psi) + i\bar\psi\gamma^\mu\partial_\mu\psi - m\bar\psi\psi$$

- The system has Poincaré symmetry (translations & Lorentz). and phase rotation symmetry $\psi \to e^{-i\alpha}\psi$.

Charge densities for translations & phase rotation are

$$T^0_{\ 0} = -i\bar\psi\boldsymbol{\gamma}\cdot\nabla\psi + m\bar\psi\psi, \quad T^0_{\ j} = i\psi^\dagger\partial_j\psi$$

$$J^0 = \psi^\dagger\psi.$$

<u>Wick rotation</u> $x^0 \to -i x^d$

$$\gamma^0 \to -i\gamma_E^d \quad , \quad \gamma_E^i := \gamma^i \quad i=1,\cdots,d-1.$$

$$\{\gamma_E^\mu, \gamma_E^\nu\} = -2\delta^{\mu,\nu} \qquad 1 \leq \mu,\nu \leq d.$$

$$\mathcal{L}_E = -i\bar{\psi}\gamma_E^\mu \partial_\mu \psi + m\bar{\psi}\psi = \bar{\psi}(-i\not{\partial}_E + m)\psi.$$

$$\langle \psi(x)\bar{\psi}(y)\rangle_E = (-i\not{\partial}_E + m)^{-1}_{xy}$$

$$= \int \frac{d^d p_E}{(2\pi)^d} \frac{e^{-ip_E(x-y)}}{-\not{p}_E + m}$$

$$\left[(-\not{p}_E + m)(\not{p}_E + m) = -\not{p}_E^2 + m^2 = p_E^2 + m^2 \right.$$

$$= \int \frac{d^d p_E}{(2\pi)^d} \frac{e^{-ip_E(x-y)}}{p_E^2 + m^2} (\not{p}_E + m)$$

Here, we regard $\langle \psi(x)\bar{\psi}(y)\rangle_E$ as $\mathrm{End}(S)$ valued function.

In components with respect to a basis $\{e^\alpha\} \subset S$, it reads

$$\langle \psi_\alpha(x)\bar{\psi}^\beta(y)\rangle_E = \int \frac{d^d p_E}{(2\pi)^d} \frac{e^{-ip_E(x-y)}}{p_E^2 + m^2} \underbrace{(\not{p}_E + m)_\alpha{}^\beta}_{(\gamma_E^\mu)_\alpha{}^\beta p_{E\mu} + m\delta_\alpha^\beta}$$

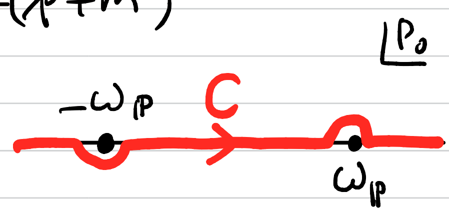
Minkowski limit

Proceeding just as in the case of the scalar field, we find that under the reverse Wick rotation

$$x^d \to i x^0, \quad p_d \to -i p_0, \quad \gamma^d \to i\gamma^0,$$

$\langle \psi(x)\bar{\psi}(y)\rangle_E$ goes to:

$$\langle \psi(x)\bar{\psi}(y)\rangle = \int_{\mathbb{R}^{d-1}\times C} \frac{d^{d-1}\mathbb{p}\, dp_0}{(2\pi)^d} \frac{i e^{-i\mathbb{p}\cdot(\mathbb{x}-\mathbb{y}) - i p_0(x^0-y^0)}}{(p_0-\omega_{\mathbb{p}})(p_0+\omega_{\mathbb{p}})}(\not{p}+m)$$



where $\omega_{\mathbb{p}} = \sqrt{\mathbb{p}^2 + m^2}$

or

$$= \int \frac{d^d p}{(2\pi)^d} \frac{i e^{-ip(x-y)}}{p^2 - m^2 + i\cdot 0}(\not{p}+m)$$

$$= \int \frac{d^{d-1}\mathbb{p}}{(2\pi)^{d-1} 2\omega_{\mathbb{p}}} e^{-i\omega_{\mathbb{p}}|x^0-y^0| - i\mathbb{p}\cdot(\mathbb{x}-\mathbb{y})}\left(\operatorname{sgn}(x^0-y^0)\,\omega_{\mathbb{p}}\gamma^0 + \boldsymbol{\gamma}\cdot\mathbb{p} + m\right)$$

Canonical quantization ($d>1$)

$$\psi(\mathbb{x}) = \int \frac{d^{d-1}\mathbb{p}}{(2\pi)^{d-1}} e^{i\mathbb{p}\cdot\mathbb{x}} \psi(\mathbb{p})$$

$$L = \int d^{d-1}\mathbb{x} \left(i\psi^\dagger (\partial_0 + \gamma^0 \boldsymbol{\gamma}\cdot\nabla)\psi - m\bar{\psi}\psi \right)$$

$$= \int \frac{d^{d-1}\mathbb{p}}{(2\pi)^{d-1}} \left(i\psi(\mathbb{p})^\dagger \dot{\psi}(\mathbb{p}) - \psi(\mathbb{p})^\dagger \gamma^0(\boldsymbol{\gamma}\cdot\mathbb{p} + m)\psi(\mathbb{p}) \right)$$

$$\Delta_{\mathbb{p}} := \gamma^0(\boldsymbol{\gamma}\cdot\mathbb{p} + m) : S \to S$$

- It is hermitian, hence diagonalizable
- $\Delta_{\mathbb{p}}^2 = \gamma^0(\boldsymbol{\gamma}\cdot\mathbb{p}+m)\gamma^0(\boldsymbol{\gamma}\cdot\mathbb{p}+m)$

$$= \underbrace{\gamma^0\gamma^0}_{1} \underbrace{(-\boldsymbol{\gamma}\cdot\mathbb{p}+m)(\boldsymbol{\gamma}\cdot\mathbb{p}+m)}_{\mathbb{p}^2+m^2} = \mathbb{p}^2+m^2 = \omega_{\mathbb{p}}^2$$

Thus, eigenvalues of $\Delta_{\mathbb{p}}$ are $\pm\omega_{\mathbb{p}}$

- $\mathrm{tr}_S \Delta_{\mathbb{p}} = 0$ since $\mathrm{tr}_S(\gamma^0\gamma^i) = \mathrm{tr}_S(\gamma^i\gamma^0) = -\mathrm{tr}_S(\gamma^0\gamma^i)$

& $\mathrm{tr}_S \gamma^0 = 0$.

Thus, half of the eigenvalues are $+\omega_{\mathbb{p}}$

and the other halves are $-\omega_{\mathbb{p}}$.

Let $S_\pm(\mathbb{P}) \subset S$ be the $\pm\omega_\mathbb{P}$ eigen space of $\Delta_\mathbb{P}$.

Then $S = S_+(\mathbb{P}) \dotplus S_-(\mathbb{P})$ orthogonal decomposition, and

$$\dim S_+(\mathbb{P}) = \dim S_-(\mathbb{P}) = \frac{1}{2}\dim S = \frac{1}{2} d_S = 2^{[d/2]-1}.$$

Let $\{u_\pm^s(\mathbb{P})\}_{s=1}^{d_S/2} \subset S_\pm(\mathbb{P})$ be an orthonormal basis.

The elements satisfy

- $u_\varepsilon^s(\mathbb{P})^\dagger u_{\varepsilon'}^{s'}(\mathbb{P}) = \delta^{ss'}\delta_{\varepsilon\varepsilon'}$
- $\sum_s u_\pm^s(\mathbb{P})\, u_\pm^s(\mathbb{P})^\dagger$ = projection operator to $S_\pm(\mathbb{P})$

$$= \frac{1}{2}\left(1 \pm \frac{1}{\omega_\mathbb{P}}\Delta_\mathbb{P}\right)$$

$$\therefore \sum_s u_\pm^s(\mathbb{P})\,\overline{u_\pm^s(\mathbb{P})} = \frac{1}{2}\left(1 \pm \frac{1}{\omega_\mathbb{P}}\Delta_\mathbb{P}\right)\gamma^0$$

$$= \frac{1}{2\omega_\mathbb{P}}\left(\omega_\mathbb{P}\gamma^0 \pm \gamma^0(\boldsymbol{\gamma}\cdot\mathbb{P} + m)\gamma^0\right)$$

$$\left[\ \gamma^0\gamma^i = -\gamma^i\gamma^0,\ \gamma^0\gamma^0 = 1\right.$$

$$= \frac{1}{2\omega_\mathbb{P}}\left(\omega_\mathbb{P}\gamma^0 \mp \boldsymbol{\gamma}\cdot\mathbb{P} \pm m\right).$$

Let us expand $\Psi(\mathbb{P})$ w.r.t. the basis $\{u_+^s(\mathbb{P})\}_{s=1}^{d_S/2} \cup \{u_-^s(\mathbb{P})\}_{s=1}^{d_S/2}$ of S as

$$\Psi(\mathbb{P}) = \sum_s \left(u_+^s(\mathbb{P})\, b_{+s}(\mathbb{P}) + u_-^s(\mathbb{P})\, b_{-s}(\mathbb{P}) \right) \cdot \sqrt{(2\pi)^{d-1}}.$$

Then

$$L = \int d^{d-1}\mathbb{P} \sum_{\varepsilon,s} \left(b_{\varepsilon s}(\mathbb{P})^\dagger \dot{b}_{\varepsilon s}(\mathbb{P}) - \varepsilon \omega_{\mathbb{P}}\, b_{\varepsilon s}(\mathbb{P})^\dagger b_{\varepsilon s}(\mathbb{P}) \right)$$

We know how to quantize such a system (Lecture 3):

$$\{ b_{\varepsilon s}(\mathbb{P}), b_{\varepsilon' s'}(\mathbb{P}')^\dagger \} = \delta_{\varepsilon\varepsilon'} \delta_{ss'} \delta^{d-1}(\mathbb{P}-\mathbb{P}'),$$

$$\{ b_{\varepsilon s}(\mathbb{P}), b_{\varepsilon' s'}(\mathbb{P}') \} = \{ b_{\varepsilon s}(\mathbb{P})^\dagger, b_{\varepsilon' s'}(\mathbb{P}')^\dagger \} = 0,$$

$$H = \int d^{d-1}\mathbb{P} \sum_{\varepsilon s} \varepsilon \omega_{\mathbb{P}}\, b_{\varepsilon s}(\mathbb{P})^\dagger b_{\varepsilon s}(\mathbb{P}).$$

$$[H, b_{\varepsilon s}(\mathbb{P})] = -\varepsilon \omega_{\mathbb{P}}\, b_{\varepsilon s}(\mathbb{P}), \quad [H, b_{\varepsilon s}(\mathbb{P})^\dagger] = \varepsilon \omega_{\mathbb{P}}\, b_{\varepsilon s}(\mathbb{P})^\dagger$$

$\therefore$ $b_{+s}(\mathbb{P})^\dagger$, $b_{-s}(\mathbb{P})$: creation operator

$b_{+s}(\mathbb{P})$, $b_{-s}(\mathbb{P})^\dagger$: annihilation operator

The state $|0\rangle$ annihilated by $b_{+s}(\mathbb{P})$ and $b_{-s}(\mathbb{P})^\dagger$ $\forall s, \forall \mathbb{P}$ is the unique ground state, with energy

$$E_0 = \int d^{d-1}\mathbb{P} \left(-\frac{d_S}{2} \omega_{\mathbb{P}} \delta^{d-1}(0) \right).$$

Other states are obtained from $|0\rangle$ by operating $b_{+s}(\mathbb{p})^\dagger$ & $b_{-s}(\mathbb{p})$, each operation increasing energy by $\omega_{\mathbb{p}}$.

<u>Remarks</u> • There is <u>no</u> negative norm states.

e.g. $|\mathbb{p};+s\rangle := b_{+s}(\mathbb{p})^\dagger|0\rangle$, $|\mathbb{p};-s\rangle := b_{-s}(\mathbb{p})|0\rangle$.

Assuming $\langle 0|0\rangle = 1$,

$$\langle \mathbb{p};+s|\mathbb{p}';+s'\rangle = \langle 0|b_{+s}(\mathbb{p})\,b_{+s'}(\mathbb{p}')^\dagger|0\rangle = \delta_{ss'}\,\delta^{d-1}(\mathbb{p}-\mathbb{p}')$$

$$\langle \mathbb{p};-s|\mathbb{p}';-s'\rangle = \langle 0|b_{-s}(\mathbb{p})^\dagger\,b_{-s'}(\mathbb{p}')|0\rangle = \delta_{ss'}\,\delta^{d-1}(\mathbb{p}-\mathbb{p}')$$

• $$Q = \int d^{d-1}\mathbb{p}\sum_{\varepsilon s} b_{\varepsilon s}(\mathbb{p})^\dagger b_{\varepsilon s}(\mathbb{p})$$

$$[Q, b_{\varepsilon s}(\mathbb{p})] = -b_{\varepsilon s}(\mathbb{p}),\quad [Q, b_{\varepsilon s}(\mathbb{p})^\dagger] = b_{\varepsilon s}(\mathbb{p})^\dagger.$$

<u>Interpretation</u>

$b_{+s}(\mathbb{p})^\dagger / b_{+s}(\mathbb{p})$ is creation/annihilation of a particle of mass m, momentum $\mathbb{p}$, charge $+1$

$b_{-s}(\mathbb{p}) / b_{-s}(\mathbb{p})^\dagger$ is creation/annihilation of a particle of mass m, momentum $-\mathbb{p}$, charge -1.

They form a representation of the subgroup of Lorentz group that fixes $(\omega_{\mathbb{p}}, \mathbb{p}) \cong \underset{m\neq 0}{SO(d-1)}$ or $\underset{m=0}{SO(d-2)}$.

$$\psi(\mathbf{x}) = \int \frac{d^{d-1}\mathbf{p}}{\sqrt{(2\pi)^{d-1}}}\, e^{i\mathbf{p}\cdot\mathbf{x}} \sum_s \Big(u_+^s(\mathbf{p})\, b_{+s}(\mathbf{p}) + u_-^s(\mathbf{p})\, b_{-s}(\mathbf{p}) \Big)$$

$$\psi(t,\mathbf{x}) = e^{itH}\, \psi(\mathbf{x})\, e^{-itH}$$

$$= \int \frac{d^{d-1}\mathbf{p}}{\sqrt{(2\pi)^{d-1}}}\, e^{i\mathbf{p}\cdot\mathbf{x}} \sum_s \Big(u_+^s(\mathbf{p})\, e^{-i\omega_{\mathbf{p}} t}\, b_{+s}(\mathbf{p}) + u_-^s(\mathbf{p})\, e^{i\omega_{\mathbf{p}} t}\, b_{-s}(\mathbf{p}) \Big)$$

$$\overline{\psi}(t,\mathbf{x}) = \int \frac{d^{d-1}\mathbf{p}}{\sqrt{(2\pi)^{d-1}}}\, e^{-i\mathbf{p}\cdot\mathbf{x}} \sum_s \Big(\overline{u_+^s(\mathbf{p})}\, e^{+i\omega_{\mathbf{p}} t}\, b_{+s}(\mathbf{p})^\dagger + \overline{u_-^s(\mathbf{p})}\, e^{-i\omega_{\mathbf{p}} t}\, b_{-s}(\mathbf{p})^\dagger \Big)$$

$$\langle 0 | \psi(x)\, \overline{\psi}(y) | 0 \rangle$$

$$= \int \frac{d^{d-1}\mathbf{p}_1\, d^{d-1}\mathbf{p}_2}{(2\pi)^{d-1}}\, e^{i\mathbf{p}_1\cdot\mathbf{x} - i\mathbf{p}_2\cdot\mathbf{y}} \sum_{s_1,s_2} e^{-i\omega_{\mathbf{p}_1} x^0 + i\omega_{\mathbf{p}_2} y^0}\, u_+^{s_1}(\mathbf{p}_1)\, \overline{u_+^{s_2}(\mathbf{p}_2)}\, \times$$

$$\underbrace{\langle 0 | b_{+s_1}(\mathbf{p}_1)\, b_{+s_2}(\mathbf{p}_2)^\dagger | 0 \rangle}_{\delta_{s_1,s_2}\, \delta^{d-1}(\mathbf{p}_1-\mathbf{p}_2)}$$

$$= \int \frac{d^{d-1}\mathbf{p}}{(2\pi)^{d-1}}\, e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y}) - i\omega_{\mathbf{p}}(x^0-y^0)}\, \underbrace{\sum_s u_+^s(\mathbf{p})\, \overline{u_+^s(\mathbf{p})}}_{\frac{1}{2\omega_{\mathbf{p}}}(\omega_{\mathbf{p}}\gamma^0 - \boldsymbol{\gamma}\cdot\mathbf{p} + m)}$$

$$\langle 0 | \overline{\psi}(y)\, \psi(x) | 0 \rangle \leftarrow$$ regarded as $\mathrm{End}(S)$ element

$$= \int \frac{d^{d-1}\mathbf{p}_1\, d^{d-1}\mathbf{p}_2}{(2\pi)^{d-1}}\, e^{i\mathbf{p}_1\cdot\mathbf{x} - i\mathbf{p}_2\cdot\mathbf{y}} \sum_{s_1,s_2} e^{-i\omega_{\mathbf{p}_2} y^0 + i\omega_{\mathbf{p}_1} x^0}\, u_-^{s_1}(\mathbf{p}_1)\, \overline{u_-^{s_2}(\mathbf{p}_2)}\, \times$$

$$\underbrace{\langle 0 | b_{-s_2}(\mathbf{p}_2)^\dagger\, b_{-s_1}(\mathbf{p}_1) | 0 \rangle}_{\delta_{s_2,s_1}\, \delta^{d-1}(\mathbf{p}_2-\mathbf{p}_1)}$$

$$= \int \frac{d^{d-1}\mathbf{p}}{(2\pi)^{d-1}}\, e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y}) - i\omega_{\mathbf{p}}(x^0-y^0)}\, \underbrace{\sum_s u_-^s(\mathbf{p})\, \overline{u_-^s(\mathbf{p})}}_{\frac{1}{2\omega_{\mathbf{p}}}(\omega_{\mathbf{p}}\gamma^0 + \boldsymbol{\gamma}\cdot\mathbf{p} - m)}$$

$$\langle 0| T\psi(x)\bar{\psi}(y)|0\rangle = \begin{cases} \langle 0|\psi(x)\bar{\psi}(y)|0\rangle & x^0 > y^0 \\ -\langle 0|\bar{\psi}(y)\psi(x)|0\rangle & y^0 > x^0 \end{cases}$$

$$= \int \frac{d^{d-1}\mathbb{p}}{(2\pi)^{d-1}} e^{-i\omega_{\mathbb{p}}|x^0-y^0| + i\mathbb{p}\cdot(\mathbb{x}-\mathbb{y})} \frac{1}{2\omega_{\mathbb{p}}}\left(\operatorname{sgn}(x^0-y^0)\,\omega_{\mathbb{p}}\gamma^0 - \boldsymbol{\gamma}\cdot\mathbb{p} + m \right)$$

$$= \int \frac{d^{d-1}\mathbb{p}}{(2\pi)^{d-1}2\omega_{\mathbb{p}}} e^{-i\omega_{\mathbb{p}}|x^0-y^0| - i\mathbb{p}\cdot(\mathbb{x}-\mathbb{y})} \left(\operatorname{sgn}(x^0-y^0)\,\omega_{\mathbb{p}}\gamma^0 + \boldsymbol{\gamma}\cdot\mathbb{p} + m \right)$$

... match with $\langle \psi(x)\bar{\psi}(y)\rangle$.

Ghost system

The gauge fixed Maxwell theory (B - eliminated) :

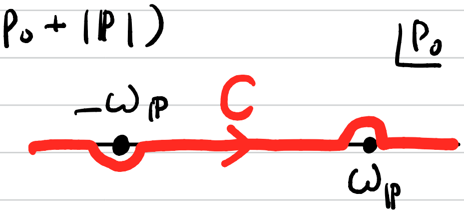
$$\tilde{\mathcal{L}}_E = \frac{1}{4e^2}\sum_{\mu,\nu} F_{\mu\nu}^2 + \frac{1}{2e^2\xi}(\partial\cdot A)^2 + \underbrace{\bar{c}\cdot\partial^2 c}_{\text{Consider this part}}$$

$$\langle c(x)\bar{c}(y)\rangle_E = \partial^{-2}_{x,y} = \int \frac{d^d p_E}{(2\pi)^d}\frac{e^{-ip_E(x-y)}}{-p_E^2}$$

reverse Wick rotation : $x^d \to ix^d$, $p_d \to -ip_0$, $\delta^{\mu\nu} \to -\eta^{\mu\nu}$

Also $\bar{c} \to -i\bar{c}$ (Sign error in Lec5)

$$\langle c(x)(-i)\bar{c}(y)\rangle = \int_{\mathbb{R}^{d-1}\times C} \frac{d^{d-1}\mathbb{P}\, dp_0}{(2\pi)^{d-1}}\frac{i\, e^{-i\mathbb{P}\cdot(\mathbb{x}-\mathbb{y})-ip_0(x^0-y^0)}}{-(p_0-|\mathbb{P}|)(p_0+|\mathbb{P}|)}$$



or

$$\langle c(x)\bar{c}(y)\rangle = \int_{\mathbb{R}^d} \frac{d^d p}{(2\pi)^d}\frac{e^{-ip(x-y)}}{p^2+i\cdot 0}$$

$$= -i\int \frac{d^{d-1}\mathbb{P}}{(2\pi)^{d-1}\,2|\mathbb{P}|}\, e^{-i|\mathbb{P}||x^0-y^0|-i\mathbb{P}\cdot(\mathbb{x}-\mathbb{y})}$$

Canonical quantization

In Minkowski space,

$$S[c,\bar{c}] = \int d^dx\,(-i\bar{c}\,\partial^2 c) = \int d^dx\; i\,\partial^\mu \bar{c}\,\partial_\mu c$$

Reality of fields : $c^* = c$, $\bar{c}^* = \bar{c}$.

Let us describe the system in momentum space.

$$c(\mathbf{x}) = \int \frac{d^{d-1}\mathbb{P}}{(2\pi)^{d-1}}\, e^{i\mathbb{P}\cdot\mathbf{x}}\, c(\mathbb{P}), \qquad c(\mathbb{P})^* = c(-\mathbb{P})$$

$$\bar{c}(\mathbf{x}) = \int \frac{d^{d-1}\mathbb{P}}{(2\pi)^{d-1}}\, e^{i\mathbb{P}\cdot\mathbf{x}}\, \bar{c}(\mathbb{P}), \qquad \bar{c}(\mathbb{P})^* = \bar{c}(-\mathbb{P})$$

$$L = \int \frac{d^{d-1}\mathbb{P}}{(2\pi)^{d-1}} \left(i\,\dot{\bar{c}}(-\mathbb{P})\dot{c}(\mathbb{P}) - i\,\mathbb{P}^2\,\bar{c}(-\mathbb{P})c(\mathbb{P}) \right)$$

$$H = \int \frac{d^{d-1}\mathbb{P}}{(2\pi)^{d-1}} \left(i\,\dot{\bar{c}}(-\mathbb{P})\dot{c}(\mathbb{P}) + i\,\mathbb{P}^2\,\bar{c}(-\mathbb{P})c(\mathbb{P}) \right)$$

A system of this type was discussed in Lecture 3, Exercise (C).

By Ward identity, we find

$$\{c(\mathbb{P}), \dot{\bar{c}}(-\mathbb{P}')\} = (2\pi)^{d-1}\delta^{d-1}(\mathbb{P}-\mathbb{P}'),$$

$$\{\bar{c}(-\mathbb{P}), \dot{c}(\mathbb{P}')\} = -(2\pi)^{d-1}\delta^{d-1}(\mathbb{P}-\mathbb{P}'),$$

all other anticommutators of $c, \bar{c}, \dot{c}, \dot{\bar{c}} = 0$.

Let us introduce

$$b(\mathbb{P}) := \sqrt{\frac{|\mathbb{P}|}{2(2\pi)^{d-1}}}\, c(\mathbb{P}) + \frac{i}{\sqrt{2(2\pi)^{d-1}|\mathbb{P}|}}\, \dot{c}(\mathbb{P}),$$

$$\bar{b}(\mathbb{P}) := -i\sqrt{\frac{|\mathbb{P}|}{2(2\pi)^{d-1}}}\, \bar{c}(\mathbb{P}) + \frac{i}{\sqrt{2(2\pi)^{d-1}|\mathbb{P}|}}\, \dot{\bar{c}}(\mathbb{P}).$$

Then

$$\{ b(\mathbb{P}), \bar{b}(\mathbb{P}')^\dagger \} = \delta^{d-1}(\mathbb{P}-\mathbb{P}')$$

$$\{ \bar{b}(\mathbb{P}), b(\mathbb{P}')^\dagger \} = \delta^{d-1}(\mathbb{P}-\mathbb{P}')$$

all other anticommutators of $b, b^\dagger, \bar{b}, \bar{b}^\dagger = 0$.

$$H = \int d^{d-1}\mathbb{P}\, |\mathbb{P}| \left(b(\mathbb{P})^\dagger \bar{b}(\mathbb{P}) + \bar{b}(\mathbb{P})^\dagger b(\mathbb{P}) - \delta^{d-1}(0) \right)$$

$$[H, b(\mathbb{P})] = -|\mathbb{P}|\, b(\mathbb{P}), \quad [H, b(\mathbb{P})^\dagger] = |\mathbb{P}|\, b(\mathbb{P})^\dagger$$

$$[H, \bar{b}(\mathbb{P})] = -|\mathbb{P}|\, \bar{b}(\mathbb{P}), \quad [H, \bar{b}(\mathbb{P})^\dagger] = |\mathbb{P}|\, \bar{b}(\mathbb{P})^\dagger$$

$\therefore$ $b(\mathbb{P})^\dagger, \bar{b}(\mathbb{P})^\dagger$: creation operators,

$b(\mathbb{P}), \bar{b}(\mathbb{P})$: annihilation operators.

The state $|0\rangle$ annihilated by $b(\mathbb{P})$ & $\bar{b}(\mathbb{P})$ $\forall \mathbb{P}$ is the unique ground state, with energy $E_0 = -\int d^{d-1}\mathbb{P}\, |\mathbb{P}|\, \delta^{d-1}(0)$.

Other states are obtained from $|0\rangle$ by operating $b(\mathbb{P})^\dagger$ & $\bar{b}(\mathbb{P})^\dagger$, each increasing energy by $|\mathbb{P}|$.

<u>Remark</u> There are zero & negative norm states.

$$|\mathbb{P},\phi\rangle := b(\mathbb{P})^\dagger|0\rangle, \quad |\phi,\mathbb{P}\rangle := \bar{b}(\mathbb{P})^\dagger|0\rangle$$

$$|\mathbb{P}_1,\mathbb{P}_2\rangle := b(\mathbb{P}_1)^\dagger\bar{b}(\mathbb{P}_2)^\dagger|0\rangle$$

$$\left.\begin{aligned}\langle\mathbb{P}_1,\phi|\mathbb{P}_2,\phi\rangle &= \langle 0|b(\mathbb{P}_1)b(\mathbb{P}_2)^\dagger|0\rangle = 0\\ \langle\phi,\mathbb{P}_1|\phi,\mathbb{P}_2\rangle &= \langle 0|\bar{b}(\mathbb{P}_1)\bar{b}(\mathbb{P}_2)^\dagger|0\rangle = 0\end{aligned}\right\}\text{ zero norm}$$

$$\begin{aligned}\langle\mathbb{P}_1,\mathbb{P}_2|\mathbb{P}_3,\mathbb{P}_4\rangle &= \langle 0|\bar{b}(\mathbb{P}_2)b(\mathbb{P}_1)b(\mathbb{P}_3)^\dagger\bar{b}(\mathbb{P}_4)^\dagger|0\rangle\\ &= -\delta^{d-1}(\mathbb{P}_1-\mathbb{P}_4)\delta^{d-1}(\mathbb{P}_2-\mathbb{P}_3) \quad \leftarrow \text{negative norm}\end{aligned}$$

<u>Interpretation?</u>

$b(\mathbb{P})^\dagger$ & $\bar{b}(\mathbb{P})^\dagger$ are interpreted as creating massless particles.

However, they create states of zero/negative norm.

Thus these particles are unphysical "ghost" particles.

<u>Computation of $\langle 0|T C(x)\bar{C}(y)|0\rangle$</u>

$$C(\mathbb{P}) = \sqrt{\frac{(2\pi)^{d-1}}{2|\mathbb{P}|}}\left(b(\mathbb{P}) + b(-\mathbb{P})^\dagger\right),$$

$$\bar{C}(\mathbb{P}) = i\sqrt{\frac{(2\pi)^{d-1}}{2|\mathbb{P}|}}\left(\bar{b}(\mathbb{P}) + \bar{b}(-\mathbb{P})^\dagger\right).$$

$$C(\mathbb{x}) = \int \frac{d^{d-1}\mathbb{P}}{\sqrt{(2\pi)^{d-1}2|\mathbb{P}|}} \left(e^{i\mathbb{P}\cdot\mathbb{x}} b(\mathbb{P}) + e^{-i\mathbb{P}\cdot\mathbb{x}} b(\mathbb{P})^\dagger \right)$$

$$\bar{C}(\mathbb{x}) = \int \frac{d^{d-1}\mathbb{P}}{\sqrt{(2\pi)^{d-1}2|\mathbb{P}|}} i \left(e^{i\mathbb{P}\cdot\mathbb{x}} \bar{b}(\mathbb{P}) - e^{-i\mathbb{P}\cdot\mathbb{x}} \bar{b}(\mathbb{P})^\dagger \right)$$

$$C(t,\mathbb{x}) = e^{itH} C(\mathbb{x}) e^{-itH}$$

$$= \int \frac{d^{d-1}\mathbb{P}}{\sqrt{(2\pi)^{d-1}2|\mathbb{P}|}} \left(e^{i\mathbb{P}\cdot\mathbb{x} - i|\mathbb{P}|t} b(\mathbb{P}) + e^{-i\mathbb{P}\cdot\mathbb{x} + i|\mathbb{P}|t} b(\mathbb{P})^\dagger \right)$$

$$\bar{C}(t,\mathbb{x}) = e^{itH} \bar{C}(\mathbb{x}) e^{-itH}$$

$$= \int \frac{d^{d-1}\mathbb{P}}{\sqrt{(2\pi)^{d-1}2|\mathbb{P}|}} i \left(e^{i\mathbb{P}\cdot\mathbb{x} - i|\mathbb{P}|t} \bar{b}(\mathbb{P}) - e^{-i\mathbb{P}\cdot\mathbb{x} + i|\mathbb{P}|t} \bar{b}(\mathbb{P})^\dagger \right)$$

$$\langle 0 | C(x)\bar{C}(y) | 0 \rangle = -i \int \frac{d^{d-1}\mathbb{P}}{(2\pi)^{d-1}2|\mathbb{P}|} e^{-i|\mathbb{P}|(x^0-y^0) + i\mathbb{P}\cdot(\mathbb{x}-\mathbb{y})}$$

$$\langle 0 | \bar{C}(y) C(x) | 0 \rangle = i \int \frac{d^{d-1}\mathbb{P}}{(2\pi)^{d-1}2|\mathbb{P}|} e^{-i|\mathbb{P}|(y^0-x^0) + i\mathbb{P}\cdot(\mathbb{y}-\mathbb{x})}$$

$$\therefore \langle 0 | T C(x)\bar{C}(y) | 0 \rangle = -i \int \frac{d^{d-1}\mathbb{P}}{(2\pi)^{d-1}2|\mathbb{P}|} e^{-i|\mathbb{P}||x^0-y^0| - i\mathbb{P}\cdot(\mathbb{x}-\mathbb{y})}$$

... match with $\langle C(x)\bar{C}(y)\rangle$.

The gauge fixed Maxwell theory (full)

$$\tilde{\mathcal{L}} = -\frac{1}{4e^2} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2e^2 \xi} (\partial \cdot A)^2 - i \bar{c}\, \partial^2 c$$

(Set $\xi = e = 1$ below)

BRST symmetry

$$\delta_B A_\mu = \partial_\mu c\,, \quad \delta_B c = 0, \quad \delta_B \bar{c} = i\, \partial \cdot A$$

$$\rightsquigarrow \quad Q_B = \int d^{d-1}\mathbf{x} \left(- F^{0i} \partial_i c - \dot{c}\, \partial \cdot A \right)$$

Quantization: (Notation change: $a_0(\mathbf{p}) \leftrightarrow a_0(-\mathbf{p})^\dagger$)

$p_{\mathbf{p}} x = |\mathbf{p}| t - \mathbf{p} \cdot \mathbf{x}$

$$A_\mu(x) = \int \frac{d^{d-1}\mathbf{p}}{\sqrt{(2\pi)^{d-1} 2|\mathbf{p}|}} \left(e^{-i p_{\mathbf{p}} x} a_\mu(\mathbf{p}) + e^{i p_{\mathbf{p}} x} a_\mu(\mathbf{p})^\dagger \right)$$

$$c(x) = \int \frac{d^{d-1}\mathbf{p}}{\sqrt{(2\pi)^{d-1} 2|\mathbf{p}|}} \left(e^{-i p_{\mathbf{p}} x} b(\mathbf{p}) + e^{i p_{\mathbf{p}} x} b(\mathbf{p})^\dagger \right)$$

$$\bar{c}(x) = \int \frac{d^{d-1}\mathbf{p}}{\sqrt{(2\pi)^{d-1} 2|\mathbf{p}|}}\, i \left(e^{-i p_{\mathbf{p}} x} \bar{b}(\mathbf{p}) - e^{i p_{\mathbf{p}} x} \bar{b}(\mathbf{p})^\dagger \right)$$

$$[a_\mu(\mathbf{p}), a_\nu(\mathbf{p}')^\dagger] = - \eta_{\mu\nu}\, \delta^{d-1}(\mathbf{p} - \mathbf{p}'),$$

$$\{ b(\mathbf{p}), \bar{b}(\mathbf{p}')^\dagger \} = \{ \bar{b}(\mathbf{p}), b(\mathbf{p}')^\dagger \} = \delta^{d-1}(\mathbf{p} - \mathbf{p}'),$$

other commutators / anticommutators $= 0$

$$H = \int d^{d-1}\mathbb{P}\, |\mathbb{P}| \Big(-\eta^{\mu\nu} a_\mu(\mathbb{P})^\dagger a_\nu(\mathbb{P}) + b(\mathbb{P})^\dagger \bar{b}(\mathbb{P}) + \bar{b}(\mathbb{P})^\dagger b(\mathbb{P})$$
$$+ \frac{d-2}{2}\delta^{d-1}(0) \Big)$$

$$[H, \mathcal{O}_{\mathbb{P}}] = -|\mathbb{P}|\mathcal{O}_{\mathbb{P}}, \quad [H, \mathcal{O}_{\mathbb{P}}^\dagger] = |\mathbb{P}|\mathcal{O}_{\mathbb{P}}^\dagger$$
for $\mathcal{O}_{\mathbb{P}} = a_\mu(\mathbb{P}), b(\mathbb{P}), \bar{b}(\mathbb{P})$.

The state $|0\rangle$ annihilated by all such $\mathcal{O}_{\mathbb{P}}$'s $\forall \mathbb{P}$ is the unque ground state. Other states are obtained from $|0\rangle$ by operating $\mathcal{O}_{\mathbb{P}}^\dagger$'s each creating energy $|\mathbb{P}|$.

$$Q_B = -\int d^{d-1}\mathbb{P} \Big\{ b(\mathbb{P}) \Big(\sum_i P_i a_i(\mathbb{P})^\dagger + |\mathbb{P}| a_0(\mathbb{P})^\dagger \Big)$$
$$+ b(\mathbb{P})^\dagger \Big(\sum_i P_i a_i(\mathbb{P}) + |\mathbb{P}| a_0(\mathbb{P}) \Big) \Big\}$$

$$[Q_B, a_i(\mathbb{P})] = P_i b(\mathbb{P}), \quad [Q_B, a_0(\mathbb{P})] = -|\mathbb{P}| b(\mathbb{P}),$$
$$[Q_B, a_i(\mathbb{P})^\dagger] = -P_i b(\mathbb{P})^\dagger, \quad [Q_B, a_0(\mathbb{P})^\dagger] = |\mathbb{P}| b(\mathbb{P})^\dagger,$$

$$\{Q_B, b(\mathbb{P})\} = \{Q_B, b(\mathbb{P})^\dagger\} = 0,$$

$$\{Q_B, \bar{b}(\mathbb{P})\} = -\sum_i P_i a_i(\mathbb{P}) - |\mathbb{P}| a_0(\mathbb{P}),$$
$$\{Q_B, \bar{b}(\mathbb{P})^\dagger\} = -\sum_i P_i a_i(\mathbb{P}) - |\mathbb{P}| a_0(\mathbb{P})^\dagger.$$

BRST cohomology

$$\mathcal{H} = \{ \text{products of } a_\mu^\dagger,\ b^\dagger,\ \bar{b}^\dagger \text{ on } |0\rangle \}$$

ghost number :	a_μ	$a_\mu^\dagger$	b	$b^\dagger$	$\bar{b}$	$\bar{b}^\dagger$	$\|0\rangle$
	0	0	1	1	-1	-1	0 (definition)

$$\mathcal{H}^i = \{ \text{state of ghost number} = i \}$$

$$\cdots \xrightarrow{Q_B} \mathcal{H}^i \xrightarrow{Q_B} \mathcal{H}^{i+1} \xrightarrow{Q_B} \mathcal{H}^{i+2} \to \cdots \qquad Q_B^2 = 0$$

$\rightsquigarrow$ BRST cohomology

$$H^i(\mathcal{H}, Q_B) = \mathrm{Ker}(Q_B : \mathcal{H}^i \to \mathcal{H}^{i+1}) / \mathrm{Im}(Q_B : \mathcal{H}^{i-1} \to \mathcal{H}^i).$$

The basic proposal was :

physical states are BRST cohomology classes

$$\mathcal{H}_{phys} = \bigoplus_i H^i(\mathcal{H}, Q_B).$$

What is $H^i(\mathcal{H}, Q_B)$ then ?

Let us compute !

Warm-up: examine low lying states.

$N_c :=$ number of creation operators

$N_c = 0$ on $|0\rangle$

$N_c = 1$ spanned by $a_\mu(\mathbb{P})^\dagger|0\rangle$, $b(\mathbb{P})^\dagger|0\rangle$, $\bar{b}(\mathbb{P})^\dagger|0\rangle$ (all $\mathbb{P}$'s).

$\mathcal{H}_{N_c} := \{ N_c \text{ creation oprs on } |0\rangle \}$

Q_B does not change N_c

$\mathcal{H}^i_{N_c} = \{ N_c \text{ creation ops on } |0\rangle, \text{ ghost number} = i \}$

$\xrightarrow{Q_B} \mathcal{H}^i_{N_c} \xrightarrow{Q_B} \mathcal{H}^{i+1}_{N_c} \xrightarrow{Q_B} \cdots$ subcomplex.

$$H^i(\mathcal{H}, Q_B) = \bigoplus_{N_c} H^i(\mathcal{H}_{N_c}, Q_B)$$

$N_c = 0$

$$\begin{array}{ccccccccc} & & \mathcal{H}^{-1}_0 & & \mathcal{H}^0_0 & & \mathcal{H}^1_0 & & \\ & & \| & & \| & & \| & & \\ \cdots 0 & \to & 0 & \longrightarrow & \mathbb{C}|0\rangle & \longrightarrow & 0 & \longrightarrow & 0 \cdots \end{array}$$

$$\therefore \; H^i(\mathcal{H}_0, Q_B) = \begin{cases} \mathbb{C}|0\rangle & i=0 \\ 0 & i \neq 0 \end{cases}$$ $\leftarrow$ vacuum

$\underline{N_c = 1}$

$$\begin{array}{ccccccccc} & \mathcal{H}_2^{-2} & & \mathcal{H}_1^{-1} & & \mathcal{H}_1^{0} & & \mathcal{H}_1^{1} & & \mathcal{H}_1^{2} \\ & \| & & \| & & \| & & \| & & \| \\ \to & 0 & \to & \{\bar{b}(\mathbb{P})^\dagger|0\rangle\}_{\mathbb{P}} & \to & \{a_\mu(\mathbb{P})^\dagger|0\rangle\}_{\mathbb{P}} & \to & \{b(\mathbb{P})^\dagger|0\rangle\}_{\mathbb{P}} & \to & 0 & \to \cdots \end{array}$$

$$Q_B\, \bar{b}(\mathbb{P})^\dagger|0\rangle = \Big(-\sum_i p_i\, a_i(\mathbb{P})^\dagger - |\mathbb{P}|\, a_0(\mathbb{P})^\dagger\Big)|0\rangle$$

$$Q_B\, \epsilon^\mu a_\mu(\mathbb{P})^\dagger|0\rangle = -\sum_i \epsilon^i p_i\, b(\mathbb{P})^\dagger|0\rangle + \epsilon^0 |\mathbb{P}|\, b(\mathbb{P})^\dagger|0\rangle$$

$$= \underbrace{\Big(-\sum_i \epsilon^i p_i + \epsilon^0 |\mathbb{P}|\Big)}_{\epsilon\cdot P_{\mathbb{P}}\ ;\ P_{\mathbb{P}} := (|\mathbb{P}|, \mathbb{P})\ \begin{cases} P_{\mathbb{P}}^0 = |\mathbb{P}| \\ P_{\mathbb{P}}^i = p_i \end{cases}} b(\mathbb{P})^\dagger|0\rangle$$

$$\therefore\ H^i(\mathcal{H}_1, Q_B) = 0 \quad \text{if } i \neq 0$$

$$H^0(\mathcal{H}_1, Q_B) = \frac{\{\epsilon^\mu(\mathbb{P})\, a_\mu(\mathbb{P})^\dagger|0\rangle \mid \epsilon(\mathbb{P})\cdot P^{\mathbb{P}} = 0\}_{\mathbb{P}}}{\epsilon(\mathbb{P}) \sim \epsilon(\mathbb{P}) + \text{const}\, P_{\mathbb{P}}}$$

$$\cong \Big\{\sum_i \epsilon^i(\mathbb{P})\, a_i(\mathbb{P})^\dagger|0\rangle \;\Big|\; \sum_i \epsilon^i(\mathbb{P})\, p_i = 0\Big\}_{\mathbb{P}}$$

a particle with a transversal polarization.

($d-2$ choices for each $\mathbb{P}$)

Let us put

$$\mathcal{H}_{\text{transv}} := \{ \text{product of transversal } a^{+}\text{'s on } |0\rangle \}$$

Then

<u>Theorem</u>

$$H^i(\mathcal{H}, Q_B) \cong \begin{cases} \mathcal{H}_{\text{transv}} & i = 0 \\ 0 & i \neq 0. \end{cases}$$

Therefore, $\mathcal{H}_{\text{phys}} \cong \mathcal{H}_{\text{transv}}$.

<u>Remarks</u>

(i) A positive definite inner product is induced on $\mathcal{H}_{\text{phys}}$.

(ii) Quantization based on Hamiltonian formulation, with Gauss law $\nabla \cdot E = 0$ and Coulomb gauge $\nabla \cdot A = 0$, directly finds $\mathcal{H}_{\text{transv}}$ as the space of states.