Massive vector field

Consider the theory of a field A_μ in d-dimensions with Lagrangian

$$\mathcal{L} = -\frac{1}{4e^2} F^{\mu\nu} F_{\mu\nu} + \frac{m^2}{2e^2} A^\mu A_\mu .$$

(i) Path integral

Compute $\langle A_\mu(x) A_\nu(y) \rangle$.

(ii) Canonical quantization

The action may be written as

$$S = \frac{1}{2e^2} \int d^dx \left\{ \sum_{i,j} A_i \left(\delta_{ij} (-\partial_0^2 + \nabla^2 - m^2) - \partial_i \partial_j \right) A_j \right.$$
$$\left. + A_0 (-\nabla^2 + m^2) A_0 + A_0 \partial_0 \nabla \cdot \mathbb{A} + \mathbb{A} \cdot \nabla \partial_0 A_0 \right\}$$

We first notice that $\dot{A}_0$ is absent in $\mathcal{L}$, i.e. does not have a kinetic term. Thus, as in Maxwell theory, A_0 is not a dynamical variable. However, unlike in Maxwell theory where A_0 enters in $\mathcal{L}$ linearly and is a Lagrange multiplier field, A_0 enters in $\mathcal{L}$ quadratically.

A_0 is an auxiliary field that can be integrated out.

Integrating out is best done in momentum space where the operator $(-\nabla^2 + m^2)$ of quadratic term is diagonalized.

Thus,

$$A_\mu(x) = \int \frac{d^{d-1}\mathbb{P}}{(2\pi)^{d-1}} e^{i\mathbb{P}\cdot x} A_\mu(\mathbb{P}) \quad : \quad A_\mu(\mathbb{P})^* = A_\mu(-\mathbb{P}).$$

We further write $A_i(\mathbb{P})$ as

$$A_i(\mathbb{P}) = \sum_{I=1}^{d-1} \epsilon_i^I(\mathbb{P})\, \phi_I(\mathbb{P}),$$

with polarization vectors $\epsilon_i^I(\mathbb{P})$, $I = 1, \cdots, d-1$.

We assume

① $\epsilon_i^I(\mathbb{P})^* = \epsilon_i^I(-\mathbb{P})$ so that $\phi_I(\mathbb{P})^* = \phi_I(-\mathbb{P})$

② Orthonormal: $\sum_i \epsilon_i^I(\mathbb{P})^* \epsilon_j^I(\mathbb{P}) = \delta^{IJ}$

③ $\epsilon_j^1(\mathbb{P}) = i\,\mathbb{P}_j / |\mathbb{P}|$.

In particular, $\sum_i \mathbb{P}_i\, \epsilon_i^\alpha(\mathbb{P}) = 0$ for $\alpha = 2, \cdots, d-1$.

Also $\quad \mathbb{P}\cdot\mathbb{A}(\mathbb{P}) = i|\mathbb{P}|\,\phi_1(\mathbb{P})$.

Lagrangian can then be written as

$$L = \frac{1}{2e^2}\int \frac{d^{d-1}\mathbb{p}}{(2\pi)^{d-1}} \Big\{ \sum_{I=1}^{d-1} \left(\dot\phi_I(-\mathbb{p})\,\phi_I(\mathbb{p}) - \phi_I(-\mathbb{p})(\mathbb{p}^2+m^2)\,\phi_I(\mathbb{p}) \right)$$

$$+ \phi_1(-\mathbb{p})\,\mathbb{p}^2\,\phi_1(\mathbb{p}) + A_0(-\mathbb{p})(\mathbb{p}^2+m^2)A_0(\mathbb{p})$$

$$- A_0(-\mathbb{p})\,|\mathbb{p}|\,\dot\phi_1(\mathbb{p}) - |\mathbb{p}|\,\dot\phi_1(-\mathbb{p})\,A_0(\mathbb{p}) \Big\}$$

$$\left(A_0(-\mathbb{p}) - \frac{|\mathbb{p}|}{\mathbb{p}^2+m^2}\dot\phi_1(-\mathbb{p}) \right)(\mathbb{p}^2+m^2)\left(A_0(\mathbb{p}) - \frac{|\mathbb{p}|}{\mathbb{p}^2+m^2}\dot\phi_1(\mathbb{p}) \right)$$

$$- \dot\phi_1(-\mathbb{p})\frac{\mathbb{p}^2}{\mathbb{p}^2+m^2}\dot\phi_1(\mathbb{p})$$

$$\simeq \frac{1}{2e^2}\int \frac{d^{d-1}\mathbb{p}}{(2\pi)^{d-1}} \Big\{ \frac{m^2}{\mathbb{p}^2+m^2}\dot\phi_1(-\mathbb{p})\,\dot\phi_1(\mathbb{p}) - m^2\phi_1(-\mathbb{p})\,\phi_1(\mathbb{p})$$

$$+ \sum_{\alpha=2}^{d-1}\left(\dot\phi_\alpha(-\mathbb{p})\,\phi_\alpha(\mathbb{p}) - (\mathbb{p}^2+m^2)\,\phi_\alpha(-\mathbb{p})\,\phi_\alpha(\mathbb{p}) \right) \Big\}$$

$$\left.\begin{aligned} \Pi_1(\mathbb{p}) &= \frac{m^2}{(2\pi)^{d-1}e^2(\mathbb{p}^2+m^2)}\dot\phi_1(-\mathbb{p}) \\ \Pi_\alpha(\mathbb{p}) &= \frac{1}{(2\pi)^{d-1}e^2}\dot\phi_\alpha(-\mathbb{p}) \end{aligned}\right\} \quad \Pi_I(\mathbb{p})^* = \Pi_I(-\mathbb{p})$$

$$H = \int d^{d-1}\mathbb{p} \sum_{I=1}^{d-1} \Pi_I(\mathbb{p})\,\dot\phi_I(\mathbb{p}) - L$$

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$$= \frac{1}{2}\int d^{d-1}\mathbb{p}\, \Bigg\{ \frac{(2\pi)^{d-1} e^2 (\mathbb{p}^2+m^2)}{m^2} \Pi_1(\mathbb{p})\Pi_1(-\mathbb{p}) + \frac{m^2}{(2\pi)^{d-1}e^2}\phi_1(-\mathbb{p})\phi_1(\mathbb{p})$$

$$+ \sum_{\alpha=2}^{d-1}\left((2\pi)^{d-1}e^2\, \Pi_\alpha(\mathbb{p})\Pi_\alpha(-\mathbb{p}) + \frac{\mathbb{p}^2+m^2}{(2\pi)^{d-1}e^2}\phi_\alpha(-\mathbb{p})\phi_\alpha(\mathbb{p}) \right) \Bigg\}$$

Canonical quantization:

$$[\phi_I(\mathbb{p}_1), \Pi_J(\mathbb{p}_2)] = i\,\delta_{IJ}\,\delta^{d-1}(\mathbb{p}_1-\mathbb{p}_2)$$

$$[\phi_I(\mathbb{p}_1), \phi_J(\mathbb{p}_2)] = [\Pi_I(\mathbb{p}_1), \Pi_J(\mathbb{p}_2)] = 0$$

$$\phi_I(\mathbb{p})^\dagger = \phi_I(-\mathbb{p}), \quad \Pi_I(\mathbb{p})^\dagger = \Pi_I(-\mathbb{p}) \qquad \omega_{\mathbb{p}} = \sqrt{\mathbb{p}^2+m^2}$$

For

$$a_1(\mathbb{p}) = \sqrt{\frac{m^2}{2(2\pi)^{d-1}e^2\omega_{\mathbb{p}}}}\,\phi_1(\mathbb{p}) + i\sqrt{\frac{(2\pi)^{d-1}e^2\omega_{\mathbb{p}}}{2m^2}}\,\Pi_1(-\mathbb{p}),$$

$$a_\alpha(\mathbb{p}) = \sqrt{\frac{\omega_{\mathbb{p}}}{2(2\pi)^{d-1}e^2}}\,\phi_\alpha(\mathbb{p}) + i\sqrt{\frac{(2\pi)^{d-1}e^2}{2\omega_{\mathbb{p}}}}\,\Pi_\alpha(-\mathbb{p}) \qquad \alpha \geq 2,$$

$$[a_I(\mathbb{p}_1), a_J(\mathbb{p}_2)^\dagger] = \delta_{I,J}\,\delta^{d-1}(\mathbb{p}_1-\mathbb{p}_2),$$

$$[a_I(\mathbb{p}_1), a_J(\mathbb{p}_2)] = [a_I(\mathbb{p}_1)^\dagger, a_J(\mathbb{p}_2)^\dagger] = 0, \quad \text{and}$$

$$H = \int d^{d-1}\mathbb{p} \sum_{I=1}^{d-1} \omega_{\mathbb{p}} \left(a_I(\mathbb{p})^\dagger a_I(\mathbb{p}) + \frac{1}{2}\delta^{d-1}(0) \right)$$

$$[H, a_I(\mathbb{p})] = -\omega_{\mathbb{p}}\, a_I(\mathbb{p}), \quad [H, a_I(\mathbb{p})^\dagger] = \omega_{\mathbb{p}}\, a_I(\mathbb{p}).$$

$\therefore$ $a_I(\mathbb{p})^\dagger / a_I(\mathbb{p})$ are creation/annihilation operators.

The system has a unique ground state $|0\rangle$, which is annihilated by all $a_I(\mathbb{p})$'s, with energy $E_0 = \int d^{d-1}\mathbb{p}\, \delta^{d-1}(0) \frac{1}{2}\omega_{\mathbb{p}}$.

Other states are obtained by operating $a_I(\mathbb{p})^\dagger$'s on $|0\rangle$, each operation increasing energy by $\omega_{\mathbb{p}}$.

$$A_0(\mathbb{x}) = e \int \frac{d^{d-1}\mathbb{p}}{\sqrt{(2\pi)^{d-1} 2\omega_{\mathbb{p}}}} \frac{i|\mathbb{p}|}{m} \left(-e^{i\mathbb{p}\cdot\mathbb{x}} a_1(\mathbb{p}) + e^{-i\mathbb{p}\cdot\mathbb{x}} a_1(\mathbb{p})^\dagger\right),$$

$$A_i(\mathbb{x}) = e \int \frac{d^{d-1}\mathbb{p}}{\sqrt{(2\pi)^{d-1} 2\omega_{\mathbb{p}}}} \sum_{I=1}^{d-1} \Big(e^{i\mathbb{p}\cdot\mathbb{x}} \hat{\epsilon}_i^I(\mathbb{p})\, a_I(\mathbb{p})$$

$$+ e^{-i\mathbb{p}\cdot\mathbb{x}} \hat{\epsilon}_i^I(\mathbb{p})^* a_I(\mathbb{p})^\dagger \Big)$$

where $\hat{\epsilon}_i^1(\mathbb{p}) = \frac{\omega_{\mathbb{p}}}{m} \epsilon_i^1(\mathbb{p})$, $\hat{\epsilon}_i^\alpha(\mathbb{p}) = \epsilon_i^\alpha(\mathbb{p})$

$$A_0(t,\mathbf{x}) = e^{itH} A_0(\mathbf{x})\, e^{-itH}$$

$$= e \int \frac{d^{d-1}\mathbb{P}}{\sqrt{(2\pi)^{d-1} 2\omega_{\mathbb{P}}}} \frac{i|\mathbb{P}|}{m} \left(-e^{i\mathbb{P}\cdot\mathbf{x} - i\omega_{\mathbb{P}} t} a_1(\mathbb{P}) + e^{-i\mathbb{P}\cdot\mathbf{x} + i\omega_{\mathbb{P}} t} a_1(\mathbb{P})^\dagger \right)$$

$$A_i(t,\mathbf{x}) = e^{itH} A_i(\mathbf{x})\, e^{-itH}$$

$$= e \int \frac{d^{d-1}\mathbb{P}}{\sqrt{(2\pi)^{d-1} 2\omega_{\mathbb{P}}}} \sum_{I=1}^{d-1} \Big(e^{i\mathbb{P}\cdot\mathbf{x} - i\omega_{\mathbb{P}} t} \hat{\epsilon}^{I}_{i}(\mathbb{P})\, a_I(\mathbb{P})$$

$$+ e^{-i\mathbb{P}\cdot\mathbf{x} + i\omega_{\mathbb{P}} t} \hat{\epsilon}^{I}_{i}(\mathbb{P})^{*} a_I(\mathbb{P})^\dagger \Big)$$

Compute $\langle 0 | T A_\mu(x) A_\nu(y) | 0 \rangle$ and compare it with the result of (i).