# Introduction to perturbation theory

Consider the theory of a single real variable $\phi$ with measure $d\phi$ and action

$$S_E(\phi) = \frac{1}{2} a \phi^2 + \frac{\lambda}{4!} \phi^4$$

partition function

$$Z = \int_{\mathbb{R}} d\phi \; e^{-S_E(\phi)} = \int_{\mathbb{R}} d\phi \; e^{-\frac{1}{2}a\phi^2} \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{\lambda}{4!}\right)^n \phi^{4n}$$

Its <u>perturbative expansion</u> is

$$Z_{pert} := \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{\lambda}{4!}\right)^n \int_{\mathbb{R}} d\phi \; e^{-\frac{1}{2}a\phi^2} \phi^{4n}$$

It is <u>not convergent</u>.

( If it had a convergence radius $R > 0$, $Z_{pert}(\lambda) = Z(\lambda)$ for $|\lambda| < R$, but $Z(\lambda)$ is not obviously analytic at $\lambda = 0$: the integral is badly divergent for real negative $\lambda$.

Instead, it is an <u>asymptotic expansion</u> of $Z(\lambda)$:

$$\lambda^{-N} \left( Z(\lambda) - Z_{pert}^{\leq N}(\lambda) \right) \to 0 \quad \text{as} \quad \lambda \searrow 0$$

truncation at order $n \leq N$

Each term of $Z_{pert}$ is a correlation function of the free theory with action $S_{E,free}(\phi) = \frac{1}{2} a \phi^2$ :

$$Z_{pert} = Z_{free} \cdot \sum_{n=0}^{\infty} \frac{1}{n!} \left( -\frac{\lambda}{4!} \right)^n \langle \phi^{4n} \rangle_{free}$$

Similarly for correlation functions

$$\langle f(\phi) \rangle = \frac{1}{Z} \int_{\mathbb{R}} d\phi \, e^{-S_E(\phi)} f(\phi) \quad \rightsquigarrow$$

$$\langle f(\phi) \rangle_{pert} := \frac{1}{Z_{pert}} \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{-\lambda}{4!} \right)^n \int_{\mathbb{R}} d\phi \, e^{-S_{E,free}(\phi)} f(\phi) \, \phi^{4n}$$

$$= \frac{\cancel{Z_{free}} \cdot \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{-\lambda}{4!} \right)^n \langle f(\phi) \, \phi^{4n} \rangle_{free}}{\cancel{Z_{free}} \cdot \sum_{n=0}^{\infty} \frac{1}{n!} \left( -\frac{\lambda}{4!} \right)^n \langle \phi^{4n} \rangle_{free}}$$

<u>Recall</u>: free correlators can be computed as the sum of Wick contractions

$$\langle \phi^4 \rangle_{free} = \phi\phi\phi\phi + \phi\phi\phi\phi + \phi\phi\phi\phi$$

$$= 3 \cdot a^{-1}$$

$$Z_{pert}/Z_{free} = 1 + \left\langle -\frac{\lambda}{4!}\phi^4 \right\rangle_{free} + \frac{1}{2!}\left\langle \left(-\frac{\lambda}{4!}\phi^4\right)^2 \right\rangle_{free} + \cdots$$

$$\left\langle -\frac{\lambda}{4!}\phi^4 \right\rangle_{free} = -\frac{\lambda}{4!}\,\phi\phi\phi\phi \times 3$$ A

$$= \frac{-\lambda}{4\cdot 2}\left(a^{-1}\right)^2$$

$$\frac{1}{2!}\left\langle \left(-\frac{\lambda}{4!}\phi^4\right)^2 \right\rangle_{free}$$

$$= \frac{1}{2!}\left(-\frac{\lambda}{4!}\right)^2 \Big\{ (\phi\phi\phi\phi)\times 3 \times (\phi\phi\phi\phi)\times 3$$ B

$$+ (\phi\phi\phi\phi)(\phi\phi\phi\phi) \times \binom{4}{2}^2 \times 2$$ C

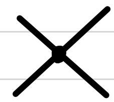
$$+ (\phi\phi\phi\phi)(\phi\phi\phi\phi) \times 4! \Big\}$$ D

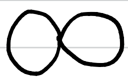
$$= \left\{ \frac{1}{2(4\cdot 2)^2} + \frac{1}{4^2} + \frac{1}{2\cdot 4!} \right\} (-\lambda)^2 \left(a^{-1}\right)^4$$

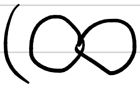
⋮

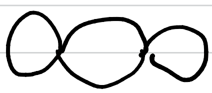
# Diagramatic presentation :

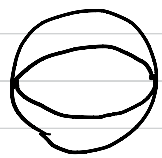
propagator $\overset{\frown}{\phi\phi} = a^{-1}$ $\rightsquigarrow$ line 

interaction $-\frac{\lambda}{4!}\phi^4$ $\rightsquigarrow$ vertex 

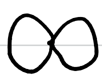
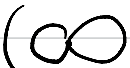
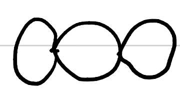
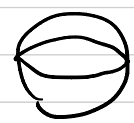
A : 

B : $\frac{1}{2!}\Big($  $\Big)^2$

C : 

D : 

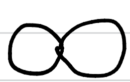
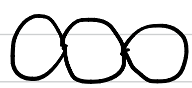
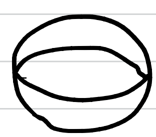
. . . .

$Z_{pert} / Z_{free} = 1 +$  $+ \frac{1}{2!}\Big($  $\Big)^2 +$  $+$  $+ \cdots$

sum of vacuum diagrams

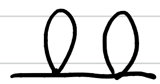
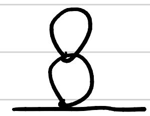
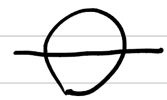
↑

no external line

$\overset{!}{=} \exp\Big($  $+$  $+$  $+ \cdots \Big)$

Sum of connected vacuum diagrams

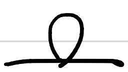
$$\langle \phi\phi \rangle_{pert} = \frac{\langle \phi\phi \rangle_{free} + \left\langle \phi\phi \left(-\frac{\lambda}{4!}\phi^4\right) \right\rangle_{free} + \cdots}{1 + \left\langle -\frac{\lambda}{4!}\phi^4 \right\rangle_{free} + \cdots}$$

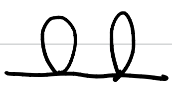
=  +  +  +  +  + ⋯

sum of diagrams with two external lines

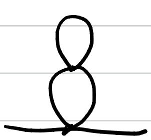
without vacuum diagram ← !! division by $Z_{pert}$

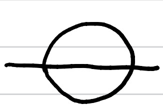
( = sum of connected diagrams $\langle \phi\phi \rangle_{conn}$ )

 $= -\frac{\lambda}{4!}\,\phi(\phi\phi\phi\phi)\phi \times 4\cdot 3 = -\frac{\lambda}{2}(a^{-1})^3$

 $= \frac{1}{2!}\left(-\frac{\lambda}{4!}\right)^2 \phi(\phi\phi\phi\phi)(\phi\phi\phi\phi)\phi \times 2\cdot 4^2\cdot 3^2 = \frac{(-\lambda)^2}{4}(a^{-1})^5$

$(\phi\phi\phi\phi)$

 $= \frac{1}{2!}\left(-\frac{\lambda}{4!}\right)^2 \phi(\phi\phi\phi\phi)\phi \times 2\cdot 4\cdot 3\cdot \binom{4}{2}\cdot 2 = \frac{(-\lambda)^2}{4}(a^{-1})^5$

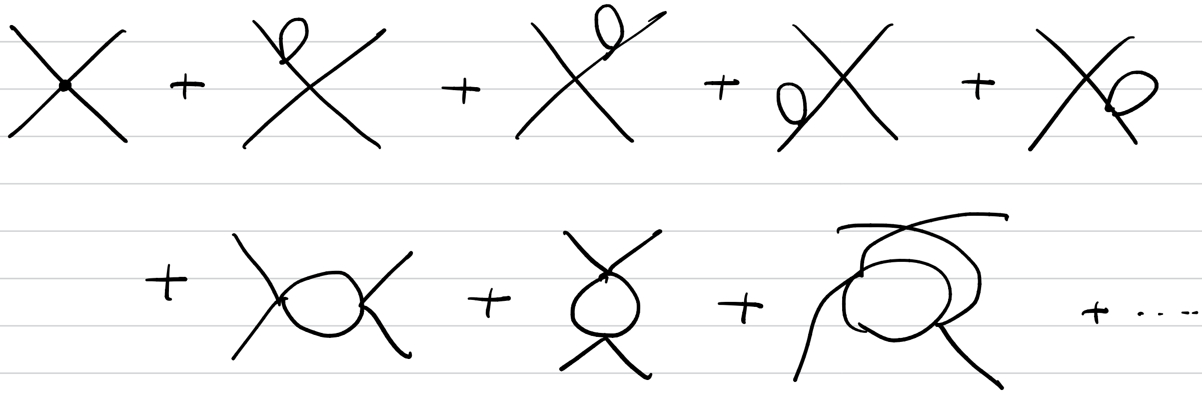
 $= \frac{1}{2!}\left(-\frac{\lambda}{4!}\right)^2 \phi(\phi\phi\phi\phi)(\phi\phi\phi\phi)\phi \cdot 2\cdot 4^2\cdot 3! = \frac{(-\lambda)^2}{6}(a^{-1})^5$

⋮

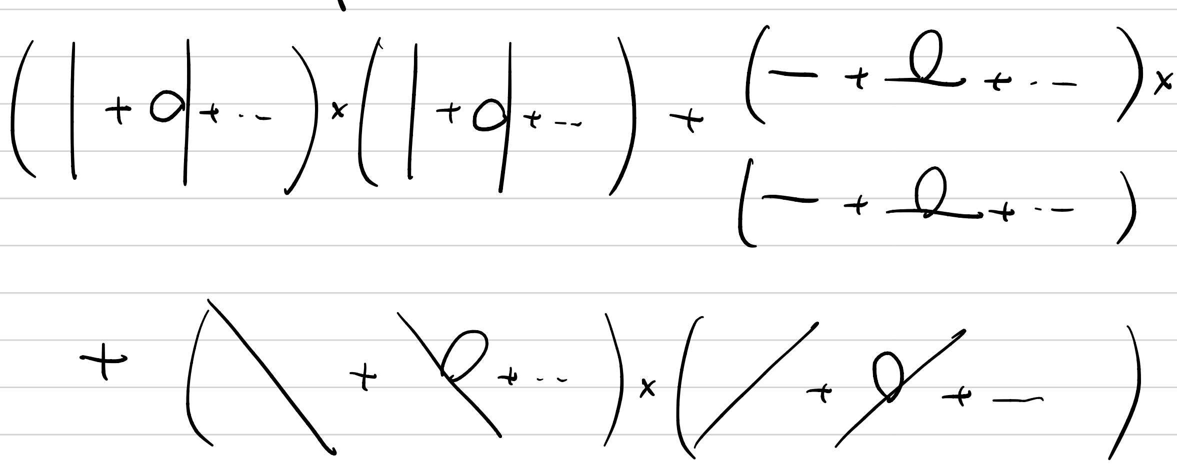
$\langle \phi\phi\phi\phi \rangle_{pert}$ = sum of diagrams with 4 external lines
without vacuum diagrams !!

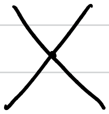
$$\overset{!!}{=} \underbrace{\langle \phi\phi\phi\phi\rangle_{conn}}_{\alpha} + \underbrace{\langle\phi\phi\rangle_{conn}\cdot\langle\phi\phi\rangle_{conn}\times 3}_{\beta}$$

$\alpha$ = connected part

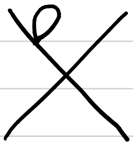


$\beta$ = disconnected part



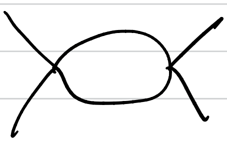


$$= -\frac{\lambda}{4!}\,\phi\,\phi\,\phi\phi\phi\phi\,\phi\,\phi \times 4! = -\lambda(a^{-1})^4,$$



$$= \frac{1}{2!}\left(-\frac{\lambda}{4!}\right)^2 \phi\,\phi\phi\phi\phi\,\phi\phi\phi\phi\,\phi\,\phi\,\phi \times 2\cdot 4\cdot(3\cdot 4)\cdot 3!$$

$$= \frac{(-\lambda)^2}{2}(a^{-1})^6,$$



$$= \frac{1}{2!}\left(-\frac{\lambda}{4!}\right)^2 \phi\,\phi\,\phi\phi\phi\phi\,\phi\phi\phi\phi\,\phi\,\phi \times 2\cdot(4\cdot 3)^2\cdot 2$$

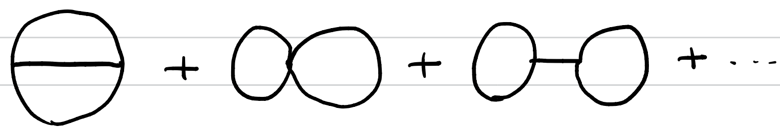
$$= \frac{(-\lambda)^2}{2}(a^{-1})^6,$$

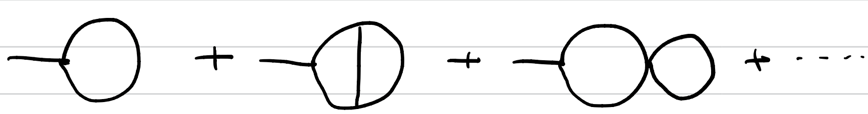
$$\vdots$$

# Generalizations

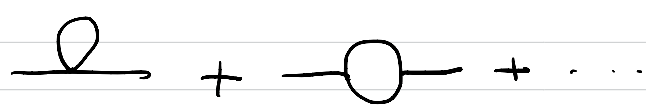
## more interactions

$$S_E(\phi) = \underbrace{\frac{1}{2} a \phi^2}_{\text{free}} + \underbrace{\frac{\lambda_3}{3!}\phi^3 + \frac{\lambda_4}{4!}\phi^4 \quad (+ \cdots)}_{\text{interactions}}$$

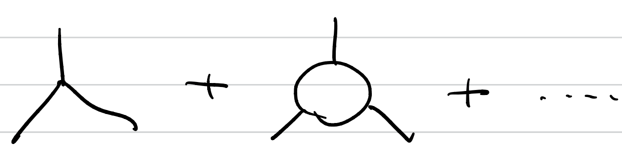
$Z_{pert}/Z_{free} = \exp\Big($  $\Big)$

$\langle \phi \rangle_{pert} =$  $\big( = \langle \phi \rangle_{conn} \big)$

$$\langle \phi\phi \rangle_{pert} = \langle \phi\phi \rangle_{conn} + \langle \phi \rangle_{conn} \cdot \langle \phi \rangle_{conn}$$



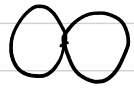
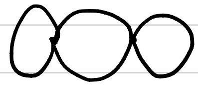
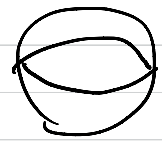
$$\langle \phi\phi\phi \rangle_{pert} = \langle \phi\phi\phi \rangle_{conn} + \langle \phi \rangle_{conn} \langle \phi\phi \rangle_{conn} \times 3 + \langle \phi \rangle_{conn}^3$$

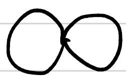


⋮

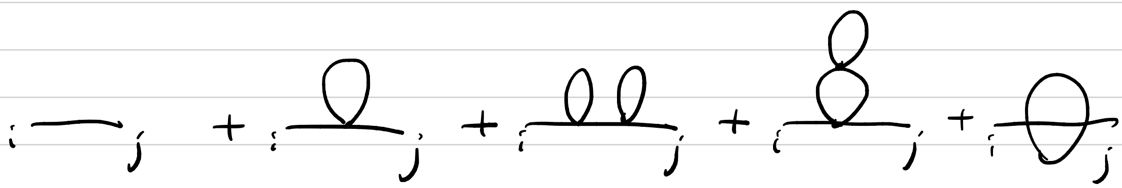
## n variables

$$S_E(\phi) = \underbrace{\frac{1}{2}\sum_{ij} \phi_i A_{ij} \phi_j}_{\text{free}} + \underbrace{\frac{\lambda}{4!}\sum_i \phi_i^4}_{\text{interaction}}$$

$Z_{pert}/Z_{free} = \exp\Big($  $+$  $+$  $+ \cdots \Big)$

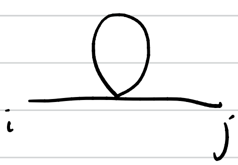
 $= -\dfrac{\lambda}{4!}\sum_i \phi_i\phi_i\phi_i\phi_i \times 3 = \dfrac{-\lambda}{4\cdot 2}\sum_i (A^{-1})_{ii}^2$

$\vdots$

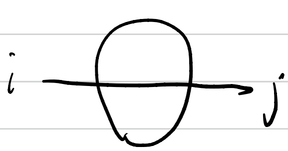
$\langle \phi_i \phi_j \rangle_{pert} =$ 

$= \langle \phi_i \phi_j \rangle_{conn}$

 $= A^{-1}_{ij}$

 $= -\dfrac{\lambda}{4!}\sum_k \phi_i(\phi_k\phi_k\phi_k\phi_k)\phi_j \times 4\cdot 3$

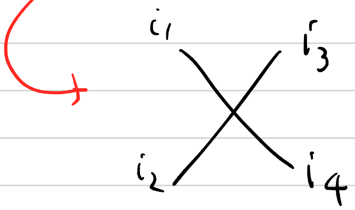
$$= \frac{-\lambda}{2}\sum_k A^{-1}_{ik} A^{-1}_{kk} A^{-1}_{kj},$$

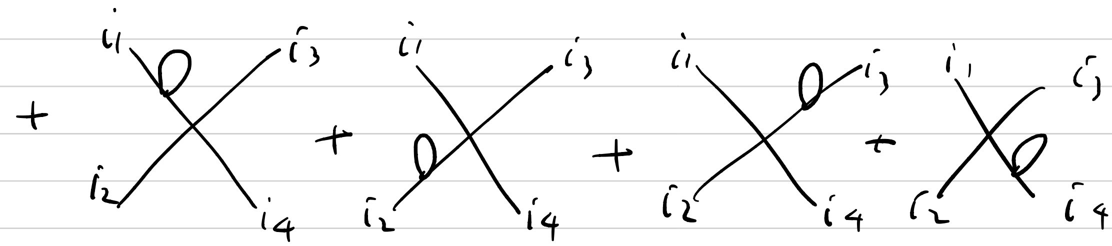
 $= \dfrac{1}{2!}\left(\dfrac{-\lambda}{4!}\right)^2 \sum_{k,l} \phi_i \phi_k\phi_k\phi_k\phi_k \phi_l\phi_l\phi_l\phi_l \phi_j \times 2\cdot 4^2 \cdot 3!$

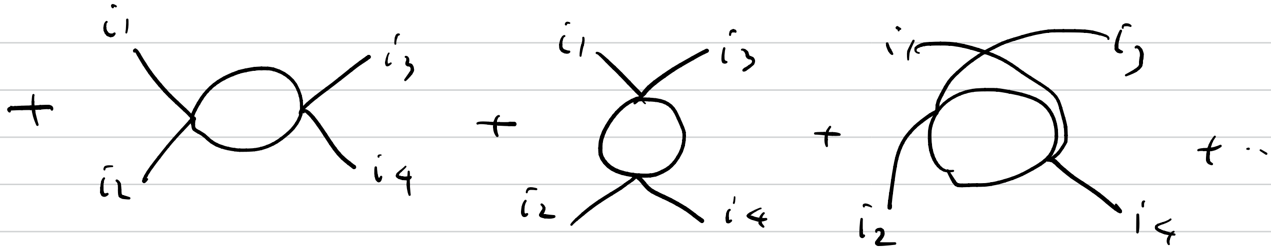
$$= \frac{(-\lambda)^2}{6}\sum_{kl} A^{-1}_{ik}(A^{-1}_{kl})^3 A^{-1}_{lj},$$

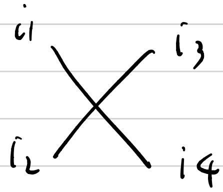
$$\langle \phi_{i_1}\phi_{i_2}\phi_{i_3}\phi_{i_4}\rangle_{pert} = \langle \phi_{i_1}\phi_{i_2}\phi_{i_3}\phi_{4}\rangle_{conn} + \langle \phi_{i_1}\phi_{i_2}\rangle_{conn}\langle \phi_{i_3}\phi_{i_4}\rangle_{conn}$$

$$+ (i_1 i_3)(i_2 i_4) + (i_1 i_4)(i_2 i_3)$$



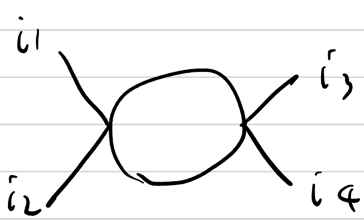




$$= -\frac{\lambda}{4!}\sum_j \phi_j\phi_j\phi_j\phi_j \times 4!$$

$\phi_{i_1}$ $\phi_{i_3}$ $\phi_{i_2}$ $\phi_{i_4}$

$$= -\lambda \sum_j A^{-1}_{i_1 j} A^{-1}_{i_2 j} A^{-1}_{i_3 j} A^{-1}_{i_4 j}$$



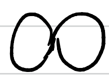
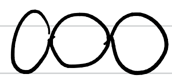
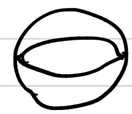

$$= \frac{1}{2!}\left(-\frac{\lambda}{4!}\right)^2 \sum_j \phi_j\phi_j\phi_j\phi_j \sum_k \phi_k\phi_k\phi_k\phi_k \times 2\cdot(4\cdot 3)^2\cdot 2$$

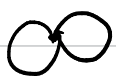
$\phi_{i_1}$ $\phi_{i_3}$ $\phi_{i_2}$ $\phi_{i_4}$

$$= \frac{(-\lambda)^2}{2}\sum_{kl} A^{-1}_{i_1 j} A^{-1}_{i_2 j} \left(A^{-1}_{jk}\right)^2 A^{-1}_{k i_3} A^{-1}_{k i_4}$$

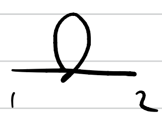
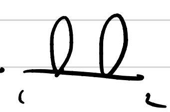
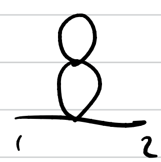
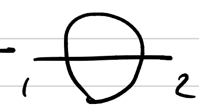
## Scalar field in d-dimensions

$$S_E[\phi] = \int d^dx \left( \underbrace{\frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2}_{\text{free}} + \underbrace{\frac{\lambda}{4!}\phi^4}_{\text{interaction}} \right) \qquad \text{"}\phi^4\text{-theory"}$$

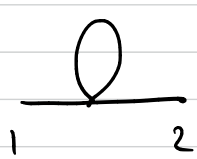
$Z_{pert}/Z_{free} = \exp\Big($  $+$  $+$  $+ \cdots \Big)$

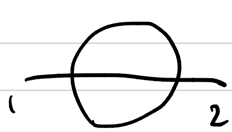
 $= \dfrac{-\lambda}{4\cdot 2}\displaystyle\int d^dx \left(\overbrace{\phi(x)\,\phi(x)}\right)^2$

⋮

$\langle \phi(x_1)\,\phi(x_2)\rangle_{pert} =$  $+$  $+$  $+$  $+$ 

 $= \overbrace{\phi(x_1)\,\phi(x_2)} = \displaystyle\int \frac{d^dp}{(2\pi)^d}\,\frac{e^{-ip(x_1-x_2)}}{p^2+m^2}$

 $= \dfrac{-\lambda}{2}\displaystyle\int d^dx\ \overbrace{\phi(x_1)\,\phi(x)}\ \overbrace{\phi(x)\phi(x)}\ \overbrace{\phi(x)\ \phi(x_2)}$

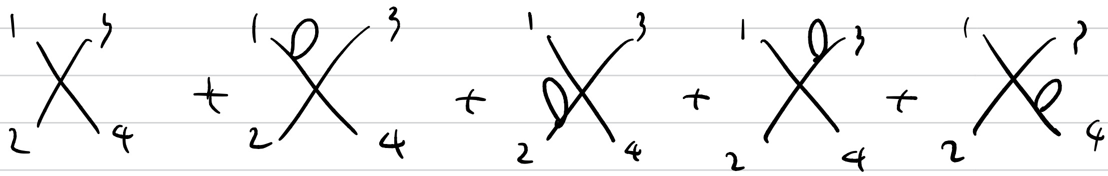
 $= \dfrac{(-\lambda)^2}{6}\displaystyle\int d^dx\, d^dy\ \overbrace{\phi(x_1)\phi(x)}\left(\overbrace{\phi(x)\,\phi(y)}\right)^3 \overbrace{\phi(y)\,\phi(x_2)}$

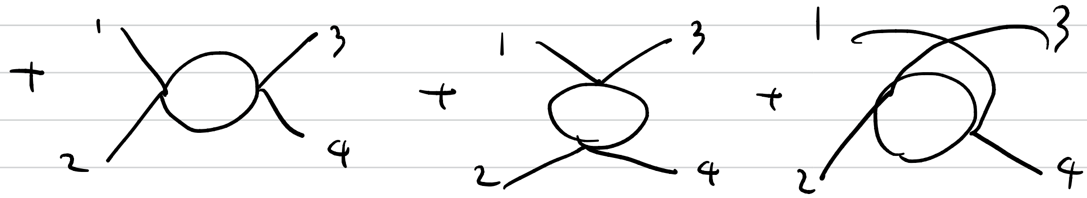
⋮

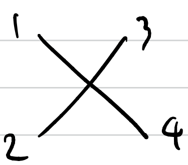
$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle_{pert} = \langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle_{conn}$$

$$+ \langle \phi(x_1)\phi(x_2)\rangle_{conn}\langle \phi(x_3)\phi(x_4)\rangle_{conn}$$

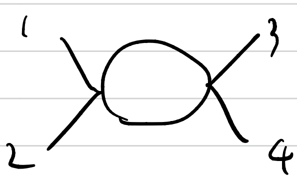
$$+ (13)(24) + (14)(23)$$







$$= -\lambda \int d^d x \prod_{i=1}^{4} \overbrace{\phi(x_i)\phi(x)}$$



$$= \frac{(-\lambda)^2}{2} \int d^d x\, d^d y\; \overbrace{\phi(x_1)\phi(x)}\;\overbrace{\phi(x_2)\phi(x)}\left(\overbrace{\phi(x)\phi(y)}\right)^2$$

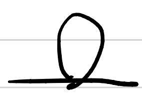
$$\times\; \overbrace{\phi(y)\phi(x_3)}\;\overbrace{\phi(y)\phi(x_4)}$$

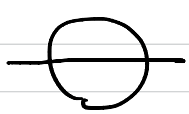
$\vdots$

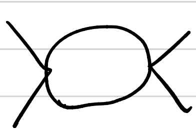
The propagator has short distance singularity if $d>1$:

$$\overline{\phi(x)\phi(y)} \overset{x\sim y}{\sim} \begin{cases} \dfrac{C}{|x-y|^{d-2}} & d>2 \\ -C\log|x-y| & d=2 \end{cases}$$

Thus some of the diagrams are divergent

-  has $\overline{\phi(x)\phi(x)} \cdots$ $\begin{cases} \log & d=2 \\ \text{linear} & d=3 \\ \text{quadratic} & d=4 \\ \vdots & \vdots \end{cases}$

-  has $\int d^dx\, \overline{\phi(x)\phi(0)}^3 \cdots$ $\begin{cases} \log & d=3 \\ \text{quadratic} & d=4 \\ \vdots \end{cases}$

-  has $\int d^dx\, \overline{\phi(x)\phi(0)}^2 \cdots$ $\begin{cases} \log & d=4 \\ \text{linear} & d=5 \\ \vdots \end{cases}$

$\boxed{\text{Correct treatment of such divergences is the subject of regularization and renormalization.}}$

# Minkowski limit

$$S[\phi] = \int d^d x \, \Big( \underbrace{\tfrac{1}{2}\partial^\mu \phi \, \partial_\mu \phi - \tfrac{m^2}{2}\phi^2}_{\text{free}} \underbrace{- \tfrac{\lambda}{4!}\phi^4}_{\text{interaction}} \Big)$$

$Z_{pert}$, $\langle \phi(x_1) \cdots \phi(x_s) \rangle_{pert}$ is obtained from the result of Euclidean theory by the replacement

$$\phi(x)\,\phi(y) \longrightarrow \int \frac{d^d p}{(2\pi)^d} \frac{i\, e^{-ip(x-y)}}{p^2 - m^2 + i\cdot 0}$$

$$-\lambda \int d^d x_E \longrightarrow -i\lambda \int d^d x$$

# Gauge theory (e.g. QCD)

variables: $A_\mu$ gauge potential of gauge group $G$

· $\psi$ Dirac fermion in a representation $R$ of $G$

$$\mathcal{L} = -\frac{1}{4e^2} F^{\mu\nu} \cdot F_{\mu\nu} + i\bar{\psi} \not{D}_A \psi - m\bar{\psi}\psi$$

gauge fixing by $\partial^\mu A_\mu = 0$

$$\rightsquigarrow \tilde{\mathcal{L}} = \mathcal{L} - \frac{1}{2e^2\xi}(\partial^\mu A_\mu)^2 + i\partial^\mu \bar{c} D_\mu c \; ; \; c, \bar{c} : \text{F.P. ghosts}$$

$$\tilde{\mathcal{L}} = \mathcal{L}_{free} + \mathcal{L}_{int} \qquad (\text{rescale } A_\mu \to eA_\mu)$$

$$\mathcal{L}_{free} = -\frac{1}{4}(\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu) - \frac{1}{2\xi}(\partial^\mu A_\mu)^2$$
$$+ i\bar{\psi}\not{\partial}\psi - m\bar{\psi}\psi + i\partial^\mu \bar{c}\, \partial_\mu c$$

$$\mathcal{L}_{int} = -\frac{e}{2}(\partial^\mu A^\nu - \partial^\nu A^\mu) \cdot [A_\mu, A_\nu] - \frac{e^2}{4}[A^\mu, A^\nu] \cdot [A_\mu, A_\nu]$$
$$+ ie\bar{\psi}\not{A}\psi + ie\,\partial^\mu \bar{c}[A_\mu, c]$$

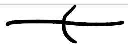
Perturbative expansion of partition / correlation functions

can be computed using

$$A_\mu = e^a A_{\mu a}\,, \quad c = e^a c_a\,, \quad \bar{c} = e^a \bar{c}_a$$
$$\psi = e^i \psi_i\,.$$

propagators

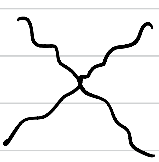
 $= \contraction{}{A}{{}_{\mu a}(x)}{A} A_{\mu a}(x) A_{\nu b}(y) = \delta_{ab} \displaystyle\int \frac{d^dp}{(2\pi)^d} \frac{-i\, e^{-ip(x-y)}}{p^2+i\cdot 0} \left( \eta_{\mu\nu} - (1-\xi) \frac{p_\mu p_\nu}{p^2+i\cdot 0} \right),$

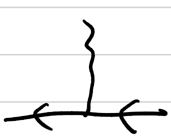
 $= \contraction{}{\psi}{{}_i(x)}{\bar\psi} \psi_i(x) \bar\psi^j(y) = \delta_i^j \displaystyle\int \frac{d^dp}{(2\pi)^d} \frac{i\, e^{-ip(x-y)}}{p^2-m^2+i\cdot 0} (\not{p}+m),$

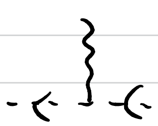
 $= \contraction{}{c}{{}_a(x)}{\bar c} c_a(x) \bar{c}_b(y) = \delta_{ab} \displaystyle\int \frac{d^dp}{(2\pi)^2} \frac{e^{-ip(x-y)}}{p^2+i\cdot 0}\,,$ and

vertices

 $= -\dfrac{ie}{2} \displaystyle\int d^dx \left( \partial^\mu A^\nu - \partial^\nu A^\mu \right) \cdot [A_\mu, A_\nu]\,,$

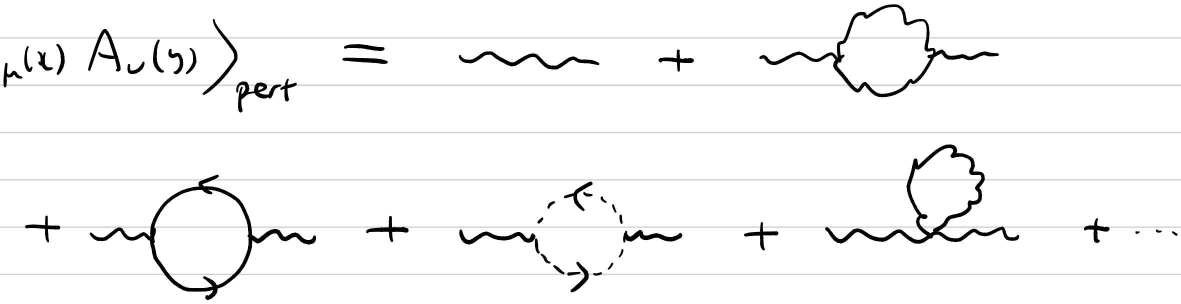
 $= -\dfrac{ie^2}{4} \displaystyle\int d^dx\, [A^\mu, A^\nu] \cdot [A_\mu, A_\nu]\,,$

 $= ie \displaystyle\int d^dx\, i \bar\psi \not{A} \psi\,,$

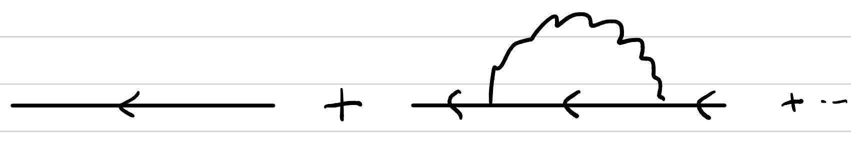
 $= ie \displaystyle\int d^dx\, i\, \partial^\mu \bar{c} \cdot [A_\mu, c]\,.$

e.g.,

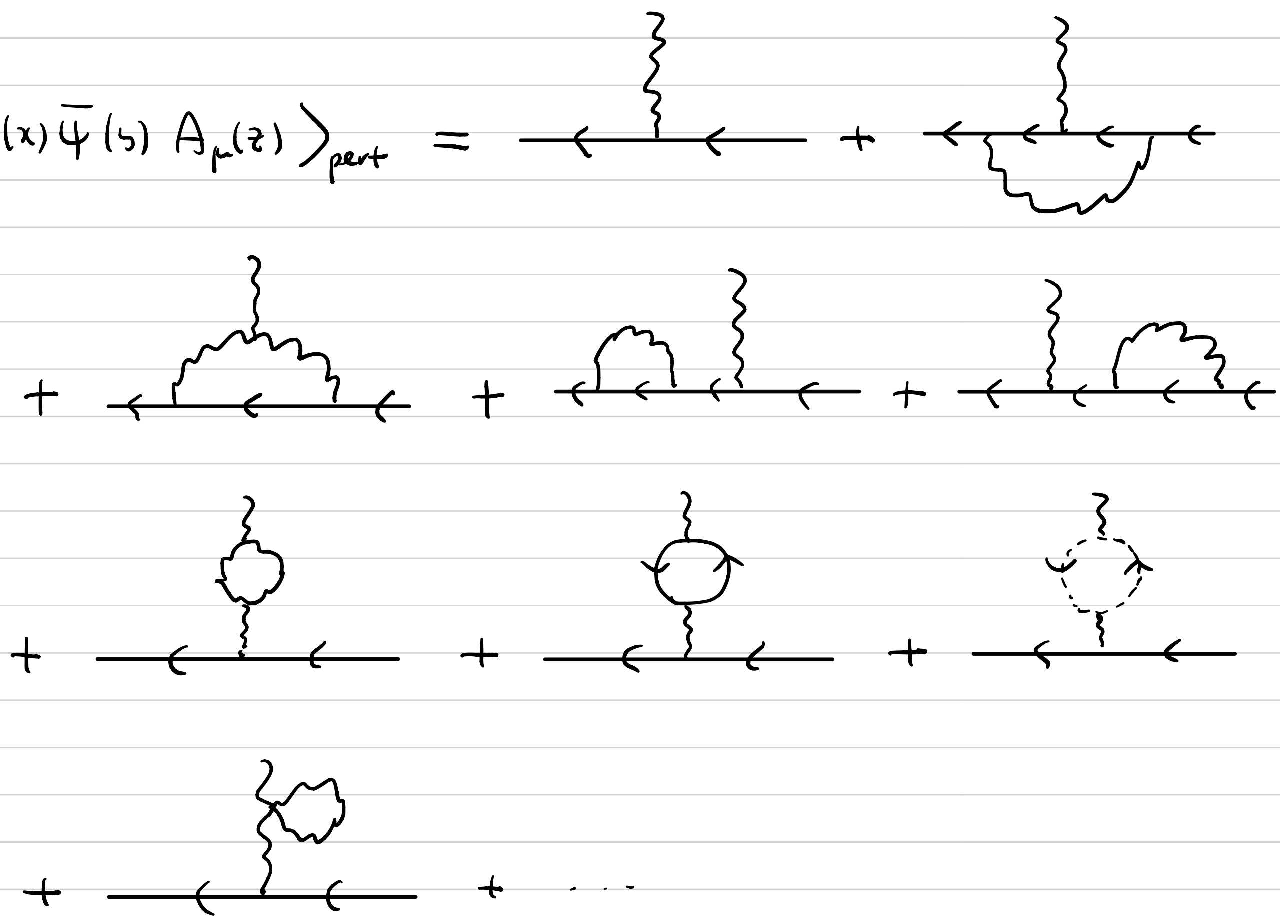
$$\langle A_\mu(x)\, A_\nu(y) \rangle_{pert} =$$



$$\langle \psi(x)\, \bar{\psi}(y) \rangle_{pert} =$$



$$\langle \psi(x)\, \bar{\psi}(y)\, A_\mu(z) \rangle_{pert} =$$



⋮

In general,

- $Z_{pert}/Z_{free}$ = sum of vacuum diagrams

$\overset{!}{=}$ exp ( sum of connected vacuum diagrams )

- For $\Phi_1, \cdots, \Phi_s$ = elementary fields inserted at points

e.g. $\phi(x_1) \cdots \phi(x_s)$ in $\phi^4$ theory

e.g. $A_{\mu_1}(x_1) \cdots, \psi(y_1), \cdots, \bar{\psi}(z_1), \cdots$ in QCD

$$\langle \Phi_1 \cdots \Phi_s \rangle_{pert} \overset{!!}{=} \sum_{\substack{\{1,\cdots,s\} \\ = I_1 \cup \cdots \cup I_\ell}} \underbrace{\pm}_{} \left\langle \prod_{i \in I_1} \Phi_{i_1} \right\rangle_{conn} \cdots \left\langle \prod_{i_\ell \in I_\ell} \Phi_{i_s} \right\rangle_{conn}$$

permutation of fermionic $\Phi_i$'s

Sum over decompositions of $\{1, \cdots, s\}$ to non-empty subsets $I_1, \cdots, I_\ell \subset \{1, \cdots, s\}$

* In particular, no term has a vacuum diagram factor.